Tight frames of multivariate orthogonal polynomials

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ABSTRACT

We show (by examples) that tight frame decompositions are useful and natural for finite dimensional Hilbert spaces which have symmetries, in particular for spaces of multivariate orthogonal polynomials.
A question

Let $u_1, u_2, u_3$ be three equally spaced unit vectors in $\mathbb{R}^2$.

For a given nonzero vector $f \in \mathbb{R}^2$, what is the sum of its orthogonal projections onto these vectors?

\[(a) \quad \sum_{j=1}^{3} \langle f, u_j \rangle u_j = 0 \quad \text{(since } u_1 + u_2 = u_3 = 0).\]

\[(b) \quad \sum_{j=1}^{3} \langle f, u_j \rangle u_j = \frac{3}{2} f, \quad \forall f \in \mathbb{R}^2.\]
Frames in finite dimensional spaces

The following sets of vectors $\{v_j\}_{j=1}^{3}$ form tight frames for $\mathbb{R}^2$

i.e., give decompositions of the form

$$f = \sum_{j=1}^{3} \langle f, v_j \rangle v_j, \quad \forall f \in \mathbb{R}^2.$$  

This is technically similar to an orthogonal expansion, except it has more terms (redundancy).
The start of a (long) story

The **Bernstein operator** $B_n : C([0, 1]) \rightarrow \Pi_n$ is defined by

\[
B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).
\]

In [Cooper, Waldron 2000] it was shown $B_n$ has the diagonal form

\[
B_n f = \sum_{k=0}^{n} \lambda_k^{(n)} p_k^{(n)}(f),
\]

where the eigenvalues $1 = \lambda_0^{(n)} = \lambda_1^{(n)} > \lambda_2^{(n)} > \cdots > \lambda_n^{(n)} > 0$ are

\[
\lambda_k^{(n)} := 1 \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right)
\]

and the corresponding eigenfunctions have the form

\[
p_k^{(n)}(x) = x^k - \frac{k}{2} x^{k-1} + \text{lower order terms}.
\]
The limiting eigenfunctions

The Bernstein operator converges as $n \to \infty$

$$B_n f = \sum_{k=0}^{n} \lambda_k^{(n)} p_k^{(n)} \mu_k^{(n)}(f)$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$f = \sum_{k=0}^{\infty} 1 \cdot p_k^* \cdot \mu_k^*(f),$$

where the “limit” eigenfunctions $p_k^*$ are related to the Jacobi polynomials (similarly for the multivariate Bernstein operator).

Fig. The first few limit eigenfunctions $p_k^*$. 
Jacobi polynomials on a simplex

Let $T = \text{conv}(V)$ be a simplex in $\mathbb{R}^d$ with $d + 1$ vertices $V$, with corresponding barycentric coordinates $\xi = (\xi_v)_{v \in V}$, and define the Jacobi inner product

$$\langle f, g \rangle_\nu := \int_T fg \xi^{\nu-1}, \quad \nu = (\nu_v)_{v \in V} > 0.$$ 

e.g., for $d = 2$, $T = \text{conv}\{e_1, e_2, 0\}$, $\nu - 1 = (\alpha, \beta, \gamma)$

$$\xi_{e_1}(x, y) = x$$

$$\xi_{e_2}(x, y) = y$$

$$\xi_0(x, y) = 1 - x - y$$

$$\langle f, g \rangle_\nu = \int_0^1 \int_0^{1-x} f(x, y)g(x, y) x^\alpha y^\beta (1 - x - y)^\gamma \, dy \, dx$$

The Jacobi polynomials of degree $k$ are

$$\mathcal{P}_k^\nu := \{ f \in \Pi_k : \langle f, p \rangle_\nu = 0, \forall p \in \Pi_{k-1} \}.$$ 

This space has

$$\dim(\mathcal{P}_k^\nu) = \binom{k + d - 1}{d - 1}.$$ 

Each polynomial in $\mathcal{P}_k^\nu$ is uniquely determined by its leading term, e.g., for $\xi_0^2 +$ lower order terms, the leading term is

$$\{(1 - x - y)^2\}_{\downarrow} = x^2 - 2xy + y^2.$$
Orthogonal and biorthogonal systems

We describe the known representations for $P_{k}^{d}$ in terms of the leading terms (for the case $d = 2, k = 2$).

**Biorthogonal system** (Appell 1920’s): partial symmetries

$$x^2, \quad xy, \quad y^2.$$  

**Orthogonal system** (Prorial 1957, et al): no symmetries

$$x^2 + y^2 + 2xy, \quad x^2 - y^2, \quad x^2 - y^2 - 4xy.$$  

For the three dimensional space of all quadratic Jacobi polynomials on the triangle, we want an orthonormal basis with leading terms determined by the six polynomials

$$x^2, \quad xy, \quad y^2, \quad x(1 - x - y), \quad y(1 - x - y), \quad (1 - x - y)^2.$$  

Let

$$\Phi := \{p_{\xi^\alpha} = \xi^\alpha + \text{l.o.t} \in P_2 : |\alpha| = 2\}$$

be these six functions. Then $\Phi$ is a frame for $P_{2}^{d}$ (i.e., it spans) but it is *not* tight. We would like to find contants $c_{\alpha} > 0$ with

$$f = \sum_{|\alpha| = 2} c_{\alpha} \langle f, p_{\xi^\alpha} \rangle p_{\xi^\alpha} = \sum_{|\alpha| = 2} \langle f, \tilde{p}_{\xi^\alpha} \rangle \tilde{p}_{\xi^\alpha}, \quad \forall f \in P_{2}^{d},$$

where $\tilde{p}_{\xi^\alpha} := \sqrt{c_{\alpha}} p_{\xi^\alpha}.$
Signed frames

**Theorem [PW].** Let \( \mathcal{H} \) be Hilbert space of dimension \( d \), and

\[
  n = \begin{cases} 
  \frac{1}{2} d(d+1), & H \text{ real}; \\
  \frac{d^2}{2}, & H \text{ complex}.
\end{cases}
\]

Then for almost every choice of unit vectors \( u_1, \ldots, u_n \) in \( \mathcal{H} \) there are unique scalars \( c_1, \ldots, c_n \) for which

\[
  f = \sum_{j=1}^{n} c_j \langle f, u_j \rangle u_j, \quad \forall f \in \mathcal{H}.
\]

The \( c_j \) can be computed explicitly, some may nonnegative, and

\[
  \sum_{j=1}^{n} c_j = d = \dim(\mathcal{H}).
\]

**Example.** For any three vectors in \( \mathbb{R}^2 \) for which none is a multiple of another, there is a unique such scaling as above.

![Tight signed frames of three vectors in \( \mathbb{R}^2 \) with the signature indicated.](image)

**Fig.** Tight signed frames of three vectors in \( \mathbb{R}^2 \) with the signature indicated.

**Example.** For our six functions \( \Phi \), \( d = \dim(\mathcal{P}_2^\nu) = 3 \), and

\[
  n = \frac{1}{2} d(d + 1) = 6.
\]

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A tight frame for the Jacobi polynomials

Let $\phi_{\alpha}^{\nu}$ be the orthogonal projection of

$$\xi^\alpha / (\nu)_\alpha, \quad |\alpha| = n$$

onto $P_n^{\nu}$, which is given by

$$\phi_{\alpha}^{\nu} := \frac{(-1)^n}{(n + |\nu| - 1)_n} F_A \left( |\alpha| + |\nu| - 1, -\alpha ; \xi \right)$$

$$= \frac{(-1)^n}{(n + |\nu| - 1)_n} \sum_{\beta \leq \alpha} \frac{(n + |\nu| - 1)_{|\beta|} (-\alpha)_\beta \xi^\beta}{\beta !},$$

with $F_A$ the Lauricella function of type $A$.

**Theorem [WXR].** The Jacobi polynomials on a simplex have the tight frame representation

$$f = (|\nu|)_n \sum_{|\alpha| = n} \frac{(\nu)_\alpha}{\alpha !} \langle f, \phi_{\alpha}^{\nu} \rangle \phi_{\alpha}^{\nu}, \quad \forall f \in P_n^{\nu},$$

where the normalisation is $\langle 1, 1 \rangle^{\nu} = 1$.

**Remark.** It can be shown that the polynomials

$$p_{\alpha}^{\nu} := (\nu)_\alpha \phi_{\alpha}^{\nu} = \xi^\alpha + \text{lower order terms}, \quad |\alpha| = n$$

have a limit $p_{\alpha}^*$ as $\nu \to 0^+$, and that $p_{\alpha}^*$ is a limit eigenfunction for the Bernstein operator $B_n$ on the simplex $T$.  

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Well distributed points on the sphere

A number of nice configurations of points on the sphere give isometric (equal length vector) tight frames, e.g.,

These turn out to be examples of the orbit of a single vector $v \in \mathbb{C}^d$ under a finite group $G$ of unitary matrices which form an irreducible representation, i.e.,

$$\text{span}\{gw : g \in G\} = \mathbb{C}^d, \quad \forall w \neq 0.$$  

**Theorem ([VW04]).** If $\text{span}\{gw\}_{g \in G} = \mathbb{C}^d$ for some vector $w$, then one can construct a vector $v$ for which

$$Gv := \{gv : g \in G\}$$

is a tight frame for $\mathbb{C}^d$.  

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A nice example

The group of symmetries of the triangle \( G = D_3 \approx S_3 \) induces a representation on the quadratic Legendre polynomials \( \mathcal{P}_2 \) on the triangle. Since there is a polynomial whose orbit spans \( \mathcal{P}_2 \), we can construct a single polynomial

\[
f = (2\sqrt{5} - 5\sqrt{2}) \left( \xi_v^2 + \xi_w^2 + \xi_u^2 - \frac{1}{2} \right) + 15\sqrt{2} \left( \xi_v^2 - \frac{4}{5} \xi_v + \frac{1}{10} \right) \in \mathcal{P}_2
\]

whose orbit under \( G \) consists of three polynomials which form an orthonormal basis for \( \mathcal{P}_2 \).

\[\text{Fig. Contour plots of } f \text{ and those of its orbit showing the triangular symmetry.}\]
Orthogonal polynomials on the disc

Let $P_n = P_n^w$ be the $n+1$ dimensional space of orthogonal polynomials of degree $n$ on the unit disc

$$D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$
given by the radially symmetric inner product

$$\langle f, g \rangle := \int_D f g w = \int_0^{2\pi} \int_0^1 (f g)(r \cos \theta, r \sin \theta) w(r) \, r \, dr \, d\theta.$$  

The Gegenbauer polynomials are given by the weight

$$w(r) := (1 - r^2)^\alpha \quad \alpha > -1.$$  

These polynomials have long been used to analyse the optical properties of a circular lens, and to reconstruct images from Radon projections, etc.

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation through the angle $\theta$, i.e.,

$$R_\theta(x, y) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{pmatrix}.$$  

Let the group of rotations of the disc (which are symmetries of the weight)

$$\text{SO}(2) = \{R_\theta : 0 \leq \theta < 2\pi\}$$

act on functions defined on the disc in the natural way, i.e.,

$$R_\theta f := f \circ R_\theta^{-1}.$$
The Logan Shepp polynomials

[Logan, Shepp 1975] showed the **Legendre polynomials** on the disc (constant weight $w = 1$) have an orthonormal basis given by the $n + 1$ polynomials

$$p_j(x, y) := \frac{1}{\sqrt{\pi}} U_n\left( x \cos \frac{j\pi}{n + 1} + y \sin \frac{j\pi}{n + 1} \right), \quad j = 0, \ldots, n,$$

where $U_n$ is the $n$–th **Chebyshev polynomial of the second kind**.

This says that an orthonormal basis can be constructed from a single simple polynomial $p_0$ (a ridge function obtained from a univariate polynomial) by rotating it through the angles

$$\frac{j\pi}{n + 1}, \quad 0 \leq j \leq n.$$

It turns out, that for *any* weight $w$ such an orthogonal expansion always exists, though the ‘simple’ polynomial $p_0$ is not in general a ridge function. Moreover, such an expansion reflects the rotational symmetry of the weight in a deeper way, e.g., for the Legendre polynomials there exists the tight frame decompositions

$$f = \frac{n + 1}{k} \sum_{j=0}^{k-1} \langle f, R_{\frac{j\pi}{k}} \rangle R_{\frac{j\pi}{k}} p_0$$

$$= \frac{n + 1}{2\pi} \int_0^{2\pi} \langle f, R_\theta p_0 \rangle R_\theta p_0 \, d\theta, \quad \forall f \in \mathcal{P}_n,$$

where $k \geq n + 1$ with $k$ not even if $k \leq 2n$.  

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A tight frame

For the weight function \( w : [0, 1] \to \mathbb{R}^+ \) and a fixed \( n \), let

\[ P_j \neq 0, \quad 0 \leq j \leq \frac{n}{2} \]

be an orthogonal polynomial of degree \( j \) for the univariate weight \( (1 + x)^{n-2j} w(\sqrt{\frac{1+x}{2}}) \) on \([-1, 1]\), and

\[ h_j := \frac{\pi}{2^{n-2j+1}} \int_{-1}^{1} P_j^2(x)(1 + x)^{n-2j} w(\sqrt{\frac{1+x}{2}}) \, dx. \]

**Theorem [W07].** Let \( v \in \mathcal{P}_n \) be the polynomial with real coefficients defined by

\[
v(x, y) := \frac{1}{\sqrt{n+1}} \sum_{0 \leq j \leq \frac{n}{2}} \frac{2}{1 + \delta_j, \frac{n}{2}} \frac{1}{\sqrt{h_j}} \text{Re}(\xi_j z^{n-2j}) P_j(2|z|^2 - 1),
\]

where \( z := x + iy \), \( \xi_j \in \mathbb{C} \), \( |\xi_j| = 1 \), with \( \xi_{\frac{n}{2}} \in \{-1, 1\} \). Then \( \{R_j^{\frac{n}{n+1}} v\}_{j=0}^{\frac{n}{2}} \) is an orthonormal basis for \( \mathcal{P}_n \), and

\[
f = \frac{n+1}{k} \sum_{j=0}^{k-1} \langle f, R_j^{\frac{n}{n+1}} v \rangle R_j^{\frac{n}{n+1}} v
\]

\[
= \frac{n+1}{2\pi} \int_0^{2\pi} \langle f, R_\theta v \rangle R_\theta v \, d\theta, \quad \forall f \in \mathcal{P}_n,
\]

whenever \( k \geq n + 1 \) and \( k \) is odd, or \( k \geq 2(n+1) \).
Zonal functions

Fig. Contour plots of the Legendre polynomial \( v \in \mathcal{P}_5 \) for the choices \( \xi_0 = 1 \) and \( \xi_1, \xi_2 \in \{-1, 1\} \). The first is the Logan-Shepp polynomial.

A function \( f \) on the ball or \( \mathbb{R}^d \) is **zonal** if it can be written in the form

\[
f(x) = g(\langle x, \xi \rangle, |x|).
\]

Compare this with

\[
f(x) = g(\langle x, \xi \rangle) \quad \text{(ridge function with direction } \xi),
\]

\[
f(x) = g(|x|) \quad \text{(radial function)}.
\]
Orthogonal polynomials on a ball

Let $\mathcal{P}_n$ be the orthogonal polynomials on a ball in $\mathbb{R}^d$.

**Theorem.** Let $p = p_\xi$ be the zonal function

$$p_\xi := \sqrt{\frac{\text{area}(S)}{\dim(\mathcal{P}_n)}} \sum_{0 \leq j \leq \frac{n}{2}} Z^{(n-2j)}_\xi \frac{P_j(|\cdot|^2)}{\|P_j\|_w} \in \mathcal{P}_n.$$  

Then

$$f = \dim(\mathcal{P}_n) \int_{SO(d)} \langle f, gp \rangle gp d\mu(g)$$

$$= \frac{\dim(\mathcal{P}_n)}{\text{area}(S)} \int_S \langle f, p_\xi \rangle p_\xi d\xi, \quad \forall f \in \mathcal{P}_n,$$

where $\mu$ denotes the normalised Haar measure on $SO(d)$.

Here $Z^{(k)}_\xi$ is the zonal harmonic of degree $k$, and $P_j$ is a univariate orthogonal polynomial of degree $j$.

**Corollary (Legendre polynomials).** For the weight $w = 1$ on the unit ball $p_\xi$ is is the ridge polynomial given by

$$p_\xi(x) = \frac{\sqrt{2n+d}}{\sqrt{\text{area}(S)} \sqrt{\dim(\mathcal{P}_n)}} C_n^{d/2}(\langle x, \xi \rangle).$$

Here $C_n^\lambda$ are Gegenbauer polynomials.