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Generalised Welch Bound Equality sequences are tight frames

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ABSTRACT

This paper shows what are called Welch bound equality (WBE) sequences by the signal processing community are precisely the isometric/equal norm/normalized/uniform tight frames which are currently being investigated for a number of applications, and in the real case are the spherical 2–designs of combinatorics. Recent applications include wavelet expansions, Grassmannian frames, frames robust to erasures, and quantum measurements.

This is done by giving an elementary proof of a generalisation of Welch's inequality to vectors which need not have equal energy, and then showing that equality occurs in this exactly when the vectors form a tight frame.

Key Words: WBE sequences, Welch bound equality sequences, isometric tight frames, normalized tight frames, spherical 2–designs, Gram matrix, wavelets

AMS (MOS) Subject Classifications: primary 05B30, 42C40, 94A12, secondary 42C15, 65D30

1. Introduction

Let $\phi_1, \ldots, \phi_n$ be $n \geq d$ unit vectors in $\mathbb{C}^d$ (signals of unit energy). Then

$$F(\phi_1, \ldots, \phi_n) := \sum_{i=1}^{n} \sum_{j=1}^{n} |\langle \phi_i, \phi_j \rangle|^2 \geq \frac{n^2}{d}, \tag{1.1}$$

which is known as the **Welch bound**, after Welch [W74] who proved

$$\max_{i \neq j} |\langle \phi_i, \phi_j \rangle|^2 \geq \frac{n^2/d - n}{n^2 - n} = \frac{n-d}{d(n-1)}. \tag{1.2}$$

Each of these give a lower bound on how small the cross correlation of a set of signals of unit energy can be.

The vectors $\phi_1, \ldots, \phi_n$ in $\mathbb{F}^d$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$) form a **tight frame** for $\mathbb{F}^d$ (cf [D92]) if

$$\|f\|^2 = C \sum_{i=1}^{n} |\langle f, \phi_i \rangle|^2, \qquad \forall f \in \mathbb{F}^d,$$

for some $C > 0$, and this is **isometric/equal norm/normalized/uniform** when all the ϕ_i have the same norm. See [CK01] for a full discussion of the 'notation battle'.

Unit vectors $\phi_1, \ldots, \phi_n$ which give equality in (1.1) are called **WBE sequences** (Welch bound equality sequences), see, e.g., [MM93] where they were used for CDMA (code–division multiple–access) systems. In Benedetto and Fickus [BF0x] it is shown that the functional F of (1.1) attains its minimum of n^2/d precisely when $\phi_1, \ldots, \phi_n$ is a tight frame (of unit vectors), thereby proving the existence of what were there called normalized tight frames, i.e., WBE sequences. The existence of isometric tight frames, equivalently WBE sequences, for each $n \geq d$ has not been well known until recently, see, e.g., the remarks in [BF0x] and [RW02]. Recent applications of such seqences include wavelet expansions (cf [D92]), Grasmannian frames (cf [HS02]), frames robust to erasures (cf [CK01]), and quantam measurements (cf [EF02]).

In the following section, we give an elementary proof of the following generalisation of (1.1) to vectors which need not have unit norm. This result could be used to extend the above applications to sequences of signals which need not have equal energy.

Theorem 1.3 (Generalised Welch inequality). *Let $\phi_1, \ldots, \phi_n$ be $n \geq d$ vectors in $\mathbb{F}^d$, which are not all zero, then*

$$\frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |\langle \phi_i, \phi_j \rangle|^2}{\left(\sum_{i=1}^{n} \langle \phi_i, \phi_i \rangle\right)^2} \geq \frac{1}{d}. \tag{1.4}$$

Clearly, (1.4) reduces to (1.1) when the vectors have unit length, but it cannot be obtained from (1.1) by making the substitution $\phi_i / \|\phi_i\|$ in the case ϕ_i is not a unit vector, and so is a new result.

In the final section, we show the condition for equality in (1.4) is that $\phi_1, \dots, \phi_n$ be a tight frame, thereby extending the result of [BF0x], and discuss some consequences.

2. Proof of the generalised Welch inequality

Proof of Theorem 1.3: Let $\Phi = [\phi_1, \dots, \phi_n]$ be the $d \times n$ matrix with the vectors ϕ_i as columns. Choose a singular value decomposition

$$\Phi = USV^*, \qquad S := \begin{bmatrix} \sqrt{\lambda_1} & & & \\ & \ddots & & \\ & & \sqrt{\lambda_m} & \\ & & & \end{bmatrix}, \quad S \in \mathbb{R}^{d\times n}$$

where $\lambda_1, \dots, \lambda_m$, $m \le d$ are the nonzero eigenvalues of the Gramian $\Phi^*\Phi$, and so satisfy

$$\lambda_1 + \cdots + \lambda_m = \operatorname{trace}(\Phi^*\Phi) = \sum_{i=1}^n \langle \phi_i, \phi_i \rangle =: K \quad \Longrightarrow \quad \min_i \lambda_i \ge \frac{K}{m}.$$

Since the Frobenius norm $\|A\|_F := (\sum_{i,j} |a_{ij}|^2)^{1/2}$ is unitarily invariant, we have

$$\sum_{i=1}^n \sum_{j=1}^n |\langle \phi_i, \phi_j \rangle|^2 = \|\Phi^*\Phi\|_F^2 = \|S^*S\|_F^2 = \lambda_1^2 + \cdots + \lambda_m^2 \ge m \left(\frac{K}{m}\right)^2 = \frac{K^2}{m} \ge \frac{K^2}{d},$$

which is (1.4), with equality if and only if $m = d$ and $\lambda_1 = \cdots = \lambda_d = K/d$. $\square$

For this inequality, the analogue of (1.2) is

$$\max_{i \ne j} |\langle \phi_i, \phi_j \rangle|^2 \ge \frac{(\sum_i \|\phi_i\|)^2/d - \sum_i \|\phi_i\|^4}{n^2 - n} > 0. \tag{2.1}$$

3. Consequences

Equality in (1.4) can be expressed in the following equivalent ways.

Theorem 3.1 (Generalised Welch equality). *Let $\phi_1, \dots \phi_n$ be $n \ge d$ vectors in $\mathbb{F}^d$, with $K := \sum_{i=1}^n \langle \phi_i, \phi_i \rangle > 0$. Then the following are equivalent*

1. *There is equality in* (1.4), *i.e.,*

$$\frac{\sum_{i=1}^n \sum_{j=1}^n |\langle \phi_i, \phi_j \rangle|^2}{\left(\sum_{i=1}^n \langle \phi_i, \phi_i \rangle\right)^2} = \frac{1}{d}.$$

2. *There is the representation*

$$f = \frac{d}{K}\sum_{i=1}^{n}\langle f,\phi_i\rangle\phi_i, \qquad \forall f\in\mathbb{F}^d.$$

3. *The vectors* $\phi_1,\ldots,\phi_n$ *form a tight frame, i.e., there exists* $C>0$*, with*

$$\|f\|^2 = C\sum_{i=1}^{n}|\langle f,\phi_i\rangle|^2, \qquad \forall f\in\mathbb{F}^d.$$

4. *The following sums hold*

$$\sum_{i=1}^{n}|\langle\phi_j,\phi_i\rangle|^2 = \frac{K}{d}\|\phi_j\|^2, \qquad \forall j.$$

Proof: $(1 \Longrightarrow 2)$. From the proof of Theorem 1.3 there is equality in (1.4) if and only if $m=d$ and $\lambda_1=\cdots=\lambda_d=K/d$, in which case $SS^*=(K/d)I_d$, and we calculate

$$\sum_{i=1}^{n}\langle f,\phi_i\rangle\phi_i = \Phi\Phi^* f = USV^*VS^*U^*f = U\left(\frac{K}{d}I_d\right)U^* = \frac{K}{d}f, \qquad \forall f\in\mathbb{F}^d.$$

$(2 \Longrightarrow 3 \Longrightarrow 4 \Longrightarrow 1)$. Take the inner product of the representation of 2 with f to obtain 3 with $C=d/K$, then choose $f=\phi_j$ to obtain 4, and finally sum over $j=1,\ldots,n$ to get 1. □

When all ϕ_i have unit norm, so $K:=\sum_{i=1}^{n}\langle\phi_i,\phi_i\rangle=n$, this reduces to the known result:

Corollary 3.2 (Welch equality). *Let* $\Phi=(\phi_1,\ldots,\phi_n)$ *be a sequence of* $n\geq d$ *unit vectors in* $\mathbb{F}^d$*. Then the following are equivalent*

1. Φ *is a WBE sequence, i.e.,*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}|\langle\phi_i,\phi_j\rangle|^2 = \frac{n^2}{d}.$$

2. *There is the representation*

$$f = \frac{d}{n}\sum_{i=1}^{n}\langle f,\phi_i\rangle\phi_i, \qquad \forall f\in\mathbb{F}^d.$$

3. Φ *is an isometric tight frame, i.e.,*

$$\|f\|^2 = \frac{d}{n}\sum_{i=1}^{n}|\langle f,\phi_i\rangle|^2, \qquad \forall f\in\mathbb{F}^d.$$

4. The following sums hold

$$\sum_{i=1}^{n} |\langle \phi_j, \phi_i \rangle|^2 = \frac{n}{d}, \qquad \forall j.$$

The Welch bound (1.1) is proved in [MM93] using the Cauchy–Schwarz inequality and it is shown that equality holds if and only if the rows of $\Phi := [\phi_1, \ldots, \phi_n]$ are orthogonal and of equal length, i.e., $\Phi\Phi^* = n/d$ (which gives 2). There the condition 4 is called the 'uniformly–good property'.

In [BF0x] the functional F of (1.1) is called the *frame potential* (for the frame force). It is shown that this continuous function on a compact set attains its minimum of n^2/d precisely when Φ is a (normalized) tight frame (its maximum is n^2). We can express Theorem 3.1 in terms of an appropriately defined potential function, thereby obtaining a physical interpretation of tight frames, which can be used in computations (cf [CFKLT03]).

Theorem 3.3. *For $n \geq d$, the frame potential function*

$$\mathcal{F} : \mathbb{F}^d \times \cdots \times \mathbb{F}^d \setminus \{0\} \to \mathbb{R}^+, \qquad \mathcal{F}(\phi_1, \ldots, \phi_n) := \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |\langle \phi_i, \phi_j \rangle|^2}{\left(\sum_{i=1}^{n} \langle \phi_i, \phi_i \rangle\right)^2}$$

attains its minimum $1/d$ when $\phi_1, \ldots, \phi_n$ is a tight frame for $\mathbb{F}^d$, and its maximum 1 when $\mathrm{span}\{\phi_1, \ldots, \phi_n\}$ is one–dimensional.

Proof: We need only consider the maximum. By Cauchy–Schwarz

$$\mathcal{F}(\phi_1, \ldots, \phi_n) := \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |\langle \phi_i, \phi_j \rangle|^2}{\left(\sum_{i=1}^{n} \langle \phi_i, \phi_i \rangle\right)^2} \leq \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \|\phi_i\|^2 \|\phi_j\|^2}{\left(\sum_{i=1}^{n} \|\phi_i\|^2\right)^2} = 1,$$

with equality precisely when $\mathrm{span}\{\phi_1, \ldots, \phi_n\}$ is one–dimensional. $\square$

In addition, from the proof of Theorem 1.3, we can obtain the bound

$$\mathcal{F}(\phi_1, \ldots, \phi_n) \geq \frac{1}{m}, \qquad m := \dim(\mathrm{span}\{\phi_1, \ldots, \phi_n\}),$$

with equality when $\phi_1 \ldots, \phi_n$ is a tight frame for its span.

Following Seidel [S01], we say a **spherical t–design** is a finite subset Φ of the unit sphere S in $\mathbb{R}^d$, for which the normalised surface integral satisfies

$$\int_S f \, d\sigma = \frac{1}{|\Phi|} \sum_{\phi \in \Phi} f(\phi),$$

for all homogeneous polynomials f of total degree t in d variables. For t even, a set $\Phi := \{\phi_1, \ldots, \phi_n\}$ of unit vectors is a spherical t–design if and only if it satisfies the so called *Waring formula*

$$\langle x, x \rangle^{t/2} = \frac{d(d+2)\cdots(d+t-2)}{1 \cdot 3 \cdot 5 \cdots (t-1)} \frac{1}{n} \sum_{i=1}^{n} \langle x, \phi_i \rangle^t, \qquad \forall x \in \mathbb{R}^d.$$

For $t = 2$, this becomes

$$\|x\|^2 = \frac{d}{n} \sum_{i=1}^{n} |\langle x, \phi_i \rangle|^2, \qquad \forall x \in \mathbb{R}^d,$$

which is equivalence 3 of Corollary 3.2. Thus a 2–design is precisely a WBE sequence of distinct vectors (equivalently an isometric tight frame) for $\mathbb{R}^d$.

Venkov [V01:Th. 8.1] has proved that if $\Phi := \{\phi_1, \ldots, \phi_n\}$ is a symmetric set of unit vectors in $\mathbb{R}^d$, i.e., $-\Phi = \Phi$, and t is even, then Φ is an r–design, $r \leq t + 1$ if and only if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \langle \phi_i, \phi_j \rangle^t = n^2 \prod_{j=0}^{\frac{t}{2}-1} \frac{2j+1}{d+2j}.$$

For $t = 2$, this becomes

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \langle \phi_i, \phi_j \rangle^2 = \frac{n^2}{d},$$

which is equivalence 1 of Corollary 3.2, i.e., Φ is a symmetric 2–design. We can summarise these results as follows.

Corollary 3.4 (t–designs). *A set $\Phi := \{\phi_1, \ldots \phi_n\}$ of unit vectors in $\mathbb{R}^d$ is a t–design for $t \leq 2$ if and only if*

$$\sum_{i=1}^{n} \phi_i = 0, \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \phi_i, \phi_j \rangle^2 = \frac{n^2}{d}.$$

Moreover, such a Φ is also a 3–design if it is symmetric.

Proof: Any Φ is a 0–design, and Φ is a 1–design if and only if

$$\sum_{i=1}^{n} \phi_i = 0,$$

which automatically holds if Φ is symmetric. By Corollary 3.2, Φ is a 2–design, and such a $0, 1, 2$–design is a 3–design if Φ is symmetric. □

This result can be found in the literature (cf [DGS77] and [S01]).

In Eldar and Forney [EF02] the relationship between tight frames and rank–one quantum measurements is investigated. It is shown that rank–one generalized quantum measurements (or POVMs) correspond to tight frames.

It is hoped, that by drawing attention to the fact that WBE sequences, isometric (normalized, uniform) tight frames and 2–designs are the same thing, that the respective communities can benefit from each others endeavours. Clearly, such an object which has appeared independently in different areas is of interest, and deserves to be understood in this wider context.

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References

[BF0x] J. J. Benedetto and M. Fickus, *Finite normalized tight frames*, Advances in Comp. Math., xxx (200x), pp. xx-xx.

[CFKLT03] P. G. Casazza, M. C. Fickus, J. Kovačević, M. T. Leon, and J. C. Tremain, *A physical interpretation for finite tight frames*, preprint, 2003.

[CK01] P. G. Casazza and J. Kovačević, *Equal–norm tight frames with erasures*, Preprint, 2001.

[D92] I. Daubechies, *Ten Lectures on Wavelets*, CBMS Conf. Series in Appl. Math., vol. 61, SIAM, Philadelphia, 1992.

[DGS77] P. Delsarte, J. M. Goethals, and J. J. Seidel, *Spherical codes and designs*, Geom. Dedicata, 6(3) (1977), pp. 363–388.

[EF02] Y. C. Eldar and G. D. Forney, *Optimal Tight Frames and Quantum Measurement*, IEEE Trans. Inform. Theory, 48 (2002), pp. 599–610.

[HS02] R. Heath and T. Strohmer, *Grassmannian Frames and Applications to Coding and Communication*, Preprint, 2002.

[MM93] J. L. Massey and T. Mittelholzer, *Welch's bound and sequence sets for code-division multiple-access systems*, Sequences II (Capocelli, R., De Santis, A., Vaccaro, U., eds), Springer-Verlag, New York, 1993, pp. 63–78.

[RW02] R. Reams and S. Waldron, *Isometric tight frames*, Electron. J. Linear Algebra, 9 (2002), pp. 122–128.

[S01] J. J. Seidel, *Definitions for spherical designs*, J. Statist. Plann. Inference, 95 no. 1–2 (2001), pp. 307–313.

[V01] B. Venkov, *Réseaux et designs sphériques (French)*, Monogr. Enseign. Math., 37 (2001), pp. 10–86.

[W74] L. R. Welch, *Lower Bounds on the Maximum Cross Correlation of Signals*, IEEE Trans. Inform. Theory, 20 (1974), pp. 397–399.