L_p-ERROR BOUNDS FOR MULTIVARIATE POLYNOMIAL INTERPOLATION SCHEMES

by

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A dissertation submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

(Mathematics)

at the

UNIVERSITY OF WISCONSIN — MADISON

1995

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The method, as in past approaches, is first to construct a formula for a right inverse R of the differential operator

$$D^{k+1} : f \mapsto D^{k+1}f := (D^\alpha f : |\alpha| = k+1),$$

and then to manipulate the expression

$$R(D^{k+1}f)$$

to obtain $L_p(\Omega)$-bounds.

New formulae for such R are presented. These are based on representations for the error in the family of polynomial interpolators which includes the maps of Kergin and Hakopian.

A multivariate form of Hardy's inequality involving the linear functional of integration against a simplex spline is given. This inequality provides a simple way to obtain $L_p(\Omega)$-bounds from the formulae for $R(D^{k+1}f)$ given here, and many others in the literature.

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Acknowledgements

It gives me great pleasure to thank my advisor Carl de Boor for being an excellent advisor during my time at Wisconsin. This thesis gives only a small indication of the profound influence his careful guidance and encouragement has had on me. I would particularly like to thank him for some of the things he has done for me in addition to the usual supervising duties. These include providing me with support over the last two years, arranging for me to attend conferences and have visitors here in Madison, and spending countless hours teaching me the finer points of `MATLAB`, `TeX`, and `vi`.

I would like to thank my other mentor, Amos Ron, especially for his capable advice during the year that Carl was in Bielefeld. Amos's encouragement to continue looking at the error in the least solution helped lead me towards this thesis, and the associated circle of ideas – whose development will keep me occupied for quite some time.

Here are just a few of the many others who have helped me over the last five years. All the participants of the Approximation Theory Seminar, which, in its recent years, has provided a testing ground for new ideas and notation second to none. Yvonne Nagel for teaching me about all things related to computing. Friends who shared the graduate school experience with me, including Joe Dolinak and Mark Wilson at Wisconsin, and Amanda Peet and Julian Wright at Stanford. People who took a personal interest in my future mathematical development, including Mike Osborne, Allan Pinkus, Sherm Riemenschneider, and the University of Auckland. And of course, my family.

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1

Introduction

The early work on Sobolev spaces contains many theorems similar to the following which can be found in Morrey [Mo66]. Let $W_p^{k+1}(\Omega)$, $1 \le p \le \infty$ be the usual Sobolev space (discussed in more detail in Section 2.1), and let Π_k denote the n-variate polynomials of degree $\le k$.

Theorem ([Mo66:Th.3.6.11,p85]). *Let Ω be a connected bounded open set in $\mathbb{R}^n$ with a Lipschitz boundary. Suppose that*

$$P : W_p^{k+1}(\Omega) \to W_p^{k+1}(\Omega)$$

is a bounded linear projector with range Π_k. Then there exists $C > 0$ such that

$$\|f - Pf\|_{W_p^{k+1}(\Omega)} \le C \, \|D^{k+1} f\|_{L_p(\Omega)}, \quad \forall f \in W_p^{k+1}(\Omega),$$

where the suggestive notation $\|D^{k+1} f\|_{L_p(\Omega)}$ indicates any of the usual (equivalent) seminorms that measure the $L_p(\Omega)$-size of the $(k+1)$st derivative of f.

Theorems of this type first occurred in the work of Sobolev, see, e.g., [So50], and in the paper of Deny and Lions [DL53] (for the case $p = 2$).

In the early 1970s it was realised that this theorem is important in numerical analysis because of the following immediate corollary.

Corollary. *Let X be a normed linear space. If Π_k is contained within the kernel of a bounded linear map*

$$L : W_p^{k+1}(\Omega) \to X,$$

then there exists $K > 0$ such that

$$\|Lf\| \le \|L\| \operatorname{dist}_{W_p^{k+1}(\Omega)}(f, \Pi_k) \le K \, \|D^{k+1} f\|_{L_p(\Omega)}, \quad \forall f \in W_p^{k+1}(\Omega).$$

The special case of this corollary when $X = \mathbb{R}$ is commonly referred to as the *Bramble-Hilbert lemma* after the paper [BH70] where it appeared.

This corollary is used to conclude that a numerical scheme, such as a finite element method (see e.g., [Ci78]), has the highest order of accuracy that its polynomial reproduction allows.

The difficulty with the theorem (and its corollary) is that it is not constructive, i.e., gives no estimate of the constant involved. Because of this fact, there have been, since the publication of [BH70], many papers dealing with 'constructive instances of the Bramble-Hilbert lemma', i.e., estimating the constants C and K of the theorem and its corollary. For example see Ciarlet and Wagschal [CW71], and Gregory [Gr75].

The standard way of doing this (see, e.g., Meinguet [Me78]) is to find a map R so that the error in approximating f by Pf can be expressed as

$$f - Pf = R(D^{k+1}f), \quad \forall f \in W_p^{k+1}(\Omega), \tag{a}$$

where D^{k+1} is the differential operator

$$D^{k+1} : f \mapsto D^{k+1}f := (D^\alpha f : |\alpha| = k+1).$$

By applying D^{k+1} to both sides of (a) it is seen that the (necessarily) linear map R is a right inverse for D^{k+1}. The hope is then to manipulate a suitable formula for $R(D^{k+1}f)$ to obtain such $L_p(\Omega)$-bounds as occur in the theorem. More generally, if L is as in the corollary, then

$$Lf = L(f - Pf) = LR(D^{k+1}f), \tag{b}$$

and one tries to bound $LR(D^{k+1}f)$ by $\|D^{k+1}f\|_{L_p(\Omega)}$.

The observation (b) can be viewed as the multivariate analogue of the Peano kernel theorem. In the Peano kernel theorem, P is taken as the Taylor interpolant of degree k at the left end point of the interval of interest, R is the formula for the error involving integration against the $(k+1)$st derivative, and L is a linear functional.

There are many possible maps R, and for each R there are many possible formulæ describing it. Thus, the difficulty with using (b) as the basis for a multivariate Peano kernel theory is deciding which R to choose, and then obtaining formulæ for R and LR which can easily be manipulated to obtain $L_p(\Omega)$ bounds. In this thesis we consider such questions.

In Chapter 1 of this thesis we consider the case when P is from a family of linear maps which includes Kergin and Hakopian interpolation. These maps are 'lifted' versions of univariate Hermite interpolation, and contain Taylor interpolation at a point as special cases. For these maps we obtain integral error formulæ of the desired form $R(D^{k+1}f)$. In contrast to most error formulæ for these maps obtained in the past (see e.g., Lai and Wang [LW84]), those given here involve only derivatives of order $k+1$.

The error formulæ given in Chapter 1, like those for many other multivariate generalisations of Hermite interpolation, express the error at the point x in terms of the linear functional of integration against a simplex spline with a knot set which includes x a certain number of times. To obtain $L_p(\Omega)$-bounds from such formulæ, in Chapter 2 we present the following multivariate form of Hardy's inequality, that for $m - n/p > 0$

$$\| \, x \mapsto \int_{[\underbrace{x,\ldots,x}_{m},\Theta]} f \, \|_p \leq \frac{\|f\|_p}{(m-1)!(m-n/p)_{\#\Theta}}, \tag{c}$$

valid for $f \in L_p(\mathbb{R}^n)$ and Θ an arbitrary finite sequence of points in $\mathbb{R}^n$. Examples treated with this inequality include the formulæ of Chapter 1 in Section 2.3, and those for 'Lagrange maps' in Section 2.4.

We conclude Chapter 2 with a discussion of why (c) plays a crucial role in obtaining $L_p(\Omega)$-bounds from pointwise integral error formulæ for multivariate generalisations of Lagrange interpolation, and why it is likely to do so for those that will be obtained in the future.

1. Integral error formulæ for the scale of mean value interpolations which includes Kergin and Hakopian interpolation

1.1. Introduction

In this chapter we study the error in a certain scale of mean value interpolations which includes Kergin and Hakopian interpolation. The literature divides into two different approaches to this problem.

The first is concerned with the convergence of the interpolants as the number of interpolation points increases. Here only Kergin interpolation has been studied. Certain conditions on the position of the interpolation points and the growth of the entire function to be interpolated are given which guarantee that the sequence of interpolants converges uniformly on compact sets. See, e.g., Bloom [Bl81].

We are interested in the second approach, which is to write the error in interpolation as integration against derivatives of high order, much as is done for univariate Hermite interpolation.

There have been several papers in this direction, including Lai and Wang [LW84] (Hakopian interpolation), [LW86] (Kergin interpolation), and Gao [Ga88] (mean value interpolation). Each of these gives formulæ for the error, complicated by the spurious use of certain *multivariate divided differences*, involving derivatives of various orders. There seems to be very little correspondence between the degree of the interpolating polynomial space and the order of the derivatives involved. This order can be as low as 0, and as high as twice the degree of the interpolating polynomial space.

In this chapter we give an integral error formula for the scale of mean value interpolations that involves only derivatives of order one higher than the degree of the interpolating polynomial space. From this we obtain sharp L_∞-estimates. These estimates imply that a numerical scheme based on mean value interpolation has the highest *order* that its polynomial reproduction allows.

The chapter is set out in the following way. To describe the scale of mean value interpolations, we use a certain linear functional $f \mapsto \int_\Theta f$ and the notion of 'lifting' univariate maps. These two notions are studied in requisite detail in Sections 1.2 and 1.3 respectively. In Section 1.4, we define the scale of mean value interpolations and give its Newton form. In Section 1.5, we give two different integral error formulæ for the scale. In Section 1.6, from these formulæ, we obtain L_∞-estimates.

Some notation

The space of n-variate polynomials of total degree k will be denoted by $\Pi_k(\mathbb{R}^n)$ and the homogeneous polynomials of degree k by $\Pi_k^0(\mathbb{R}^n)$. The differential operator induced by $g \in \Pi_k(\mathbb{R}^n)$ will be written $g(D)$.

We find it convenient to make no distinction between the matrix $[\theta_1, \ldots, \theta_k]$ and the k-**sequence** $\theta_1, \ldots, \theta_k$ of its columns. Since $[\theta_1, \ldots, \theta_k]f$ is a standard notation for the *divided difference* of f at $\Theta = [\theta_1, \ldots, \theta_k]$, we use for the latter the nonstandard notation

$$\delta_\Theta f = \delta_{[\theta_1, \ldots, \theta_k]} f.$$

Note the special case

$$\delta_{[x]} f = f(x).$$

Similarly, to avoid any confusion, the closed interval with endpoints a and b will be denoted by $[a\,..\,b]$.

The notation $\tilde{\Theta} \subset \Theta$ means that $\tilde{\Theta}$ is a subsequence of Θ, $\Theta\backslash\tilde{\Theta}$ denotes the complementary subsequence. The derivative of f in the directions Θ is denoted

$$D_\Theta f := D_{\theta_1} \cdots D_{\theta_k} f.$$

The subsequence consisting of the first j terms of Θ is denoted Θ_j, and

$$x - \Theta := [x - \theta_1, \ldots, x - \theta_k].$$

Thus, with $\Theta := [\theta_1, \ldots, \theta_7]$, we have, for example, that

$$D_{[x-\Theta\backslash\Theta_5, x-\theta_3]} f = D_{x-\theta_6} D_{x-\theta_7} D_{x-\theta_3} f.$$

The diameter and convex hull of a sequence Θ will be that of the corresponding set and will be denoted by $\operatorname{diam}\Theta$ and $\operatorname{conv}\Theta$ respectively. Let $\|\cdot\|$ be the *Euclidean norm.* To measure the size of the k-th derivative of f at $x \in \mathbb{R}^n$, we use the seminorm

$$|D^k f|(x) := \sup_{\substack{u_1,\ldots,u_k \in \mathbb{R}^n \\ \|u_i\| \le 1}} |D_{u_1} \cdots D_{u_k} f(x)|.$$

Notice that

$$|D_{u_1} \cdots D_{u_k} f(x)| \le |D^k f|(x)\, \|u_1\| \cdots \|u_k\|. \tag{1.1.1}$$

To measure the size of the k-th derivative of f over $K \subset \mathbb{R}^n$, we use

$$|f|_{k,\infty,K} := \sup_{x \in K} |D^k f|(x). \tag{1.1.2}$$

Because of (1.1.1), the co-ordinate-independent seminorm $|\cdot|_{k,\infty,K}$ is more appropriate to the analysis that follows than other equivalent seminorms, such as

$$f \mapsto \max_{|\alpha|=k} \|D^\alpha f\|_{L_\infty(K)}.$$

1.2. The linear functional $f \mapsto \int_\Theta f$

The construction of the maps of Kergin and Hakopian depends intimately on the following linear functional called the **divided difference functional on** $\mathbb{R}^n$ by Micchelli in [M79], and analysed there and in [M80].

Definition 1.2.1. *For any* $\Theta \in \mathbb{R}^{n\times(k+1)}$, *let*

$$f \mapsto \int_\Theta f := \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{k-1}} f(\theta_0 + s_1(\theta_1 - \theta_0) + \cdots + s_k(\theta_k - \theta_{k-1}))\, ds_k \cdots ds_2\, ds_1,$$

with the convention that $\int_{[\,]} f := 0$.

In addition to Kergin and Hakopian interpolation, the linear functional $f \mapsto \int_\Theta f$ also occurs when discussing other *multivariate* generalisations of Lagrange interpolation, e.g., the *Lagrange maps* of Section 2.4. It was used as early as 1859, when in [He1859] Hermite proved the **Hermite-Genocchi formula**, namely that for $\Theta \in \mathbb{R}^{1\times(k+1)}$ and $f \in C^k(\operatorname{conv}\Theta)$

$$\delta_\Theta f = \int_\Theta D^k f.$$

In this section we outline those properties of $f \mapsto \int_\Theta f$ needed in the subsequent sections. Many of these properties are apparent from the following observation.

Observation 1.2.2. *If S is any k-simplex in $\mathbb{R}^m$ and $A : \mathbb{R}^m \to \mathbb{R}^n$ is any affine map taking the $k+1$ vertices of S onto the $k+1$ points in Θ, then*

$$\int_\Theta f = \frac{1}{k!\,\mathrm{vol}_k(S)} \int_S f \circ A,$$

with $\mathrm{vol}_k(S)$ the (k-dimensional) volume of S.

In Definition 1.2.1

$$A : \mathbb{R}^k \to \mathbb{R}^n : (s_1, \ldots, s_k) \mapsto \theta_0 + s_1(\theta_1 - \theta_0) + \ldots + s_k(\theta_k - \theta_{k-1}),$$

$$S := \{(s_1, \ldots, s_k) \in \mathbb{R}^k : 0 \le s_k \le \cdots \le s_2 \le s_1 \le 1\}.$$

In [M80], Micchelli uses a different choice of S and A, namely

$$A : \mathbb{R}^{k+1} \to \mathbb{R}^n : (v_0, \ldots, v_k) \mapsto v_0\theta_0 + \cdots + v_k\theta_k,$$

$$S := \{(v_0, \ldots, v_k) \in \mathbb{R}^{k+1} : v_j \ge 0,\ \sum_{j=0}^{k} v_j = 1\}.$$

This is the form that Genocchi used in [Ge1878] when giving his version of the Hermite-Genocchi formula.

Properties 1.2.3.

(a) *The value of $\int_\Theta f$ does not depend on the ordering of the points in Θ.*

(b) *The distribution*

$$M_\Theta : C_0^\infty(\mathbb{R}^n) \to \mathbb{R} : f \mapsto k! \int_\Theta f$$

is the **(normalised) simplex spline** *with knots Θ.*

(c) *If $f \in C(\mathrm{conv}\,\Theta)$, then $\int_\Theta f$ is defined and, for some $\xi \in \mathrm{conv}\,\Theta$,*

$$\int_\Theta f = \frac{1}{k!} f(\xi).$$

(d) *If* $g : \mathbb{R}^s \to \mathbb{R}$, *and* $B : \mathbb{R}^n \to \mathbb{R}^s$ *is an affine map, then*

$$\int_\Theta (g \circ B) = \int_{B\Theta} g.$$

Some technical details

Remark 1.2.4. In view of Property (a),

$$\Theta \mapsto \int_\Theta f$$

could be thought of as a map defined on finite multisets in $\mathbb{R}^n$ rather than on sequences. However, adopting this definition leads to certain unnecessary complications. For example, to discuss the continuity of $\Theta \mapsto \int_\Theta f$, it would be necessary to endow the set of multisets of $k+1$ points in $\mathbb{R}^n$ with the appropriate topology. Thus, in the interest of simplicity, $\Theta \mapsto \int_\Theta f$ remains a map on sequences – but with the reader encouraged to think of it, as does the author, as a map on multisets. □

Remark 1.2.5. The simplex spline M_Θ of (b) has support $\operatorname{conv}\Theta$. It can be represented by the nonnegative bounded function

$$\operatorname{conv}\Theta \to \mathbb{R} : t \mapsto M(t|\Theta) := \frac{\operatorname{vol}_{k-d}(A^{-1}t \cap S)}{|\det A| \operatorname{vol}_k(S)}, \quad d := \dim \operatorname{conv}\Theta,$$

in the sense that

$$M_\Theta f = \int_{\operatorname{conv}\Theta} M(\cdot|\Theta) f. \tag{1.2.6}$$

In particular, if the points of Θ are affinely independent, then

$$k! \int_\Theta f = \frac{1}{\operatorname{vol}_k(\operatorname{conv}\Theta)} \int_{\operatorname{conv}\Theta} f = \text{average value of } f \text{ on } \operatorname{conv}\Theta. \tag{1.2.7}$$

Thus, $\int_\Theta f$ is defined (as a real number) if and only if $M(\cdot|\Theta)f \in L_1(\operatorname{conv}\Theta)$, in which case

$$|\int_\Theta f\,| \le \int_\Theta |f|. \tag{1.2.8}$$

If f is nonnegative on $\operatorname{conv}\Theta$, then $\int_\Theta f \in [0\,..\,\infty]$ is defined (by Definition 1.2.1). Therefore, we will write (1.2.8) for all f which are defined on $\operatorname{conv}\Theta$ – with the understanding that $\int_\Theta f$ is defined if and only if $\int_\Theta |f| < \infty$ or f is nonnegative. In the univariate case, that is when $n = 1$, $M(\cdot|\Theta)$ is the **(normalised) B-spline** with knots Θ. For additional details about M_Θ and $M(\cdot|\Theta)$, see, e.g., Micchelli [M79]. □

Lastly, by (1.2.6), we can describe the continuity of $\Theta \mapsto \int_\Theta f$ as follows.

Proposition 1.2.9.

(a) *For $f \in C(\mathbb{R}^n)$, the map*

$$\mathbb{R}^{n\times(k+1)} \to \mathbb{R} : \Theta \mapsto \int_\Theta f$$

is continuous.

(b) *For $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, the map*

$$\{\Theta \in \mathbb{R}^{n\times(k+1)} : \operatorname{vol}_n(\operatorname{conv}\Theta) > 0\} \to \mathbb{R} : \Theta \mapsto \int_\Theta f$$

is continuous.

1.3. Liftable maps

In this section, we discuss univariate maps which may be *lifted* to multivariate ones. These 'liftable' maps are crucial to both the construction and description of the error in a family of linear projectors which includes the Kergin and

Hakopian maps. The main papers on 'lifting' are [CMS80$_1$], [CMS80$_2$], [CGMS83] and [HM87].

We denote the linear functional on $\mathbb{R}^n$, induced by scalar product with $\lambda \in \mathbb{R}^n$, by

$$\lambda^* : \mathbb{R}^n \to \mathbb{R} : x \mapsto \lambda^* x := \sum_{i=1}^{n} \lambda(i) x(i).$$

A **plane wave** (or **ridge function**) is any map

$$g \circ \lambda^* : \mathbb{R}^n \to \mathbb{R},$$

where $g : \mathbb{R} \to \mathbb{R}$ and $\lambda \in \mathbb{R}^n$. If $g \in C^1(\mathbb{R})$, then we can differentiate $g \circ \lambda^*$, thereby obtaining

$$D_y(g \circ \lambda^*) = (\lambda^* y)(Dg) \circ \lambda^*. \tag{1.3.1}$$

This 'lifts' differentiation to $\mathbb{R}^n$.

In [CMS80$_1$] only the lifting of polynomial-valued maps is discussed. To 'lift' the error in such maps, we need a more general definition. The only real difficulty involved in giving such a definition is in choosing the domain of the lifted map appropriately to make certain that the plane waves are fundamental in it, thereby implying the uniqueness of the 'lift', as is done in the following definition.

Definition 1.3.2. *Let $L : \Theta \mapsto L_\Theta$ associate with each k-sequence Θ in $\mathbb{R}$ a continuous linear map $L_\Theta : C^s(\mathbb{R}) \to C(\mathbb{R})$. We say that a continuous linear map $\mathcal{L}_\Theta : C^s(\mathbb{R}^n) \to C(\mathbb{R}^n)$ is the* **lift of** L **to** Θ *in $\mathbb{R}^n$ if it satisfies*

$$\mathcal{L}_\Theta(g \circ \lambda^*) = (L_{\lambda^*\Theta} g) \circ \lambda^*, \quad \forall \lambda \in \mathbb{R}^n,\ \forall g \in C^s(\mathbb{R}). \tag{1.3.3}$$

If there exists a lift $\mathcal{L}_\Theta$ of L to each k-sequence Θ in $\mathbb{R}^n$, then we say that L is **liftable** *(***to** $\mathbb{R}^n$*), and call $\mathcal{L} : \Theta \mapsto \mathcal{L}_\Theta$ the* **lift of** L *(***to** $\mathbb{R}^n$*).*

Notice that (1.3.3) overdetermines the map $\mathcal{L}_\Theta$, and so the use of the definite article in the above definition is justified. Furthermore, by the fundamentality of the polynomial plane waves (which span $\Pi(\mathbb{R}^n)$) in $C^s(\mathbb{R}^n)$, if L can be lifted to $\mathcal{L}_\Theta$, then $\mathcal{L}_\Theta$ is uniquely determined by (1.3.3). To avoid confusion, we will use calligraphic letters to denote the lift of a univariate map and, from now on, reserve k for the number of points such a map is based on.

The geometric intent of lifting is that the 'lift' of a function which varies in one direction, i.e., a plane wave, should be a plane wave (varying in the same direction) obtained in a natural way from the univariate map to be lifted.

The basic tool for recognising liftable maps and presenting their lifts is to write them as a sum of 'elementary liftable maps', which are defined as follows.

Definition 1.3.4. *Let $s, m \geq 0$. Fix $a_j \in \mathbb{R}^{k+1}\backslash 0$, $j = 1, \ldots, s$ and $B \in \mathbb{R}^{(k+1)\times(m+1)}$. For each k-sequence Θ in $\mathbb{R}$, let $L_\Theta : C^s(\mathbb{R}) \to C(\mathbb{R})$ be the continuous linear map given by*

$$L_\Theta f(x) := \left(\prod_{j=1}^{s} [x, \Theta] a_j\right) \int_{[x,\Theta]B} D^s f = \int_{[x,\Theta]B} \left(\prod_{j=1}^{s} D_{[x,\Theta]a_j}\right) f. \tag{1.3.5}$$

We call $L : \Theta \mapsto L_\Theta$ an **elementary (k-point) liftable map (of order** *s***)**.

Here and below, in line with our earlier identification of vector sequences and matrices, $[x, \Theta]B$ is the matrix whose j-th column is the vector

$$xB(1, j) + \theta_1 B(2, j) + \cdots + \theta_k B(k + 1, j).$$

In other words, $[x, \Theta]B$ is an $(m + 1)$-sequence.

The equality in (1.3.5) expresses $L_\Theta f(x)$ in a form which has a natural multivariate analogue. In this way, the definition is tailor-made to make it obvious that such a map is liftable, as we prove next.

Theorem 1.3.6. *Each elementary liftable map of order s, as in Definition 1.3.4, is liftable to $\mathrm{I\!R}^n$. Its lift $\mathcal{L} : \Theta \mapsto \mathcal{L}_\Theta$, with $\mathcal{L}_\Theta : C^s(\mathrm{I\!R}^n) \to C(\mathrm{I\!R}^n)$, is given by*

$$\mathcal{L}_\Theta f(x) := \int_{[x,\Theta]B} \left(\prod_{j=1}^{s} D_{[x,\Theta]a_j} \right) f. \tag{1.3.7}$$

In the special case that $B(1,\cdot) = 0$, the range of $\mathcal{L}_\Theta$ is contained in $\Pi_s(\mathrm{I\!R}^n)$.

Proof. The continuity of L_Θ required in Definition 1.3.4 and the continuity of $\mathcal{L}_\Theta$ asserted in Theorem 1.3.6, follow from the inequality

$$\|\mathcal{L}_\Theta f\|_{L_\infty(K)} \le \frac{1}{m!} \left(\max_{x\in K} \prod_{j=1}^{s} \|[x,\Theta]a_j\| \right) |f|_{s,\infty,\mathrm{conv}([x,\Theta]B)},$$

where $K \subset \mathrm{I\!R}^n$ is compact. This is proved by applying, to (1.3.7), Property 1.2.3 (c) followed by (1.1.1) and (1.1.2).

Given the continuity of the maps L_Θ and $\mathcal{L}_\Theta$, to show that $\mathcal{L}$ is the lift of L, it is sufficient to prove that

$$\mathcal{L}_\Theta(g \circ \lambda^*) = (L_{\lambda^*\Theta}\, g) \circ \lambda^*, \quad \forall \lambda \in \mathrm{I\!R}^n,\ \forall g \in C^s(\mathrm{I\!R}),\ \forall \Theta \in (\mathrm{I\!R}^n)^k.$$

By applying (1.3.1) s times, it follows that

$$(\mathcal{L}_\Theta(g \circ \lambda^*))(x) = \int_{[x,\Theta]B} \left(\prod_{j=1}^{s} \lambda^*[x,\Theta]a_j \right) (D^s g) \circ \lambda^*.$$

To the right-hand side of this, we apply Property 1.2.3 (d) (with λ^* the affine map) and the identity $\lambda^*[x,\Theta] = [\lambda^* x, \lambda^*\Theta]$ to obtain that

$$\int_{[\lambda^* x,\lambda^*\Theta]B} \left(\prod_{j=1}^{s} [\lambda^* x, \lambda^*\Theta]a_j \right) (D^s g) = (L_{\lambda^*\Theta}\, g)(\lambda^* x).$$

□

Example 1.3.8. In [HM87] it is shown that (sadly) the divided difference cannot be lifted; however we may lift the following divided difference identity

$$\delta_{[\Theta,v,w]} g = \frac{\delta_{[\Theta,v]} g - \delta_{[\Theta,w]} g}{v - w}, \quad v \ne w. \tag{1.3.9}$$

By the Hermite-Genocchi formula, (1.3.9) may be rewritten as

$$(v-w)\int_{[\Theta,v,w]} Df = \int_{[\Theta,v]} f - \int_{[\Theta,w]} f,$$

where $f := D^k g$ and $k = \#\Theta$. By Theorem 1.3.6, this lifts to

$$\int_{[\Theta,v,w]} D_{v-w} f = \int_{[\Theta,v]} f - \int_{[\Theta,w]} f, \tag{1.3.10}$$

for all sufficiently smooth f, where Θ is any finite sequence in $\mathbb{R}^n$ and $v, w \in \mathbb{R}^n$.

An elementary liftable map depends continuously on Θ, in the following sense.

Theorem 1.3.11. *Let $\mathcal{L}$ be the lift to $\mathbb{R}^n$ of an elementary k-point liftable map of order s. For all $f \in C^s(\mathbb{R}^n)$, the map*

$$(\mathbb{R}^n)^k \to C(\mathbb{R}^n) : \Theta \mapsto \mathcal{L}_\Theta f$$

is continuous.

Proof. By Property 1.2.3 (e), the map

$$(x, \Theta) \mapsto \mathcal{L}_\Theta f(x)$$

is continuous. □

The literature contains no discussion of the 'continuous' dependence of $\mathcal{L}_\Theta$ on Θ. In [CMS80$_1$] it is shown that a *complex regular Birkhoff interpolation* procedure is liftable by writing it as a sum of what we have called here elementary liftable maps. Thus, we have the following.

Corollary 1.3.12. *Let B be the complex regular Birkhoff interpolation procedure and $\mathcal{B}$ its lift to $\mathbb{R}^n$. For each $f \in C^s(\mathbb{R}^n)$, the map*

$$\Theta \mapsto \mathcal{B}_\Theta f$$

is continuous.

In the case $n = 1$, i.e., when $\mathcal{B}_\Theta = B_\Theta$, this continuity result was proved in [DLR82] by using 'de-coalescence' of the interpolation matrix.

Another immediate consequence is the continuous dependence of the Hermite interpolant on its points of interpolation. However, that is a direct consequence of the well-known continuity of $\Theta \mapsto \delta_\Theta f$.

Related considerations

In [CMS80$_1$], [HM87], there is a discussion about lifting the family of distributions

$$\mathbb{R}^{k+1} \to S : (x, \Theta) \mapsto \delta_{[x]} L_\Theta,$$

where S is some suitable space of distributions, e.g., $C_0^\infty(\mathbb{R})$, or, in our case, $E'^s(\mathbb{R})$ (the space of compactly supported distributions of order s).

Lifting such a family is shown there to be equivalent to inverting its *Radon transform.* Without going too far into details, we mention that, for an elementary liftable map of the form (1.3.5), its Radon transform $\mathtt{H}$ is given by

$$\mathtt{H}(f) := \int_B \Big(\prod_{j=1}^{s} D_{a_j}\Big) f,$$

and so $\mathcal{L}_\Theta$ may be expressed as

$$\mathcal{L}_\Theta f(x) = \mathtt{H}(f \circ [x, \Theta]).$$

One useful consequence of the Radon transform theory is the following **compatibility condition**: if L is liftable, then $(x, \Theta) \mapsto L_\Theta((\cdot)^i)(x)$ is homogeneous of degree i. Moreover, by Property 1.2.3 (d), if L is an elementary liftable map and f is a homogeneous polynomial of degree i, then $(x, \Theta) \mapsto \mathcal{L}_\Theta f(x)$ is homogeneous of degree i.

1.4. The scale of mean value interpolations

In this section we describe a family $H^{(m)}$, $m < k$, of liftable maps that were lifted in [G83] to obtain multivariate polynomial interpolation schemes. Special cases of these multivariate schemes, referred to in [BHS93:p203] as *the scale of mean value interpolations*, are the well-known maps of Kergin and Hakopian.

We will need the following facts about linear interpolation.

Linear interpolation

Let F be a finite-dimensional space and Λ a finite-dimensional space of linear functionals defined at least on F. We say that the corresponding **linear interpolation problem**, LIP(F, Λ) for short, is **correct** if for every g upon which Λ is defined there is a unique $f \in F$ which agrees with g on Λ, i.e.,

$$\lambda(f) = \lambda(g), \quad \forall \lambda \in \Lambda.$$

The linear map $L : g \mapsto f$ is called the associated (**linear**) **projector** with **interpolants** F and **interpolation conditions** Λ. Each linear projector with finite-dimensional range F is the solution of a LIP(F, Λ) for some unique choice of the interpolation conditions Λ.

Notice that the correctness of $\mathrm{LIP}(F, \Lambda)$ depends only on the action of Λ on F.

The map $H^{(m)}$

Let $D^{-m} f$ be any function with $D^m(D^{-m} f) = f$. If

$$P : C^s(\mathbb{R}) \to \Pi_n(\mathbb{R})$$

is any linear projector, then for $m \leq n$

$$f \mapsto D^m P(D^{-m} f),$$

is a linear projector into $\Pi_{n-m}(\mathbb{R})$ which is defined on $C^{s-m}(\mathbb{R})$.

We are interested in the case where P is H_Θ, which is, by definition, the *Hermite interpolation operator* at Θ, a k-sequence in $\mathbb{R}$.

Definition 1.4.1. *For* $0 \leq m < k = \#\Theta$, *the* **generalised Hermite map**

$$H^{(m)} : \Theta \mapsto H_\Theta^{(m)}$$

is given by the linear projectors

$$H_\Theta^{(m)} : C^{k-m-1}(\mathbb{R}) \to \Pi_{k-m-1}(\mathbb{R}) : f \mapsto D^m(H_\Theta D^{-m} f).$$

For convenience, $H^{(k)} := 0$.

Observe that $H_\Theta^{(0)} = H_\Theta$, which in part justifies the term 'generalised Hermite map'. The generalised Hermite maps $H_\Theta^{(m)}$ occurred in the approximation theory literature before they were lifted by Goodman in [G83]; see e.g., de Boor [B75] where they were used to bound spline interpolation.

The interpolants for $H_\Theta^{(m)}$ are $\Pi_{k-m-1}(\mathbb{R})$, and the interpolation conditions are

$$\mathrm{span}\{f \mapsto \int_{\tilde\Theta} D^{\#\tilde\Theta-m-1} f : \tilde\Theta \subset \Theta,\ \#\tilde\Theta \geq m+1\}.$$

For Θ a finite sequence in $\mathbb{R}$, let

$$\omega_\Theta(x) := \prod_{\theta\in\Theta}(x-\theta).$$

Note that if $j \leq \#\Theta$, then

$$D^j\omega_\Theta = j! \sum_{\substack{\tilde\Theta\subset\Theta \\ \#\tilde\Theta=j}} \omega_{\Theta\backslash\tilde\Theta}. \tag{1.4.2}$$

If $\Theta = [\theta_1, \ldots, \theta_k]$, then we may write the 'Newton form' of $H_\Theta^{(m)}$ as

$$H_\Theta^{(m)} f(x) = \sum_{j=m+1}^{k} \delta_{\Theta_j}(D^{-m}f)\, D^m\omega_{\Theta_{j-1}}(x), \quad m < k. \tag{1.4.3}$$

The term 'Newton form' used here is justified not only by the fact that (1.4.3) is obtained by differentiating the Newton form of $H_\Theta(D^{-m}f)$, but by the observation that

$$H_{\Theta_{k+1}}^{(m)} f = H_{\Theta_k}^{(m)} f + \delta_{\Theta_{k+1}}(D^{-m}f)\, D^m\omega_{\Theta_k}, \quad m < k+1.$$

$\mathcal{H}^{(m)}$ the lift of $H^{(m)}$

We now show that $H^{(m)}$ is liftable to $\mathbb{R}^n$. The lifts $\mathcal{H}^{(m)}$, $m < k$, form what we call, with [BHS93], the **scale of mean value interpolations**.

By using (1.4.2) and the Hermite-Genocchi formula, the 'Newton form' (1.4.3) may be written as the following sum of elementary liftable maps:

$$H_\Theta^{(m)} f(x) = m! \sum_{j=m+1}^{k} \sum_{\substack{\tilde\Theta\subset\Theta_{j-1} \\ \#\tilde\Theta=m}} \Big(\prod_{\theta\in\Theta_{j-1}\backslash\tilde\Theta}(x-\theta)\Big) \int_{\Theta_j} D^{j-m-1} f. \tag{1.4.4}$$

We refer to this as the **Newton form** of $H_\Theta^{(m)}$.

Thus, by Theorem (1.3.6), the map $H^{(m)}$ can be lifted to $\mathcal{H}^{(m)}$, where

$$\mathcal{H}_\Theta^{(m)} : C^{k-m-1}(\mathbb{R}^n) \to \Pi_{k-m-1}(\mathbb{R}^n),$$

with its **Newton form** given by

$$\mathcal{H}_\Theta^{(m)} f(x) = m! \sum_{j=m+1}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{j-1} \\ \#\tilde\Theta = m}} \int_{\Theta_j} D_{x-\Theta_{j-1}\backslash\tilde\Theta} f. \tag{1.4.5}$$

This formula (1.4.5) is due to Goodman [G83]. He shows that each $\mathcal{H}_\Theta^{(m)}$ is a linear projector with range $\Pi_{k-m-1}(\mathbb{R}^n)$ and (lifted) interpolation conditions

$$\mathrm{span}\{f \mapsto \int_{\tilde\Theta} g(D)f : \tilde\Theta \subset \Theta,\ \#\tilde\Theta \geq m+1,\ g \in \Pi^0_{\#\tilde\Theta-m-1}(\mathbb{R}^n)\}. \tag{1.4.6}$$

Special cases

The map $\mathcal{H}_\Theta^{(0)}$ is the **Kergin map**, see [K80] and [M80]. The Newton form of Kergin's map,

$$\mathcal{H}_\Theta^{(0)} f(x) = f(\theta_1) + \int_{[\theta_1,\theta_2]} D_{x-\theta_1} f + \cdots + \int_{[\theta_1,\ldots,\theta_k]} D_{x-\theta_1} \cdots D_{x-\theta_{k-1}} f,$$

is given in [M80] and [MM80]. Notice that the interpolation conditions of this map include evaluation at the points Θ. Thus Kergin's map is a multivariate generalisation of Lagrange interpolation.

The map $\mathcal{H}_\Theta^{(1)}$ was introduced in [CMS80$_2$] where it was referred to as the **area matching map**. Presumably the term 'area matching' came from the fact that if the points in $\Theta := [\theta_1, \ldots, \theta_k]$ in $\mathbb{R}$ are distinct, then the interpolation conditions of $H_\Theta^{(1)}$ are

$$\mathrm{span}\{f \mapsto \int_{\theta_i}^{\theta_{i+1}} f : i = 1, \ldots, k-1\}.$$

If the $k \geq n$ points in Θ are in general position in $\mathbb{R}^n$, then $\mathcal{H}_\Theta^{(n-1)}$ is the **Hakopian map**, see [H81] and [H82$_2$]. For this map, the interpolation conditions may be written as

$$\mathrm{span}\{f \mapsto \int_{\tilde{\Theta}} f : \tilde{\Theta} \subset \Theta,\ \#\tilde{\Theta} = n\}.$$

Thus, $\mathcal{H}_\Theta^{(n-1)}$ has an extension (the map originally given by Hakopian) to $C(\mathbb{R}^n)$ and interpolants $\Pi_{k-n}(\mathbb{R}^n)$. Though not immediately apparent from (1.4.6), the interpolation conditions for Hakopian's map include evaluation at the points Θ. Thus it, like Kergin's map, provides a multivariate generalisation of Lagrange interpolation.

For additional discussion on expressing the interpolation conditions for $\mathcal{H}_\Theta^{(m)}$ in terms of derivatives of lower orders than given in (1.4.6), see [DM83].

1.5. Integral error formulæ

Observe that

$$f - H_\Theta^{(m)} f = D^m\big(D^{-m} f - H_\Theta(D^{-m} f)\big). \tag{1.5.1}$$

Thus, to obtain an error formula for $\mathcal{H}^{(m)}$, one might hope to lift the error formula for Hermite interpolation. In this section, this is done in two ways. The first and more natural way introduces derivatives of higher order than one might like. In the second, this deficiency is remedied by taking advantage of a little-known formula for the derivative of the error in Hermite interpolation.

The first error formula

Using the differentiation rule for divided differences

$$\frac{d^i}{dx^i}\delta_{[x,\Theta]}f = i!\,\delta_{[\underbrace{x,\ldots,x}_{i+1},\Theta]}f, \tag{1.5.2}$$

the Hermite error formula

$$D^{-m}f(x) - H_\Theta(D^{-m}f)(x) = \omega_\Theta(x)\,\delta_{[x,\Theta]}(D^{-m}f) \tag{1.5.3}$$

can be differentiated (m times) to obtain, by (1.5.1), that

$$f(x) - H_\Theta^{(m)}f(x) = \sum_{j=0}^{m}\binom{m}{j} D^j\omega_\Theta(x)\,(m-j)!\,\delta_{[\underbrace{x,\ldots,x}_{m-j+1},\Theta]}(D^{-m}f). \tag{1.5.4}$$

Using (1.4.2) and the Hermite-Genocchi formula, we may write (1.5.4) as

$$f(x) - H_\Theta^{(m)}f(x) = m!\sum_{j=0}^{m}\sum_{\substack{\tilde\Theta\subset\Theta\\ \#\tilde\Theta=j}} \omega_{\Theta\setminus\tilde\Theta}(x)\int_{[\underbrace{x,\ldots,x}_{m-j+1},\Theta]}(D^{k-j}f), \quad \forall f\in C^k(\mathbb{R}). \tag{1.5.5}$$

The formula (1.5.5) expresses the error, $f\mapsto f - H_\Theta^{(m)}f$, as a sum of elementary liftable maps of orders $k-m,\ldots,k$. Thus, using Theorem 1.3.6, this can be lifted, thereby giving the following.

First error formula. *If $m<k$ and $f\in C^k(\mathbb{R}^n)$, then*

$$f(x) - \mathcal{H}_\Theta^{(m)}f(x) = m!\sum_{j=0}^{m}\sum_{\substack{\tilde\Theta\subset\Theta\\ \#\tilde\Theta=j}} \int_{[\underbrace{x,\ldots,x}_{m-j+1},\Theta]} D_{x-\Theta\setminus\tilde\Theta}f. \tag{1.5.6}$$

For Kergin interpolation, i.e., when $m=0$, this formula reduces to

$$f(x) - \mathcal{H}_\Theta^{(0)} = \int_{[x,\Theta]} D_{x-\Theta}f, \tag{1.5.7}$$

which was given in Micchelli [M80].

The only other mention of this formula in the literature is for Hakopian interpolation, i.e., when $m = n-1$, and occurs in the book [BHS93:p200]. There (1.5.6) is stated incorrectly, and without proof, as

$$f(x) - \mathcal{H}_\Theta^{(n-1)} f(x) = \sum_{j=0}^{n-1} \sum_{\substack{\tilde\Theta\subset\Theta \\ \#\tilde\Theta = j}} \binom{n-1}{j} \int_{[\underbrace{x,\ldots,x}_{m-j+1},\Theta]} D_{x-\Theta\backslash\tilde\Theta} f.$$

In other words, the constant $\binom{n-1}{j}$ there should be replaced by $(n-1)!$.

The interpolants for $\mathcal{H}_\Theta^{(m)}$ are $\Pi_{k-m-1}(\mathbb{R}^n)$. The error formula (1.5.6) involves derivatives of orders $k-m, \ldots, k$. For $m > 0$, it would be desirable to not have the higher derivatives $k-m+1, \ldots, k$ occurring. We now give such a formula.

The second error formula

The higher derivatives in (1.5.6) are introduced when (1.5.2) is used to differentiate $x \mapsto \delta_{[x,\Theta]}(D^{-m} f)$ in (1.5.3). To avoid this problem, we use the following formula for the derivative in Hermite interpolation. It was given independently by Dokken and Lyche [DoLy78], [DoLy79] and by Wang [W78], [W79].

Theorem 1.5.8 ([DoLy78],[W78]). *If* $\Theta = [\theta_1, \ldots, \theta_k]$, $0 \le j < k$ *and* $f \in C^k(\mathbb{R})$, *then*

$$D^j(f - H_\Theta f)(x) = j! \sum_{i=k-j}^{k} \frac{(x-\theta_i)}{(j+i-k)!} D^{j+i-k} \omega_{\Theta_{i-1}}(x)\, \delta_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} f.$$

Applying to (1.5.1), Theorem 1.5.8 followed by the Hermite-Genocchi formula, we obtain that, for $f \in C^{k-m}(\mathbb{R})$,

$$f(x) - H_\Theta^{(m)} f(x) = m! \sum_{i=k-m}^{k} \frac{(x-\theta_i)}{(m+i-k)!} D^{m+i-k} \omega_{\Theta_{i-1}}(x) \int_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} D^{k-m} f. \tag{1.5.9}$$

This formula (1.5.9) is a sum of elementary liftable maps, each of order $k-m$. Its lift, using Theorem 1.3.6, gives the following error formula for $\mathcal{H}_\Theta^{(m)}$.

Second error formula. *If $m < k$ and $f \in C^{k-m}(\mathbb{R}^n)$, then*

$$f(x) - \mathcal{H}_\Theta^{(m)} f(x) = m! \sum_{i=k-m}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{i-1} \\ \#\tilde\Theta = m+i-k}} \int_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} D_{[x-\Theta_{i-1}\backslash\tilde\Theta, x-\theta_i]} f. \tag{1.5.10}$$

This formula involves only derivatives of f of order $k-m$.

Those worried that the formula (1.5.10) is not symmetric in the points of Θ could, if desired, take the average over all possible orderings for Θ to obtain such a symmetric formula. More to the point, it would be desirable to find the 'simplest' symmetric form of Theorem 1.5.8.

Derivatives of the error

The univariate identity

$$D^j(H_\Theta^{(m)} f) = H_\Theta^{(m+j)}(D^j f)$$

can be 'lifted' to the following; see, e.g., [BHS93:p205].

Proposition 1.5.11. *If $m < k$, $j < k-m$, $g \in \Pi_j^0(\mathbb{R}^n)$ and $f \in C^{k-m-1}(\mathbb{R}^n)$, then*

$$g(D)(\mathcal{H}_\Theta^{(m)} f) = \mathcal{H}_\Theta^{(m+j)}(g(D)f).$$

This allows us, in a very natural way, to use an error formula for $\mathcal{H}_\Theta^{(m)}$ to describe the *derivatives of* the error in $\mathcal{H}_\Theta^{(m)}$. In particular, with the second error formula (1.5.10), we obtain the following.

Theorem 1.5.12. *If $m < k$, $j < k - m$, $g \in \Pi_j^0$ and $f \in C^{k-m}(\mathbb{R}^n)$, then*

$$g(D)(f - \mathcal{H}_\Theta^{(m)} f)(x)$$
$$= (m+j)! \sum_{i=k-m-j}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{i-1} \\ \#\tilde\Theta = m+j+i-k}} \int_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} D_{[x-\Theta_{i-1}\backslash\tilde\Theta, x-\theta_i]} g(D) f.$$

This formula involves only derivatives of f of order $k - m$.

Proof. By Proposition 1.5.11,

$$g(D)(f - \mathcal{H}_\Theta^{(m)} f) = (g(D)f) - \mathcal{H}_\Theta^{(m+j)}(g(D)f).$$

Since $g(D)f \in C^{k-(m+j)}(\mathbb{R}^n)$, we may apply the second error formula (1.5.10) to the error in $\mathcal{H}_\Theta^{(m+j)}$ at $g(D)f$, thereby obtaining the given formula. □

This theorem is the major result of this chapter. Special cases of it include the second error formula (1.5.10) and Theorem 1.5.8. It expresses the error in $\mathcal{H}_\Theta^{(m)} f$, and its derivatives, in terms of integration against the derivative of order one higher than the degree of the interpolating polynomial space. This is precisely the estimate that numerical analysts want, to guarantee that their scheme, e.g., a $\mathcal{H}_\Theta^{(m)}$ finite element (see, e.g., [L92:p164]), has the maximum possible *order*.

From this Theorem, L_∞-estimates for the error can easily be obtained. This is done in Section 1.6.

Comparison with the results of Lai-Wang and Gao

The results of [LW84], [LW86] and [Ga88] are written in terms of the **mul-**

tivariate divided differences

$$[\theta_1,\ldots,\theta_{|\alpha|}]^{\alpha} f := \int_{[\theta_1,\ldots,\theta_{|\alpha|}]} D^{\alpha} f, \quad \forall \alpha \in \mathbb{Z}_+^s. \tag{1.5.13}$$

The simplest of these results to state is the following error formula for Kergin interpolation.

Theorem 1.5.14 ([LW86:Th.3.1]). *If $\alpha \in \mathbb{Z}_+^s$ with $|\alpha| \le j < k-1$, then*

$$\begin{aligned} D^{\alpha}(f - \mathcal{H}_{\Theta}^{(0)} f)(x) = & \sum_{r=0}^{|\alpha|} \sum_{\substack{\gamma \le \alpha \\ |\gamma| = r}} \sum_{\substack{\beta \ge \alpha - \gamma \\ |\beta| = j - r}} r! \binom{\alpha}{\gamma} D^{\alpha-\gamma} \omega_{\beta}(x) \sum_{i=1}^{n} (x - \theta_{j-r+1})_i \\ & [\underbrace{x,\ldots,x}_{r+1}, \theta_1,\ldots,\theta_{j-r+1}]^{\beta+\gamma+e^i} f \\ & - \sum_{r=j+1}^{k-1} \sum_{\substack{\gamma \ge \alpha \\ |\gamma| = r}} D^{\alpha} \omega_{\gamma}(x)\ [\theta_1,\ldots,\theta_{r+1}]^{\gamma} f, \end{aligned} \tag{1.5.15}$$

where

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n},$$

and

$$\omega_{\gamma}(x) := \sum_{e^{i_1} + \cdots + e^{i_{|\gamma|}} = \gamma} (x - \theta_1)_{i_1} \cdots (x - \theta_{|\gamma|})_{i_{|\gamma|}}.$$

The above uses standard multi-index notation. The i-th component of $x \in \mathbb{R}^n$ is x_i, and e^i is the i-th unit vector in $\mathbb{R}^n$.

Formula (1.5.15) of Theorem 1.5.14 involves derivatives of f of orders $j+1,\ldots,k-1$; whereas the formula (1.5.10) involves only derivatives of order k.

Also, in the case of greatest interest for this formula, namely when $j+1 = k-1$ and $\alpha = 0$, formula (1.5.15) reduces, in the univariate case, to

$$f(x) - H_{\Theta} f(x) = \omega_{\Theta_{k-1}}(x) \int_{[x,\theta_1,\ldots,\theta_{k-1}]} D^{k-1} f - \omega_{\Theta_{k-1}}(x) \int_{[\theta_1,\ldots,\theta_k]} D^{k-1} f. \tag{1.5.16}$$

Since formula (1.5.16) is a sum of elementary liftable maps, and follows from one application of (1.3.10) to the Hermite error formula

$$f(x) - H_\Theta f(x) = (x-\theta_1)\cdots(x-\theta_k)\int_{[x-\theta_1,\cdots,x-\theta_k]} D^k f,$$

we obtain at once the case $j+1 = k-1$ and $\alpha = 0$ of Theorem 1.5.14 by lifting (1.5.16), and in the following form:

$$f(x) - \mathcal{H}_\Theta^{(0)} f(x) = \int_{[x,\theta_1,\ldots\theta_{k-1}]} D_{x-\theta_1}\cdots D_{x-\theta_{k-1}} f - \int_{[\theta_1,\ldots\theta_k]} D_{x-\theta_1}\cdots D_{x-\theta_{k-1}} f. \tag{1.5.17}$$

If one now expands (1.5.17) in multivariate divided differences, then one obtains (1.5.15) for this case. However, it is not clear what has been gained in the process.

Similar considerations, can, and should, be given to other formulas in [LW84], [LW86] and [Ga88].

Additional comments

The only justification for the term 'multivariate divided difference' for (1.5.13) that the author can see, is the identity (1.3.10), which is due to Micchelli (see [M80:Th.6]), and (in its many guises) pervades the multivariate spline literature. With that justification, the term might as well be applied to any linear combination of functionals

$$f \mapsto \int_\Theta g(D)f, \quad \Theta \in (\mathbb{R}^n)^k,\ g \in \Pi_j(\mathbb{R}^n),$$

that can be expressed as a linear combination of other such functionals involving lower order derivatives of f.

1.6. L_∞-estimates

In this final section, we obtain L_∞-estimates from the formulæ of Section 1.5. Our choice of the seminorm $|\cdot|_{k,\infty,K}$ defined in (1.1.2) makes this a straightforward task. Let

$$h_{x,\Theta} := \max_{\theta\in\Theta} \|x-\theta\| \le \operatorname{diam}[x,\Theta].$$

From the first error formula (1.5.6), we obtain the following L_∞-estimate.

Proposition 1.6.1. *If $m < k$ and $f \in C^k(\mathbb{R}^n)$, then*

$$|f(x) - \mathcal{H}_\Theta^{(m)} f(x)| \le \sum_{j=0}^{m} \operatorname{const}_{j,k,m} (h_{x,\Theta})^{k-j} |f|_{k-j,\infty,\operatorname{conv}[x,\Theta]},$$

where

$$\operatorname{const}_{j,k,m} := \frac{m!}{(k+m-j)!}\binom{k}{j}.$$

Proof. To the first error formula (1.5.6), apply Property 1.2.3 (c), then use (1.1.1) and (1.1.2) to obtain

$$|f(x) - \mathcal{H}_\Theta^{(m)} f(x)| \le m! \sum_{j=0}^{m} \sum_{\substack{\tilde\Theta\subseteq\Theta \\ \#\tilde\Theta=j}} \frac{1}{(k+m-j)!}(h_{x,\Theta})^{k-j} |f|_{k-j,\infty,\operatorname{conv}[x,\Theta]}.$$

Lastly, observe that

$$\#\{\tilde\Theta \subset \Theta : \#\tilde\Theta = j\} = \binom{k}{j}.$$

□

From Theorem 1.5.12, we obtain the main result of this section.

Theorem 1.6.2. *If $m < k$, $j < k-m$ and $f \in C^{k-m}(\mathbb{R}^n)$, then*

$$|D^j(f - \mathcal{H}_\Theta^{(m)} f)|(x) \le \frac{1}{(k-m-j)!}(h_{x,\Theta})^{k-m-j} |f|_{k-m,\infty,\operatorname{conv}[x,\Theta]}. \qquad (1.6.3)$$

The constant is the best possible in the sense that if $\Theta = [\theta, \ldots, \theta]$, *then it cannot be improved.*

Proof. To prove the inequality, begin as in the proof of Proposition 1.6.1, then use the identity:

$$\frac{(m+j)!}{k!} \sum_{i=k-m-j}^{k} \binom{i-1}{m+j+i-k} = \frac{1}{(k-m-j)!}.$$

Suppose $\Theta = [\theta, \ldots, \theta]$. By (1.4.6) we have that $\mathcal{H}_\Theta^{(m)} f$ is the *Taylor interpolant* from $\Pi_{k-m-1}(\mathbb{R}^n)$ to f at θ. Let $u := (x-\theta)/\|x-\theta\|$. Note that $h_{x,\Theta} = \|x-\theta\|$. Then for the plane wave

$$f := (\cdot - u^*\theta)^{k-m} \circ u^* \in \Pi_{k-m}(\mathbb{R}^n),$$

$\mathcal{H}_\Theta^{(m)} f = 0$, and we have, by (1.3.1), that

$$\begin{aligned}\frac{|D^j(f - \mathcal{H}_\Theta^{(m)} f)|(x)}{|f|_{k-m,\infty,\mathrm{conv}[x,\Theta]}} &\geq \frac{|D_u^j f(x)|}{(k-m)!} \\ &= \frac{(k-m)\cdots(k-m-j+1)}{(k-m)!}(\cdot - u^*\theta)^{k-m-j} \circ (u^*x) \\ &= \frac{1}{(k-m-j)!}(h_{x,\Theta})^{k-m-j}.\end{aligned}$$

Thus, in the case $\Theta = [\theta, \ldots, \theta]$, the constant is the best possible. □

When $m = 0$, Proposition 1.6.1 and Theorem 1.6.2 (with $j = 0$) reduce to

$$|f(x) - \mathcal{H}_\Theta^{(0)} f(x)| \leq \frac{1}{k!}(h_{x,\Theta})^k |f|_{k,\infty,\mathrm{conv}[x,\Theta]},$$

which was given in [M80]. For $m > 0$, none of the above L_∞-estimates are in the literature.

Remark 1.6.4. In [Bo83:Th.2.5] Bos gives the following estimate for Kergin interpolation on the disc. Let Θ consist of k points equally spaced on the disc $\{x \in \mathbb{R}^2 : \|x\| = h\}$, where $h > 0$. Then for $f \in C^k(\mathbb{R}^2)$

$$\max_{\|x\|\le h} |f(x) - \mathcal{H}_\Theta^{(0)} f(x)| \le \frac{1}{k!}\frac{4}{2^k} h^k |f|_{k,\infty,\{x:\|x\|\le h\}}.$$

This indicates that it may be possible to reduce the size of the constant in (1.6.3) for restricted values of $h_{x,\Theta}$. However, in view of the sharpness for the case of *Taylor interpolation* (when $\Theta = [\theta, \ldots, \theta]$) and the continuity of $\Theta \mapsto \mathcal{H}_\Theta^{(m)} f$ (by Theorem 1.3.11), for unrestricted values of $h_{x,\Theta}$ the constant is the best possible in all cases.

It is not possible to apply Properties 1.2.3 (c) to the integral error formulæ of this chapter to obtain L_p-estimates for $1 \le p < \infty$. A partial solution to this impasse, which uses a multivariate form of Hardy's inequality, is given in Chapter 2.

2. A multivariate form of Hardy's inequality and L_p-error bounds for multivariate Lagrange interpolation schemes

2.1. Introduction

The central result of this chapter is the inequality, that for $m - n/p > 0$

$$\| x \mapsto \int_{[\underbrace{x,\ldots,x}_{m},\Theta]} f \,\|_{L_p(\Omega)} \le \frac{1}{(m-1)!(m-n/p)_{\#\Theta}} \|f\|_{L_p(\Omega)}, \quad \forall f \in L_p(\Omega), \tag{2.1.1}$$

where Θ is a finite sequence of points in $\mathbb{R}^n$, and Ω is a suitable domain in $\mathbb{R}^n$. This inequality is a *multivariate* generalisation of **Hardy's inequality**, that for $p > 1$

$$\| x \mapsto \frac{1}{x}\int_0^x f \,\|_{L_p(0,\infty)} \le \frac{p}{p-1} \|f\|_{L_p(0,\infty)}, \quad \forall f \in L_p(0,\infty). \tag{2.1.2}$$

Thus, we will refer to (2.1.1) as the **multivariate form of Hardy's inequality**.

Our interest in (2.1.1) comes from a desire to obtain L_p-bounds from the many integral error formulæ for *multivariate* generalisations of Lagrange interpolation that involve the linear functional

$$f \mapsto \int_{[\underbrace{x,\ldots,x}_{m},\Theta]} f. \tag{2.1.3}$$

The chapter is set out in the following way. In the remainder of this section, the notation, and facts about Sobolev spaces that we will need are discussed. In Section 2.2, the multivariate form of Hardy's inequality is proved. In Section 2.3, the multivariate form of Hardy's inequality is applied to obtain L_p-bounds for the error in the scale of mean value interpolations, which includes Kergin and Hakopian interpolation. In Section 2.4, in a similar vein, L_p-bounds for the error in *Lagrange maps* are obtained. In Section 2.5, we discuss why the multivariate form of Hardy's inequality is applicable to the many error formulæ for multivariate Lagrange interpolation schemes, and is likely to be so for others obtained in the future.

Some additional notation

Let $\Omega \subset \mathbb{R}^n$, with $\bar{\Omega}$ its closure. The letters i, j, k, l, m, n will be reserved for integers, and $1 \le p \le \infty$.

Many of the constants in this chapter involve the **shifted factorial function**

$$(a)_n := (a)(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad (2.1.4)$$

where Γ is the **Gamma function**. The Gamma function satisfies the relation: $\Gamma(a+1) = a\Gamma(a)$, $\forall a > 0$, and has $\Gamma(1) = 1$. Some of our calculations require the **Beta integrals**

$$\int_0^1 t^{a-1}(1-t)^{b-1}\, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad a, b > 0, \qquad (2.1.5)$$

and the **hypergeometric function**

$${}_2F_1\left({a, b \atop c}; x \right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} x^n. \qquad (2.1.6)$$

The standard reference to these is the monograph [E53].

Geometry of the domain Ω

We say that $\Omega \subset \mathbb{R}^n$ is **starshaped with respect to** S a set (resp. sequence) in $\mathbb{R}^n$ when Ω contains the convex hull of $S \cup \{x\}$ for any $x \in \Omega$. This condition is weaker than Ω being convex.

In our results, it will be required that $\bar{\Omega}$ be starshaped with respect to $\Theta \in \mathbb{R}^{n\times k}$, where Ω is an open set in $\mathbb{R}^n$. This condition is required of $\bar{\Omega}$, rather than of Ω, so as to include cases where some points in Θ lie on the boundary of Ω. One such example of interest is the *Lagrange* finite element given by linear interpolation at Θ, the vertices of a n-simplex, see, e.g. Ciarlet [Ci78:p46]. In this case, $\bar{\Omega} = \operatorname{conv}\Theta$ and none of the points of Θ lies in the open simplex Ω.

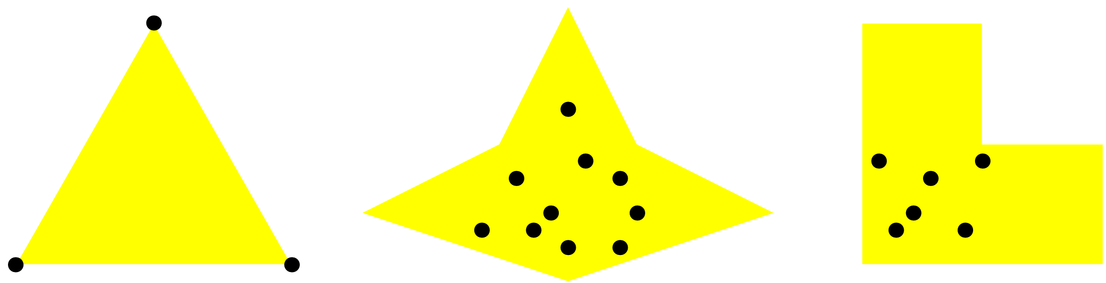

Fig 2.1.1 Examples of domains Ω (shaded) for which $\bar{\Omega}$ is starshaped with respect to the points in Θ ($\bullet$)

We now show that being starshaped with respect to a finite sequence is equivalent to being starshaped with respect to its convex hull.

Proposition 2.1.7. *If $\Omega \subset \mathbb{R}^n$ and $\Theta \in \mathbb{R}^{n\times k}$, then the following are equivalent:*

(a) *Ω is starshaped with respect to Θ.*

(b) *Ω is starshaped with respect to $\operatorname{conv}\Theta$.*

Proof. Only the implication (a) $\Longrightarrow$ (b) requires proof. Suppose (a). To obtain (b) it suffices to prove that if Ω is starshaped with respect to points u and v, then $\text{conv}\{u, v, x\} \subset \Omega$, $\forall x \in \Omega$, i.e., Ω is starshaped with respect to $\text{conv}\{u, v\}$.

Assume wlog that u, v, x are affinely independent and $z \in \text{conv}\{u, v, x\}$. Let w be the point of intersection of the line through u and z with the interval $\text{conv}\{x, v\}$. Since Ω is starshaped with respect to v, one has that $w \in \Omega$. Thus, since Ω is starshaped with respect to u, one has that $z \in \text{conv}\{u, w\} \subset \Omega$. $\square$

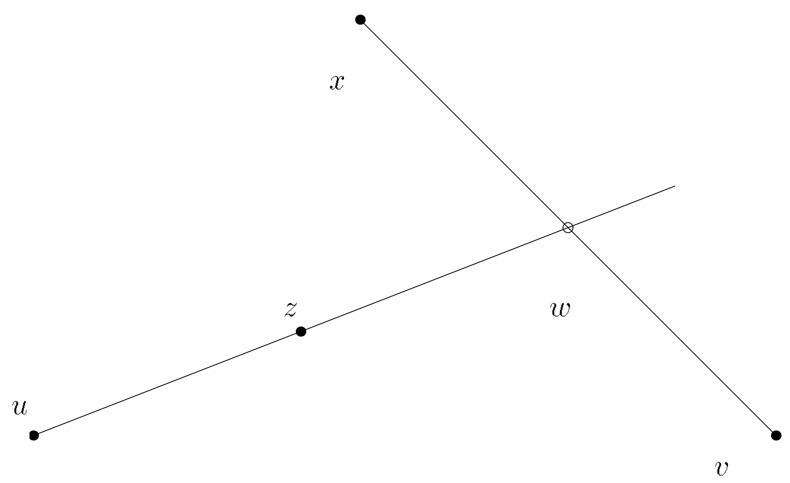

Fig 2.1.2 The proof of Proposition 2.1.7

This equivalence ensures that if $\bar{\Omega}$ is starshaped with respect to Θ, then $f \in L_p(\Omega)$ is defined over the region of integration in (2.1.3) for all $x \in \Omega$.

Sobolev spaces

Let $W_p^{(k)}(\Omega)$ be the **Sobolev space** consisting of those functions defined on Ω (a bounded open set in $\mathbb{R}^n$ with a *Lipschitz* boundary) with derivatives up to order k in $L_p(\Omega)$, and equipped with the usual topology; see, e.g., Adams [Ad75].

It is convenient to include in the definition the condition that Ω have a Lipschitz boundary, so that Sobolev's embedding theorem can be applied. The full statement of Sobolev's embedding theorem can be found in any text on Sobolev spaces, see, e.g., [Ad75:p97]; however we will need only the following consequence of it. If $j - n/p > 0$, then

$$W_p^{k+j}(\Omega) \subset C^k(\bar{\Omega}).$$

To measure the size of its k-th derivative, it is convenient to associate with each $f \in W_p^{(k)}(\Omega)$ the function $|D^k f| \in L_p(\Omega)$, given by the rule

$$|D^k f|(x) := \sup_{\substack{\Theta \in \mathbb{R}^{n\times k} \\ \|\theta_i\| \le 1}} |D_\Theta f(x)| = \sup_{\substack{\theta \in \mathbb{R}^n \\ \|\theta\|=1}} |D_\theta^k f(x)|, \tag{2.1.8}$$

where the derivatives $D_\Theta f$ are computed from any (fixed) choice of representatives for the partial derivatives $D^\alpha f \in L_p(\Omega)$, $|\alpha| = k$. The equality of the two suprema is proved in Chen and Ditzian [CD90]. This definition of $|D^k f|$ is consistent with its alternative interpretation in the univariate case. From (2.1.8), it is easy to see that $|D^k f|$ is well-defined and satisfies

$$|D_\Theta f| \le |D^k f| \, \|\theta_1\| \cdots \|\theta_k\|, \tag{2.1.9}$$

for all $\Theta \in \mathbb{R}^{n\times k}$. The inequality (2.1.9) holds a.e. To emphasize that $D_\Theta f$, $|D^k f| \in L_p(\Omega)$, we will say that (2.1.9) holds in $L_p(\Omega)$. The $L_p(\Omega)$-norm of $|D^k f|$ gives a seminorm on $W_p^{(k)}(\Omega)$,

$$f \mapsto |f|_{k,p,\Omega} := \| \, |D^k f| \, \|_{L_p(\Omega)}. \tag{2.1.10}$$

Because of (2.1.9), this coordinate-independent seminorm (2.1.10) is more appropriate for the analysis that follows than other equivalent seminorms, such as

$$f \mapsto \| \, (\|D^\alpha f\|_{L_p(\Omega)} : |\alpha| = k) \, \|_p.$$

2.2. The main results: the multivariate form of Hardy's inequality

In this section we prove the multivariate form of Hardy's inequality. This inequality is useful for obtaining L_p-bounds from integral error formulæ for various multivariate interpolation schemes.

First we need a technical lemma.

Lemma 2.2.1. *Let m, k be integers, and $\mu \in \mathbb{R}$. If $1 \le m \le k$ and $m + \mu > 0$, then*

$$\int_0^1 \int_0^{s_1} \cdots \int_0^{s_{k-1}} (1 - s_m)^\mu \, ds_k \cdots ds_1 = \frac{\Gamma(m+\mu)}{\Gamma(m)\Gamma(k+1+\mu)}.$$

Proof. This can be proved by successively evaluating the univariate integrals. Instead we give the following proof – a neat application of the properties of $f \mapsto \int_\Theta f$. From Definition 1.2.1, we see that

$$\int_0^1 \int_0^{s_1} \cdots \int_0^{s_{k-1}} (1 - s_m)^\mu \, ds_k \cdots ds_1 = \int_\Theta (\cdot)^\mu,$$

where

$$\Theta := [\underbrace{0, \ldots, 0}_{m}, \underbrace{1, \ldots, 1}_{k+1-m}].$$

For this Θ, the nontrivial part of $M(\cdot|\Theta)$ is a polynomial of order k on $[0\,.\,.\,1]$, with $(m-1)$-fold, $(k-m)$-fold zeros at 0, 1 respectively. Since $\int M(\cdot|\Theta) = 1$, (2.1.5) implies that

$$M(t|\Theta) = \frac{\Gamma(k+1)}{\Gamma(m)\Gamma(k+1-m)} t^{m-1} (1-t)^{k-m}, \quad 0 \le t \le 1.$$

From (1.2.6) and (2.1.5) we conclude that

$$\begin{aligned}\int_{\Theta}(\cdot)^{\mu} =& \frac{1}{\Gamma(k+1)}\int_0^1 (\cdot)^{\mu} M(\cdot|\Theta)\\ =& \frac{1}{\Gamma(m)\Gamma(k+1-m)}\int_0^1 t^{\mu}t^{m-1}(1-t)^{k-m}\,dt\\ =& \frac{\Gamma(m+\mu)}{\Gamma(m)\Gamma(k+1+\mu)}.\end{aligned}$$

Here the condition that $m+\mu>0$ is needed to ensure that the Beta integral is finite. □

The multivariate form of Hardy's inequality

Now we prove the multivariate form of Hardy's inequality.

Theorem 2.2.2. *Let Θ be a finite sequence in $\mathbb{R}^n$, and let Ω be an open set in $\mathbb{R}^n$ for which $\bar{\Omega}$ is starshaped with respect to Θ. If $m-n/p>0$, then the rule*

$$L_{m,\Theta}f(x) := \int_{[\underbrace{x,\ldots,x}_{m},\Theta]} f \tag{2.2.3}$$

induces a monotone bounded linear map $L_{m,\Theta}: L_p(\Omega)\to L_p(\Omega)$ with norm

$$\|L_{m,\Theta}\| \le \frac{1}{(m-1)!(m-n/p)_{\#\Theta}} \to \infty \quad \text{as} \quad m-n/p\to 0^+. \tag{2.2.4}$$

This upper bound for $\|L_{m,\Theta}\|$ is sharp when $p=\infty$.

Proof. Suppose that $m-n/p>0$. Then $m>0$, and we let $k+1 := m+\#\Theta$. Let $\mathbf{L}_p(\Omega)$ be the semi-normed linear space consisting of those (measurable) functions f defined on Ω with $\|f\|_{L_p(\Omega)}<\infty$, together with the semi-norm $\|\cdot\|_{L_p(\Omega)}$. Let Z be the set of those $f\in\mathbf{L}_p(\Omega)$ for which $\|f\|_{L_p(\Omega)}=0$. By Proposition 2.1.7, the condition that $\bar{\Omega}$ be starshaped with respect to Θ ensures that it is starshaped with respect to $\operatorname{conv}\Theta$. In particular, for any $x \in \Omega$, the region of integration in (2.2.3) is contained within $\bar{\Omega}$ (upto a null set := set of measure zero). However, *a priori*, we do not know whether (2.2.3) defines a function $L_{m,\Theta} f \in L_p(\Omega)$ for every $f \in L_p(\Omega)$, i.e., equivalently, that the linear map $\mathcal{L}_{m,\Theta} : \mathbf{L}_p(\Omega) \to \mathbf{L}_p(\Omega)$ given by (2.2.3) maps Z to Z. In view of Remark 1.2.5, to show this, together with the bound for $\|L_{m,\Theta}\|$, it is sufficient to prove the inequality

$$\|L_{m,\Theta} f\|_{L_p(\Omega)} \le \frac{\Gamma(m - n/p)}{\Gamma(m)\Gamma(k + 1 - n/p)} \|f\|_{L_p(\Omega)}, \tag{2.2.5}$$

for all $f \in \mathbf{L}_p(\Omega)$ which are nonnegative. In this case, $L_{m,\Theta} f$ is a well-defined nonnegative function, which could possibly take on the value ∞.

We now prove (2.2.5). Let $f \in \mathbf{L}_p(\Omega)$ be nonnegative, and write

$$[\underbrace{x, \ldots, x}_{m}, \Theta] = [\underbrace{x, \ldots, x}_{m}, \theta_m, \theta_{m+1}, \ldots, \theta_k].$$

By Definition 1.2.1,

$$L_{m,\Theta} f(x) = \int_S f(A_x s)\, ds, \tag{2.2.6}$$

where $s := (s_1, \ldots, s_k)$ and

$$\int_S := \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{k-1}}, \qquad ds := ds_k \cdots ds_1,$$
$$A_x s := x + s_m(\theta_m - x) + s_{m+1}(\theta_{m+1} - \theta_m) + \cdots + s_k(\theta_k - \theta_{k-1}).$$

Applying Minkowski's inequality for integrals (see, e.g., Folland [Fo84:p186]) to the sum $\int_S$ of functions $x \mapsto f(A_x s)$ we obtain, by (2.2.6), that

$$\|L_{m,\Theta} f\|_{L_p(\Omega)} \le \int_S \|x \mapsto f(A_x s)\|_{L_p(\Omega)}\, ds. \tag{2.2.7}$$

The case $1 \le p < \infty$. We may write (2.2.7) as

$$\|L_{m,\Theta} f\|_{L_p(\Omega)} \le \int_S \left(\int_\Omega f(A_x s)^p \, dx \right)^{1/p} ds.$$

In the inner integral, make the change of variables $y = A_x s$. For this choice, the new region of integration is contained in Ω, and $dy = (1 - s_m)^n dx$. Thus, by the change of variables formula, see, e.g., Rudin [Ru87:p153], we obtain that

$$\begin{aligned}\int_S \left(\int_\Omega f(A_x s)^p \, dx \right)^{1/p} ds &\le \int_S \left(\int_\Omega \frac{f(y)^p \, dy}{(1 - s_m)^n} \right)^{1/p} ds \\ &= \left(\int_S (1 - s_m)^{-n/p} \, ds \right) \|f\|_{L_p(\Omega)}.\end{aligned}$$

Finally, by Lemma 2.2.1 with $m + \mu = m - n/p > 0$, we have

$$\int_S (1 - s_m)^{-n/p} \, ds = \frac{\Gamma(m - n/p)}{\Gamma(m)\Gamma(k + 1 - n/p)},$$

giving (2.2.4) for $1 \le p < \infty$.

The case $p = \infty$. Since $x \mapsto A_x s$ maps null sets to null sets, we obtain from (2.2.7) that

$$\|L_{m,\Theta} f\|_{L_\infty(\Omega)} \le \int_S \|f\|_{L_\infty(\Omega)} \, ds = \frac{1}{k!} \|f\|_{L_\infty(\Omega)},$$

with equality when f is constant. Here we used

$$\int_S ds = \frac{1}{k!} = \frac{\Gamma(m)}{\Gamma(m)\Gamma(k + 1)},$$

which follows from Observation 1.2.2, or by Lemma 2.2.1 with $\mu = 0$. This completes the case $p = \infty$. □

Remark 2.2.8. If $\operatorname{vol}_n(\operatorname{conv} \Theta) > 0$, then, by Remark 1.2.5, it follows that the value of $L_{m,\Theta} f(x)$ is the same for all representatives of $f \in L_p(\Omega)$. Indeed, by

Proposition 1.2.9, for all $f \in L_p(\Omega)$ we have that $L_{m,\Theta} f \in C(\bar{\Omega})$, regardless of whether or not $m - n/p > 0$.

On the other hand, when $\mathrm{vol}_n(\mathrm{conv}\,\Theta) = 0$, then the function $L_{m,\Theta} f$ need not be so well-behaved. For example, if $n > 1$ and Θ consists of a single point θ, then $f \in L_p(\Omega)$ can be altered on a null set so that $L_{m,\Theta} f$ takes on arbitrary preassigned values on any countable dense subset of Ω. For the details of one such construction, see the end of this section.

The function $L_{m,[\theta]} f$ is more than simply an interesting example. It occurs in the *multipoint Taylor error formula* for multivariate Lagrange interpolation given by Ciarlet and Raviart [CR72]. From the multipoint Taylor formula, Arcangeli and Gout [AG76] obtained L_p-bounds for multivariate Lagrange interpolation, long used by those working in finite elements, but known to few approximation theorists. For this reason, these bounds are discussed in some detail in Section 2.4. □

Special case: Hardy's inequality

In the very special case $n = 1$, $m = 1$, and $\Theta = [0]$, one has, by (1.2.7), that

$$L_{m,\Theta} f(x) = \frac{1}{x} \int_0^x f. \tag{2.2.9}$$

With the choice $\Omega = (0, \infty)$, (2.2.4) is Hardy's inequality (2.1.2). This well-known inequality was first proved by Hardy [Ha28], see also [HLP67:§9.8].

For a comprehensive survey of the literature connected with Hardy's inequality, see Chapter IV: Hardy's, Carleman's and related inequalities, of the monograph [FMP91]. The only *multivariate* occurrence of Theorem 2.2.2 that the author is aware of is, implicitly, in Arcangeli and Gout [AG76] for the case when Θ consists of a single point. The bulk of the 174 references for chapter IV of [FMP91] deals with *univariate* generalisations of Hardy's inequality − some of which are special cases of Theorem 2.2.2.

In this thesis we will not be concerned with the sharpness of (2.2.4). However, for those so interested we mention the following point of departure. For the map (2.2.9), with $\Omega = (0, \infty)$,

$$\|L_{m,\Theta}\| = \frac{p}{p-1}.$$

See, e.g., Shum [Sh71], [Ru87:ex.14,p72], and [Jo93:p275,p289].

Further L_p-bounds

Next we use Theorem 2.2.2 to give a bound particularly suited for obtaining L_p-bounds from integral error formulæ, such as those given in Sections 2.3 and 2.4.

Theorem 2.2.10. *Fix $a_1, \ldots, a_s \in \mathbb{R}^{k+1} \setminus 0$, where $s \geq 0$. Let $\Theta \in \mathbb{R}^{n \times k}$, and let Ω be a bounded open set in $\mathbb{R}^n$ for which $\bar{\Omega}$ is starshaped with respect to Θ. If $m - n/p > 0$, then the rule*

$$\mathcal{L}f(x) := \int_{[\underbrace{x, \ldots, x}_{m}, \Theta]} \left(\prod_{j=1}^{s} D_{[x,\Theta]a_j} \right) f \tag{2.2.11}$$

induces a bounded linear map $\mathcal{L} : W_p^s(\Omega) \to L_p(\Omega)$, with

$$\|\mathcal{L}f\|_{L_p(\Omega)} \leq \left(\max_{x \in \bar{\Omega}} \prod_{j=1}^{s} \|[x,\Theta]a_j\| \right) \frac{1}{(m-1)!(m-n/p)_{\#\Theta}} |f|_{s,p,\Omega}. \tag{2.2.12}$$

In addition, when $p = \infty$, we have the pointwise estimate

$$|\mathcal{L}f(x)| \leq \frac{1}{(\#\Theta + m - 1)!} \left(\prod_{j=1}^{s} \|[x,\Theta]a_j\| \right) |f|_{s,\infty,\Omega}, \qquad \text{a.e. } x \in \Omega. \tag{2.2.13}$$

Proof. Let $x \in \Omega$, and $f \in W_p^s(\Omega)$. By (2.1.9),

$$\left| \left(\prod_{j=1}^{s} D_{[x,\Theta]a_j} \right) f \right| \leq \left(\prod_{j=1}^{s} \|[x,\Theta]a_j\| \right) |D^s f|, \tag{2.2.14}$$

in $L_p(\Omega)$. Here $|D^s f| \in L_p(\Omega)$ is defined by (2.1.8). Thus,

$$A_x f := \left(\prod_{j=1}^{s} D_{[x,\Theta]a_j} \right) f$$

defines a bounded linear map $A_x : W_p^s(\Omega) \to L_p(\Omega)$, with

$$|A_x f| \leq K \, |D^s f|, \tag{2.2.15}$$

in $L_p(\Omega)$, where

$$K := K(a_1, \ldots, a_s, \Omega) := \max_{x \in \bar{\Omega}} \prod_{j=1}^{s} \|[x,\Theta]a_j\|.$$

Notice that

$$\mathcal{L}f(x) = \left(L_{m,\Theta} \, A_x f \right)(x).$$

Thus, (2.2.15) and the *monotonicity* of $L_{m,\Theta} : L_p(\Omega) \to L_p(\Omega)$ implies

$$|\mathcal{L}f| \leq L_{m,\Theta}(K \, |D^s f|),$$

in $L_p(\Omega)$. Take the $L_p(\Omega)$-norm of this inequality, then apply Theorem 2.2.2, to obtain

$$\|\mathcal{L}f\|_{L_p(\Omega)} \leq \frac{1}{(m-1)!(m-n/p)_{\#\Theta}} K \, \| \, |D^s f| \, \|_{L_p(\Omega)}.$$

Since

$$\| \, |D^s f| \, \|_{L_p(\Omega)} = | f |_{s,p,\Omega},$$

this proves (2.2.12).

Similarly, from (2.2.14) and Theorem 2.2.2, we have for a.e. $x \in \Omega$, that

$$\begin{aligned} |\mathcal{L}f(x)| &\le \Bigl(\prod_{j=1}^{s} \|[x,\Theta]a_j\|\Bigr) \|L_{m,\Theta}(|D^s f|)\|_{L_\infty(\Omega)} \\ &\le \Bigl(\prod_{j=1}^{s} \|[x,\Theta]a_j\|\Bigr) \frac{1}{(\#\Theta + m - 1)!} |f|_{s,\infty,\Omega}, \end{aligned}$$

which is (2.2.13). □

In the special case when $s = 0$, Theorem 2.2.10 reduces to Theorem 2.2.2. Theorem 2.2.10, together with Property 1.2.3 (d), can be used to obtain bounds for maps more general than (2.2.11). One such example is the *lift* of an *elementary liftable map*, see Section 1.3.

An example

Finally, the example promised in Remark 2.2.8.

Let $n > 1$ and Θ consist of the single point θ. Suppose that $\bar{\Omega}$ is starshaped with respect to θ, and that B is a countable dense subset of Ω. It is possible to change $f \in L_p(\Omega)$ on the intersection of Ω with the cone C with vertex θ and base B, which is a null set, so that $L_{m,[\theta]}f$, as computed from (2.2.3), takes on arbitrary preassigned values on B.

The cone C consists of the union of rays r emanating from θ and passing through a point $b \in B$. Let r be such a ray, and order the points from B lying on r as $b_1, b_2, \ldots$, so that b_i is closer to θ than b_{i+1}. By Remark 1.2.5,

$$L_{m,[\theta]}f(b_i) = \int M(\cdot|\underbrace{b_i,\ldots,b_i}_{m},\theta)\, f$$

with the integration above being over the interval $[\theta\,..\,b_i] := \operatorname{conv}\{\theta, b_i\}$ weighted by a nonnegative polynomial. Thus, by redefining f to be an appropriate constant over each of the intervals $[\theta \,..\, b_1]$, $[b_1 \,..\, b_2]$, $[b_2 \,..\, b_3], \ldots$, one can make $L_{m,[\theta]} f(b_i)$ take on any preassigned values.

2.3. Application: L_p-error bounds for Kergin and Hakopian interpolation

In this section we use Theorem 2.2.10 to obtain L_p-error bounds for the *scale of mean value interpolations*, which includes the *Kergin* and *Hakopian maps*.

The scale of mean value interpolations

Throughout this section, $\Theta \in \mathbb{R}^{n\times k}$. For $0 \le m < k$, we have the **mean value interpolation** (see Section 1.4)

$$\mathcal{H}_\Theta^{(m)} : \{f : f \text{ is } C^{k-m-1} \text{ on } \operatorname{conv}\Theta\} \to \Pi_{k-m-1}(\mathbb{R}^n),$$

which is given by

$$\mathcal{H}_\Theta^{(m)} f(x) := m! \sum_{j=m+1}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{j-1} \\ \#\tilde\Theta = m}} \int_{\Theta_j} D_{x-\Theta_{j-1}\backslash\tilde\Theta} f.$$

For the remainder of this section, Ω will be a bounded open set in $\mathbb{R}^n$ with a Lipschitz boundary. From Theorem 1.5.12, one obtains the following integral error formulæ for the scale of mean value interpolations.

Theorem 2.3.1. *Suppose that $\bar\Omega$ is starshaped with respect to Θ. If $0 \le j < k-m$, $q \in \Pi_j^0(\mathbb{R}^n)$, $p > n$, and $f \in W_p^{(k-m)}(\Omega)$, then*

$$q(D)\big(f - \mathcal{H}_\Theta^{(m)} f\big)(x) = (m+j)! \sum_{i=k-m-j}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{i-1} \\ \#\tilde\Theta = m+j+i-k}} \int_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} D_{[x-\Theta_{i-1}\backslash\tilde\Theta,\, x-\theta_i]}\, q(D) f. \tag{2.3.2}$$

This formula involves only derivatives of f of order $k-m$.

Remark 2.3.3. In Theorem 1.5.12 the formula (2.3.2) was proved only for $f \in C^{k-m}(\mathbb{R}^n)$, without any reference to p. We now outline how it can be extended to $f \in W_p^{(k-m)}(\Omega)$. By Sobolev's embedding theorem, the condition $p > n$ implies that

$$W_p^{(k-m)}(\Omega) \subset C^{k-m-1}(\bar{\Omega}) \subset C(\bar{\Omega}).$$

Thus, $\mathcal{H}_\Theta^{(m)} f$ is defined for all $f \in W_p^{(k-m)}(\Omega)$. To extend (2.3.2) to $f \in W_p^{(k-m)}(\Omega)$ use the density of $C_0^\infty(\Omega)$ in $W_p^{(k-m)}(\Omega)$. $\square$

L_p-bounds for the scale of mean value interpolations

Next we apply Theorem 2.2.10 to (2.3.2) to obtain L_p-bounds for the scale of mean value interpolations. Let

$$h_{x,\Theta} := \sup_{\theta \in \Theta} \|x - \theta\|, \quad h_{\Omega,\Theta} := \sup_{x \in \Omega} h_{x,\Theta} \leq \operatorname{diam} \Omega.$$

Theorem 2.3.4. *Suppose that $\bar{\Omega}$ is starshaped with respect to Θ. If $0 \leq j < k-m$, $p > n$, and $f \in W_p^{(k-m)}(\Omega)$, then*

$$|f - \mathcal{H}_\Theta^{(m)} f|_{j,p,\Omega} \leq C_{n,p,j,k,m} (h_{\Omega,\Theta})^{k-m-j} |f|_{k-m,p,\Omega}, \tag{2.3.5}$$

where

$$C_{n,p,j,k,m} := \frac{1}{(1-n/p)_{k-m-j}}.$$

The constant $C_{n,p,j,k,m} \to \infty$ as $p \to n^+$. Additionally, if $p = \infty$, then we have the pointwise estimate that, for all $x \in \bar{\Omega}$,

$$|D^j(f - \mathcal{H}_\Theta^{(m)} f)|(x) \leq \frac{1}{(k-m-j)!} (h_{x,\Theta})^{k-m-j} |f|_{k-m,\infty,\Omega}.$$

Proof. Choose $q \in \Pi_j^0(\mathbb{R}^n)$ so that

$$q(D) = D_{u_1} \cdots D_{u_j},$$

where $u_1, \ldots, u_j \in \mathbb{R}^n$ with $\|u_i\| \le 1$. By Theorem 2.2.10, we have for each of the terms in (2.3.2) that

$$\|x \mapsto \int_{[\underbrace{x,\ldots,x}_{k+1-i},\Theta_i]} D_{[x-\Theta_{i-1}\backslash\tilde{\Theta},x-\theta_i]} q(D)f\|_{L_p(\Omega)}$$
$$\le \frac{\Gamma(k+1-i-n/p)}{\Gamma(k+1-i)\Gamma(k+1-n/p)}(h_{x,\Theta})^{k-m-j} |f|_{k-m,\infty,\Omega}.$$

Notice that in the above, the constants

$$\max_{x\in\bar{\Omega}} \prod_{\theta\in[\Theta_{i-1}\backslash\tilde{\Theta},\theta_i]} \|x-\theta\|$$

were replaced by the possibly larger, but far less complicated constant $(h_{\Omega,\Theta})^{k-m-j}$. This gives the first inequality with

$$C_{n,p,j,k,m} := \frac{(m+j)!}{\Gamma(k+1-n/p)} \sum_{i=k-m-j}^{k} \binom{i-1}{m+j+i-k} \frac{\Gamma(k+1-i-n/p)}{(k-i)!}$$
$$= \frac{(k-1)!}{(k-m-j-1)!(1-n/p)}\ {}_2F_1\Big(\begin{matrix}-m-j,\ 1-n/p\\ 1-k\end{matrix};1\Big).$$

By the Chu-Vandermonde convolution identity:

$${}_2F_1\Big(\begin{matrix}-n, b\\ c\end{matrix};1\Big) = \frac{(c-b)_n}{(c)_n},$$

which is the special case $a = -n$ of equation (14) in [E53:p61], it follows that

$$C_{n,p,j,k,m} = \frac{1}{(1-n/p)_{k-m-j}}.$$

The second inequality, which is Theorem 1.6.2, follows from the pointwise estimate (2.2.13). □

By considering the special case of Taylor interpolation at a point by polynomials of degree $\le k$, one obtains the following estimate of the distance of smooth functions from Π_k.

Corollary. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded, open, starshaped region that has a Lipschitz boundary. Then for $p > n$ and $0 \le j \le k+1$,*

$$\begin{aligned}\operatorname{dist}_{|\cdot|_{j,p,\Omega}}(f, \Pi_k) &:= \inf_{g \in \Pi_k} |f-g|_{j,p,\Omega} \\ &\le \frac{1}{(1-n/p)_{k+1-j}} (\operatorname{diam}\Omega)^{k+1-j} |f|_{k+1,p,\Omega}, \quad \forall f \in W_p^{k+1}(\Omega).\end{aligned} \tag{2.3.6}$$

Note that

$$\frac{1}{(1-n/p)_{k+1-j}} \to \infty, \qquad \text{as } p \to n^+.$$

That an inequality of the form (2.3.6) exists for $j = 0$, where the constant $1/(1-n/p)_{k+1-j}$ is replaced by some unknown constant depending only on n, k and p, is the content of the paper by Dechevski and Quak [DQ90]. From this they obtain the corresponding 'improved' version of the Bramble-Hilbert lemma (see [BH70]).

A related result of Lai and Wang

The only related result in the literature is an L_p-bound for the error in Hakopian interpolation given by Lai and Wang [LW84]. In that paper they show the following.

Theorem 2.3.7 ([LW84:Th.1]). *Let $|\alpha| \le k-n$. Then for any positive integer*

$\ell \leq k + |\alpha| - n + 1$, *we have*

$$
\begin{aligned}
&D^\alpha(f - \mathcal{H}_\Theta^{(n-1)})(x) \\
&=(|\alpha| + n - 1) \sum_{\mu_1=1}^{|\alpha|+n} \sum_{i_1=1}^{n} (x - \theta_{|\alpha|+n-\mu_1+1})_{i_1} \sum_{\mu_2=1}^{\mu_1} \sum_{i_2=1}^{n} (x - \theta_{|\alpha|+n-\mu_2+2})_{i_2} \times \\
&\quad \cdots \times \sum_{\mu_\ell=1}^{\mu_{\ell-1}} \sum_{i_\ell=1}^{n} (x - \theta_{|\alpha|+n-\mu_\ell+\ell})_{i_\ell} \int_{[\underbrace{x,\ldots,x}_{\mu_\ell},\theta_1,\ldots,\theta_{|\alpha|+n-\mu_\ell+\ell}]} D^{\alpha+\sum_{j=1}^{\ell} e^{i_j}} f \\
&\quad - \sum_{j=|\alpha|+n-1+\ell}^{k-1} \sum_{|\gamma|=j-n+1} D^\alpha \omega_\gamma(x) \int_{[\theta_1,\ldots,\theta_j]} D^\gamma f.
\end{aligned}
\tag{2.3.8}
$$

To (2.3.8), Lai and Wang apply the integral form of Minkowski's inequality in the form

$$
\|x \mapsto \int_{[\underbrace{x,\ldots,x}_{\mu},\theta_1,\ldots,\theta_{k+1-\mu}]} D^\beta f\|_{L_p(G)} \leq C_2 \, \|D^\beta f\|_{L_p(G)}, \quad \mu = 1, \ldots, |\alpha| + n,
\tag{2.3.9}
$$

to obtain the following.

Theorem 2.3.10 ([LW84:Th.2]). *Let G be a convex set containing Θ, with diameter h. If $p > n$, $|\alpha| \leq k - n$, and $f \in W_p^{(k-n+1)}(G)$, then*

$$
\|D^\alpha(f - \mathcal{H}_\Theta^{(n-1)} f)\|_{L_p(G)} \leq C \, h^{k-n+1-|\alpha|} \max_{|\beta|=k-n+1} \|D^\beta f\|_{L_p(G)},
\tag{2.3.11}
$$

where C a constant independent of f.

Since $f \mapsto \max_{|\beta|=k+1-n} \|D^\beta f\|_{L_p(\Omega)}$, and $f \mapsto |f|_{k+1-n,p,\Omega}$ are equivalent seminorms, Theorem 2.3.10 follows from Theorem 2.3.4. Had Lai and Wang attempted to compute the C_2 of (2.3.9) using the multivariate form of Hardy's inequality, they would have obtained

$$C_2 \le \frac{\Gamma(\mu - n/p)}{\Gamma(\mu)\Gamma(k+1)}.$$

Thus, their constant C in (2.3.11) would have the same qualitative behaviour as our own $C_{n,p,j,k,m}$ of (2.3.5), namely that $C \to \infty$ as $p \to n^+$.

The behaviour of $C_{n,p,j,k,m}$ as a function of its parameters

In Section 1.6 it is shown that, in an appropriate sense, the constant $C_{n,p,j,k,m}$ of (2.3.5) is best possible when $p = \infty$. The question then arises whether or not the over-estimation committed in using the multivariate form of Hardy's inequality to obtain $C_{n,p,j,k,m}$ is significant for $p < \infty$. In particular, does the best possible constant C in the inequality

$$|f - \mathcal{H}_\Theta^{(m)} f|_{j,p,\Omega} \le C\,(h_{\Omega,\Theta})^{k-m-j}\,|f|_{k-m,p,\Omega} \tag{2.3.12}$$

become unbounded as $p \to n^+\Gamma$ In the univariate case, at least, the answer is *no* – the best possible constant in (2.3.12) does not become unbounded.

Before we show this, let us clarify a little the role that the condition $p > n$ plays in Theorems 2.3.4 and 2.3.10. The condition $p > n$ is necessary if these results are to be stated in terms of the Sobolev space $W_p^{(k-m)}(\Omega)$ – in particular so that $\mathcal{H}_\Theta^{(m)} f$ is defined for $f \in W_p^{(k-m)}(\Omega)$. However, it makes good sense to ask what is the best constant C for which (2.3.12) holds for all sufficiently smooth functions f – say, e.g., $f \in C^{k-m}(\bar{\Omega})$. The condition $p > n$ is again needed when one seeks to apply the multivariate form of Hardy's inequality to the integral error formulæ (2.3.2) and (2.3.8).

We end this section by showing that in the univariate case, i.e., when $n = 1$, there is a best possible constant C in (2.3.12) for all sufficiently smooth f, which can be bounded independently of $1 \le p \le \infty$. The crucial step in the argument to follow is the use of the B-spline L_p-estimate that

$$\|M(\cdot|\Theta)\|_{L_p(\mathbb{R})} \le \left(\frac{\#\Theta - 1}{\operatorname{diam}\Theta}\right)^{1-1/p} \tag{2.3.13}$$

when $\operatorname{diam}\Theta > 0$, see de Boor [B73].

The univariate case of the map $\mathcal{H}_\Theta^{(m)}$, will be emphasised by writing it as $H_\Theta^{(m)}$. This map has the simple form

$$H_\Theta^{(m)} f = D^m (H_\Theta D^{-m} f),$$

where H_Θ is the *Hermite interpolator* at the points Θ, and $D^{-m} f$ is any function for which $D^m(D^{-m} f) = f$.

Theorem 2.3.14. *Let Θ be a k-sequence in the interval $[a \,..\, b]$. If $0 \le j < k - m$, and $f \in C^{k-m}[a \,..\, b]$, then*

$$\|D^j(f - H_\Theta^{(m)} f)\|_{L_p[a..b]} \le \frac{(m+j)!}{(k-m-j)!} \frac{k^{1/q}}{k!} (b-a)^{k-m+\frac{1}{p}-\frac{1}{q}} \|D^{k-m} f\|_{L_q[a..b]}.$$

Here $1 \le p, q \le \infty$.

Proof. Fix $x \in [a \,..\, b]$. For Θ a finite sequence in $\mathbb{R}$, let

$$\omega_\Theta(x) := \prod_{\theta \in \Theta} (x - \theta).$$

With this notation, replacing each occurrence in (2.3.2) of a linear functional of the form $f \mapsto \int_\Theta f$ by integration against a B-spline, we obtain that

$$D^j(f - H_\Theta^{(m)} f)(x) = (m+j)! \sum_{i=k-m-j}^{k} \sum_{\substack{\tilde\Theta \subset \Theta_{i-1} \\ \#\tilde\Theta = m+j+i-k}} \omega_{\Theta_{i-1} \backslash \tilde\Theta}(x)\,(x-\theta_i)\, \frac{1}{k!} \int D^{k-m} f \, M(\cdot|x, \Theta_i).$$

By Hölder's inequality, and (2.3.13), we have that

$$\left|\int D^{k-m} f\, M(\cdot|x,\Theta_i)\right| \le \left(\frac{k}{\operatorname{diam}[x,\Theta_i]}\right)^{1/q} \|D^{k-m} f\|_{L_q[a..b]}.$$

Since

$$\left|\frac{\omega_{\Theta_{i-1}\backslash\tilde{\Theta}}(x)\,(x-\theta_i)}{(\operatorname{diam}[x,\Theta_i])^{1/q}}\right| \le (b-a)^{k-m-1/q},$$

we obtain that

$$\begin{aligned}
&|D^j(f - H_\Theta^{(m)} f)(x)| \\
&\quad \le (m+j)! \sum_{i=k-m-j}^{k} \binom{i-1}{m+j+i-k} \frac{k^{1/q}}{k!}(b-a)^{k-m-1/q} \|D^{k-m} f\|_{L_q[a..b]} \\
&\quad = \frac{(m+j)!}{(k-m-j)!} \frac{k^{1/q}}{k!}(b-a)^{k-m-1/q} \|D^{k-m} f\|_{L_q[a..b]}.
\end{aligned}$$

Finally, take $\|\cdot\|_{L_q[a..b]}$ of both sides. □

To adapt this argument to the multivariate case, it is necessary to have the *simplex spline* analog of the B-spline L_p-estimate (2.3.13). This is provided by Dahmen [D79], who shows that when $\operatorname{vol}_n(\operatorname{conv}\Theta) > 0$,

$$\|M(\cdot|\Theta)\|_{L_p(\mathbb{R}^n)} \le \frac{k!(k+1)!}{n!(n+1)!(n-k)!} \left(\frac{1}{\operatorname{vol}_n(\operatorname{conv}\Theta)}\right)^{1-1/p}, \tag{2.3.15}$$

with $k+1 := \#\Theta$. Yet, with this in hand, it does not seem possible to apply the argument of Theorem 2.3.14 in any satifactory form.

Remark 2.3.16. Incidentally, the constant in (2.3.15) is not the best possible. Already, by using the fact that $\int M(\cdot|\Theta) = 1$, together with the case $p = \infty$ of (2.3.15), one obtains

$$\|M(\cdot|\Theta)\|_{L_p(\mathbb{R}^n)} \le \left(\frac{k!(k+1)!}{n!(n+1)!(n-k)!} \frac{1}{\operatorname{vol}_n(\operatorname{conv}\Theta)}\right)^{1-1/p}.$$

In the univariate case this over-estimates (2.3.13) by a factor of $((k+1)!/2)^{1-1/p}$.

The key step in proving (2.3.13) is the bound

$$M(\cdot|\Theta) \le \frac{k}{\operatorname{diam}\Theta}, \tag{2.3.17}$$

which follows from the partition of unity property of B-splines. Thus, a close examination of the simplex spline analog of the B-spline partition of unity, given recently by Dahmen, Micchelli and Seidel [DMS92], should give tighter bounds than those of (2.3.15). However, we make no attempt here to give such an argument. □

Remark 2.3.18. There are other integral error formulæ for the scale of mean value interpolations, to which Theorem 2.2.10 can be applied to give L_p-bounds. These include Lai and Wang [LW86] (Kergin interpolation), Gao [Ga88], and Hakopian [BHS93:p200] (Hakopian interpolation). See Section 1.5 for a discussion of the relative merits of each of these formulæ. □

2.4. Application: L_p-error bounds for multivariate Lagrange interpolation

In this section we use Theorem 2.2.10 to obtain L_p-error bounds for *multivariate Lagrange interpolation schemes.*

Lagrange maps

A linear interpolation problem for which the space of interpolation conditions is spanned by *point evaluations* at Θ, a finite sequence in $\mathbb{R}^n$, is called a **Lagrange interpolation problem**. If P is the space of interpolants for such a problem and the problem is correct, then the associated linear projector, called the **Lagrange map**, will be denoted by $L_{P,\Theta}$. The **Lagrange form** of a Lagrange map is given by

$$L_{P,\Theta} f = \sum_{\theta\in\Theta} f(\theta)\ell_\theta. \tag{2.4.1}$$

Here (2.4.1) uniquely defines

$$\ell_\theta := \ell_{\theta,P,\Theta} \in P,$$

the **Lagrange function** for $\theta \in \Theta$. In other words, $(\delta_{[\theta]})_{\theta\in\Theta}$ is dual (bi-orthonormal) to $(\ell_\theta)_{\theta\in\Theta}$.

Lagrange maps into a space containing polynomials of degree k are frequently used to interpolate to scattered data, see, e.g., Alfeld [Al89]. Particular examples receiving much attention lately are maps where the interpolants include *radial basis functions* or *multivariate splines*, and de Boor and Ron's *least solution* for the polynomial interpolation problem [BR90] (also see [BR92] for its generalisation). In addition there are of course the maps of Kergin and Hakopian.

For such maps, there is the *multipoint Taylor formula* for the error. This formula was initiated by the work of Ciarlet and Wagschal [CW71]; most of the relevant papers are in French, and it is little known outside the area of finite elements. It is for these reasons, and because our Theorem 2.2.10 implies L_p-estimates from the multipoint Taylor formula, that we discuss the formula here.

The multipoint Taylor formula

Multipoint Taylor formula 2.4.2 ([CR72]). *Let Θ be a finite sequence in $\mathbb{R}^n$, and let Ω be an open set in $\mathbb{R}^n$ for which $\bar{\Omega}$ is starshaped with respect to Θ.*

If $L_{P,\Theta}$ is a Lagrange map with $\Pi_k(\mathbb{R}^n) \subset P \subset C^k(\bar{\Omega})$, then for $f \in C^{k+1}(\bar{\Omega})$, $q \in \Pi_k(\mathbb{R}^n)$, and $x \in \bar{\Omega}$, its error satisfies:

$$\big(q(D)(L_{P,\Theta}f - f)\big)(x) = \sum_{\theta\in\Theta} \Big(\int_{[\underbrace{x,\ldots,x}_{k+1},\theta]} D_{\theta-x}^{k+1} f \Big) (q(D)\ell_\theta)(x). \qquad (2.4.3)$$

The term *multipoint Taylor formula* comes from the fact that

$$\theta \mapsto \int_{[\underbrace{x,\ldots,x}_{k+1},\theta]} D_{\theta-x}^{k+1} f$$

is the error in *Taylor interpolation* of degree k at the point x, a special case of the error in Kergin interpolation. The proof of (2.4.3) further justifies the use of this term.

The region of integration in (2.4.3) consists of line segments from x to $\theta \in \Theta$.

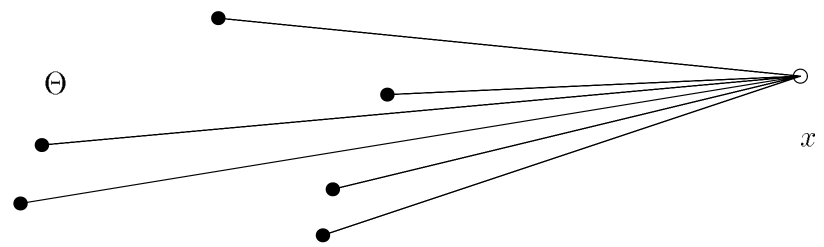

Fig 2.5.1 The region of integration in (2.4.3) for Θ consisting of 6 points

From the multipoint Taylor formula, Arcangeli and Gout [AG76] obtain L_p-bounds for the error in a Lagrange map. These bounds are precisely those obtained by applying Theorem 2.2.10 to (2.4.3). The crucial step in the argument presented in [AG76:Prop.1-1] is the use of the multivariate form of Hardy's inequality for the map

$$x \mapsto L_{k+1,[v]}f(x) := \int_{[\underbrace{x,\ldots,x}_{k+1},v]} f. \qquad (2.4.4)$$

This inequality is not explicitly stated, though the proof of their (weaker) Proposition 1-1 would imply it.

Remark 2.4.5. The key step in the proof of Proposition 1-1 in [AG76] is an application of Hölder's inequality to the splitting

$$\int_{[\underbrace{x,\ldots,x}_{k+1},v]} f = \frac{1}{k!}\int_0^1 (1-t)^{-1/q-\varepsilon}\left((1-t)^{k+1/q-\varepsilon} f(x+t(v-x))\right) dt,$$

where $\varepsilon := (k+1-n/p)/q$, and $1/p+1/q=1$, as opposed to our use of the integral form of Minkowski's inequality. □

Having identified the precise role of the multivariate form of Hardy's inequality in [AG76] it is possible to use it to run through Arcangeli and Gout's calculation for a much more general class of norms, including those most often used in numerical analysis. The resulting bounds, given below, have smaller (and simpler) constants than those one might hope to obtain by applying the inequalities for similar norms to the results of [AG76].

For the remainder of this section Ω will denote a bounded open set in $\mathbb{R}^n$ with a Lipschitz boundary, and Θ a finite sequence in $\mathbb{R}^n$. Recall

$$h_{\Omega,\Theta} = \sup_{\theta\in\Theta}\sup_{x\in\Omega}\|x-\theta\| \leq \operatorname{diam}\Omega.$$

Corollary 2.4.6. *Suppose that $\bar{\Omega}$ is starshaped with respect to Θ, and that $L_{P,\Theta}$ is a Lagrange map with $\Pi_k(\mathbb{R}^n)\subset P\subset C^k(\Omega)$. If $k+1-n/p>0$, and $f\in W_p^{(k+1)}(\Omega)$, then*

$$|f-L_{P,\Theta}f|_{p,\Omega} \leq \frac{1}{k!(k+1-n/p)}\left(\sum_{\theta\in\Theta}|\ell_\theta|_{\infty,\Omega}\right)|f|_{k+1,p,\Omega}\,(h_{\Omega,\Theta})^{k+1}. \qquad (2.4.7)$$

Here $|\cdot|_{p,\Omega}$ *is any seminorm on* $W_p^k(\Omega)$ *of the form*

$$|f|_{p,\Omega} := \|\,(\|g_i(D)f\|_{L_p(\Omega)})_{i=1}^m\,\|_{\mathbb{R}^m},$$

where the $g_i \in \Pi_k(\mathbb{R}^n)$ *are fixed, and* $\|\cdot\|_{\mathbb{R}^m}$ *is any norm on* $\mathbb{R}^m$ *— or* $|\cdot|_{p,\Omega}$ *is* $|\cdot|_{i,p,\Omega}$ *for some* $0 \le i \le k$*.*

Proof. By Sobolev's embedding theorem, the condition $k+1-n/p > 0$ implies

$$W_p^{(k+1)}(\Omega) \subset C(\bar{\Omega}),$$

and so the Lagrange map $L_{P,\Theta}$ is well defined. As in Remark 2.3.3, (2.4.3) can be extended to $f \in W_p^{(k+1)}(\Omega)$. Fix $f \in W_p^{(k+1)}(\Omega)$, and $x \in \Omega$. Let $h := h_{\Omega,\Theta}$. By (2.1.9),

$$|D_{\theta-x}^{k+1} f| \le |D^{k+1}f|\,\|\theta - x\|^{k+1} \le |D^{k+1}f|\,h^{k+1},$$

in $L_p(\Omega)$. Thus, with $g_i \in \Pi_k(\mathbb{R}^n)$, we have for a.e. $x \in \Omega$ that

$$|(g_i(D)(f - L_{P,\Theta}f))(x)| \le \sum_{\theta\in\Theta}\left(\int_{[\underbrace{x,\ldots,x}_{k+1},\theta]} |D^{k+1}f|\right)\|g_i(D)\ell_\theta\|_{L_\infty(\Omega)}\,h^{k+1}.$$

To this, the condition $k+1-n/p > 0$ allows us to apply the multivariate form of Hardy's inequality to obtain

$$\|g_i(D)(f - L_{P,\Theta}f)\| \le \frac{1}{k!(k+1-n/p)}\left(\sum_{\theta\in\Theta}\|g_i(D)\ell_\theta\|_{L_\infty}\right)|f|_{k+1,p,\Omega}\,h^{k+1}.$$

Finally, take the $\|\cdot\|_{\mathbb{R}^m}$ norm of the inequality (for m-vectors) given above. □

In [AG76:Th.1-1] Corollary 2.4.6 is proved only in the case when $|\cdot|_{p,\Omega}$ is of the form $|f|_{i,p,\Omega}$ for some $0 \le i \le k$, with $h_{\Omega,\Theta}$ replaced by $\operatorname{diam}\Omega$. In that paper some bounds on the size of the Lagrange functions ℓ_θ, together with relevant applications are given.

The condition in Corollary 2.4.6 that $k+1-n/p>0$ plays an analogous role to the condition in Theorem 2.3.4 that $n>p$. Namely, it is required so that the results can be stated in terms of Sobolev spaces, and to apply the multivariate form of Hardy's inequality. Additionally, by Theorem 2.3.14, the unboundedness of the constant in (2.4.7) as $k+1-n/p\to 0^+$ is, in the univariate case, not a true reflection of the behaviour of the error.

With the multivariate form of Hardy's inequality in hand, it is also possible to obtain pointwise error bounds for Lagrange maps.

Corollary 2.4.8. *Suppose that $\bar{\Omega}$ is starshaped with respect to Θ, and that $L_{P,\Theta}$ is a Lagrange map with $\Pi_k(\mathbb{R}^n)\subset P\subset C^k(\Omega)$. With $f\in W_\infty^{(k+1)}\subset C(\bar{\Omega})$, and $x\in\bar{\Omega}$ we have the (coordinate-independent) pointwise error bound*

$$|f(x)-L_{P,\Theta}f(x)|\le\frac{1}{(k+1)!}|f|_{k+1,\infty,\Omega}\sum_{\theta\in\Theta}\|\theta-x\|^{k+1}|\ell_\theta(x)|,\tag{2.4.9}$$

and the (coordinate-dependent) pointwise error bound

$$|f(x)-L_{P,\Theta}f(x)|\le\sum_{\theta\in\Theta}\sum_{|\alpha|=k+1}\frac{1}{\alpha!}\|D^\alpha f\|_{L_\infty(\Omega)}\,|(\theta-x)^\alpha\ell_\theta(x)|.\tag{2.4.10}$$

Proof. The proof runs along the same lines as that for Corollary 2.4.6, except that for (2.4.10) we first expand $D_{\theta-x}^{k+1}f$ as

$$D_{\theta-x}^{k+1}f=\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}(\theta-x)^\alpha D^\alpha f,$$

by using the multinomial identity. □

Neither of (2.4.9) or (2.4.10) occurs in the literature. For $f \in C^{k+1}(\Omega)$, they can be obtained more simply, by applying the mean value theorem, as given by Properties 1.2.3 (c), to the integrals occurring in (2.4.3).

Remark 2.4.11. The results of [AG76] have been extended in the following ways. In [Go77], Gout treats the error in certain forms of *Hermite interpolation* − that is where, in addition to function values, certain derivatives are matched at the points in Θ. In [AS84], Arcangeli and Sanchez bound the error in a Lagrange map for functions from *fractional order* Sobolev spaces. □

The error formula of Sauer and Xu

There is another error formula, for the error in a Lagrange map with range (interpolants) $\Pi_k(\mathbb{R}^n)$, that has been given recently by Sauer and Xu, see [SX94].

Sauer and Xu order the $\dim \Pi_k(\mathbb{R}^n)$ points in Θ so that each Lagrange interpolation problem with points Θ^j (by definition the initial segment of Θ consisting of the first $\dim \Pi_j(\mathbb{R}^n)$ terms) and interpolants $\Pi_j(\mathbb{R}^n)$ is correct for $j = 0, \ldots, k$. They consider the collection $\boldsymbol{\Psi}$ of all $(k+1)$-sequences $\Psi = [\psi_0, \ldots, \psi_k]$, called *paths* by them, with $\psi_j \in \Theta^j \backslash \Theta^{j-1}$, all j. Given this notation, Sauer and Xu state their result in the following form.

Theorem 2.4.12 ([SX94:Th.3.6]). *Suppose that* $L_{P,\Theta} := L_{\Pi_k(\mathbb{R}^n),\Theta}$ *is a Lagrange map, and* $f \in C^{k+1}(\mathbb{R}^n)$. *Then*

$$L_{P,\Theta} f(x) - f(x) = \sum_{\Psi \in \boldsymbol{\Psi}} p_\Psi(x) \int_{[x,\Psi]} D_{x-\psi_k} D_{\psi_k - \psi_{k-1}} \cdots D_{\psi_2 - \psi_1} D_{\psi_1 - \psi_0} f, \tag{2.4.13}$$

where $p_\Psi \in \Pi_k(\mathbb{R}^n)$ *is given by*

$$p_\Psi(x) := (k+1)!\, \ell_{\psi_k,\Pi_k(\mathbb{R}^n),\Theta}(x) \prod_{i=1}^{k} \ell_{\psi_i,\Pi_i(\mathbb{R}^n),\Theta^i}(\psi_{i+1}).$$

The region of integration in each term of (2.4.13) is the convex hull of x and Ψ.

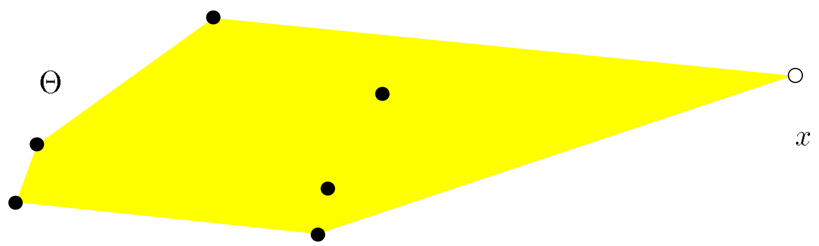

Fig 2.5.2 The region of integration in (2.4.13) for Θ consisting of 6 points

From (2.4.13) the following pointwise estimate is obtained.

Corollary 2.4.14 ([SX94:Cor.3.11]). *Suppose, in addition to the hypotheses of Theorem 2.4.12, that* $\bar\Omega$ *is starshaped with respect to* Θ*. Then, for all* $x \in \bar\Omega$*,*

$$|f(x) - L_{P,\Theta} f(x)| \le \frac{1}{(k+1)!} \sum_{\Psi\in\boldsymbol{\Psi}} \|D_{x-\psi_k} D_{\psi_k-\psi_{k-1}} \cdots D_{\psi_1-\psi_0} f\|_{L_\infty(\Omega)} |p_\Psi(x)|. \tag{2.4.15}$$

The bound (2.4.15) is of a similar form to those of (2.4.9) and (2.4.10). For a more direct comparison, one obtains from (2.4.3) the bound

$$|f(x) - L_{P,\Theta} f(x)| \le \frac{1}{(k+1)!} \sum_{\theta\in\Theta} \|D^{k+1}_{\theta-x} f\|_{L_\infty(\Omega)} |\ell_\theta(x)|. \tag{2.4.16}$$

This bound has $\#\Theta = \sum_{j=0}^{k} \#\Theta^j$ terms, as opposed to $\#\boldsymbol{\Psi} = \prod_{j=0}^{k} \#\Theta^j$ for (2.4.15), and requires no ordering of Θ. For the purposes of comparison, in the bivariate case, i.e., when $n = 2$, one has that $\#\Theta = (k+2)(k+1)/2$, while $\#\boldsymbol{\Psi} = (k+1)!$. In addition, analogous bounds to (2.4.16) can be obtained, from (2.4.3), for the derivatives of the error in $L_{P,\Theta}$.

To obtain L_p-bounds from (2.4.13) it is necessary to bound

$$x \mapsto L_{1,\Psi} f(x) := \int_{[x,\Psi]} f \tag{2.4.17}$$

in terms of $\|f\|_{L_p(\Omega)}$. This can be done by using the multivariate form of Hardy's inequality. Thus, we have the following instance of Theorem 2.2.10.

Corollary 2.4.18. *Suppose the hypotheses of Corollary 2.4.14. If* $1 - n/p > 0$, *then*

$$\|f - L_{P,\Theta} f\|_{L_p(\Omega)} \le \frac{\Gamma(1-n/p)}{\Gamma(k+2-n/p)} \Bigl(\sum_{\Psi \in \boldsymbol{\Psi}} \|p_\Psi\|_{L_\infty(\Omega)} \Bigr) |f|_{k+1,p,\Omega} (h_{\Omega,\Theta})^{k+1}.$$

The condition $1-n/p > 0$ is needed so that the multivariate form of Hardy's inequality can be applied to (2.4.17). By comparison, to obtain (2.4.7) from (2.4.4), only the weaker condition that $k+1-n/p > 0$ was needed.

2.5. Other error bounds

All of the integral error formulæ for Lagrange maps given in the literature, including those of Section 5, can be obtained from

$$f(x) - L_{P,\Theta} f(x) = \sum_{\theta \in \Theta} \Bigl(\int_{[x]} f - \int_{[\theta]} f \Bigr) \ell_\theta(x),$$

which is valid whenever P contains the constants, by appropriately using the identity

$$\int_{[\Theta,v]} f - \int_{[\Theta,w]} f = \int_{[\Theta,v,w]} D_{v-w} f, \tag{2.5.1}$$

and the integration by parts formula.

For example, in Gregory [Gr75] the integration by parts formula is used to give a *Taylor type* expansion for f. From this is obtained an integral error formula for *linear interpolation* on a triangle, i.e., when Θ consists of 3 affinely independent points in $\mathbb{R}^2$, and the interpolants are the linear polynomials $P := \Pi_1(\mathbb{R}^2)$. Such an argument is frequently referred to as a *Sard kernel theory* argument, as developed by Sard [Sa63]. The resulting formula is complicated – it has 4 line integrals and 5 area integrals. Another example is given by Hakopian [$\mathrm{H82}_1$], who uses (2.5.1) to obtain an integral error formula for *tensor product* Lagrange interpolation.

In view of their derivations, all of these integral error formulæ involve terms which consist of a function (obtained appropriately from the Lagrange functions) multiplied by the integral of some derivative against a simplex spline. Thus, it is possible to apply the multivariate form of Hardy's inequality to all such formulæ (and those likely to be obtained in the future) to obtain L_p-bounds – with the caution that, as pointed out for the examples in Sections 2.3 and 2.4, for small p this may not accurately reflect the behaviour of the error.

Exactly how to use (2.5.1) and the integration by parts formula to obtain the best possible error formula for a given purpose is far from clear. In a future paper the author considers the simplest case, that of linear interpolation on a triangle. There, the formulæ of Ciarlet and Wagschal [CW71], Gregory [Gr75], Sauer and Xu [SX94], amongst others, are discussed.

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