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On the spacing of Fekete points for a sphere, ball or simplex

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ABSTRACT

Suppose that $K \subset \mathbb{R}^d$ is either the unit ball, the unit sphere or the standard simplex. We show that there are constants $c_1, c_2 > 0$ such that for a set of Fekete points (maximizing the Vandermonde determinant) of degree n, $F_n \subset K$,

$$\frac{c_1}{n} \leqslant \min_{\substack{b \in F_n \\ b \neq a}} \operatorname{dist}(a, b) \leqslant \frac{c_2}{n}$$

for all $a \in F_n$. Here $\operatorname{dist}(a, b)$ is a natural distance on K that will be described in the text.

1. INTRODUCTION

Suppose that $K \subset \mathbb{R}^d$ is a compact set. The polynomials of degree at most n in d real variables, when restricted to K, form a certain vector space which we will denote by $\mathcal{P}_n(K)$. It has therefore a dimension $N_n := \dim(\mathcal{P}_n(K))$. The polynomial interpolation problem for K is then, given a set of N_n distinct points $A_n \subset K$ and a function $f : K \to \mathbb{R}$, to find a polynomial $p \in \mathcal{P}_n(K)$ such that

$$p(a) = f(a), \quad \forall a \in A_n. \tag{1}$$

In one dimension ($d = 1$), there is always a unique solution to the problem (1). However, in higher dimensions ($d > 1$), depending on the geometry of the interpolation

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points A_n, it may be that it is not possible to find a solution to (1). To see why this is so, consider a basis

$$B_n = \{P_1, P_2, \ldots, P_{N_n}\}$$

of $\mathcal{P}_n(K)$. Then any polynomial $p \in \mathcal{P}_n(K)$ may be written in the form

$$p = \sum_{j=1}^{N_n} c_j P_j$$

for some constants $c_j \in \mathbb{R}$. Hence the conditions (1) may be expressed as

$$\text{(2)} \qquad p(a) = \sum_{j=1}^{N_n} c_j P_j(a) = f(a), \quad a \in A_n,$$

which are exactly N_n linear equations in N_n unknowns c_j. In matrix form this becomes

$$[P(a)]_{a \in A_n, P \in B_n} c = F,$$

where $c \in \mathbb{R}^{N_n}$ is the vector formed of the c_j and F is the vector of function values $f(a)$, $a \in A_n$. This linear system has a unique solution precisely when the so-called Vandermonde determinant

$$\mathrm{vdm}(A_n; B_n) := \det\bigl([P(a)]_{a \in A_n, P \in B_n}\bigr) \neq 0.$$

If this is the case, then the interpolation problem (1) is said to be *correct* (or sometimes *univsolvent*).

Note that $\mathrm{vdm}(A_n; B_n) = 0$ precisely when the interpolation points A_n all lie on an algebraic variety of degree n and hence the generic situation is that the interpolation problem is indeed correct. We will assume that this is the case throughout this note. Note further that correctness depends only on the set of interpolation points A_n and *not* on the particular basis B_n chosen.

Supposing then that the interpolation problem (1) is correct, we may write the interpolating polynomial in so-called Lagrange form as follows. For $a \in A_n$ set

$$\text{(3)} \qquad \ell_a(x) := \frac{\mathrm{vdm}(A_n \backslash \{a\} \cup \{x\}; B_n)}{\mathrm{vdm}(A_n; B_n)}.$$

A brief explanation of this formula is in order. The numerator is but the Vandermonde determinant with the interpolation point $a \in A_n$ replaced by the variable $x \in \mathbb{R}^d$.

Then, expanding $\mathrm{vdm}(A_n \backslash \{a\} \cup \{x\}; B_n)$ along the row corresponding to x, we see that ℓ_a is a linear combination of the P_j and hence $\ell_a \in \mathcal{P}_n(K)$. Further, it is easy to see that $\ell_a(b) = \delta_{ab}$, the Kronecker delta, for $b \in A_n$. The ℓ_a are called the

Fundamental Lagrange Interpolating Polynomials and using them we may write the interpolant of (1) as

$$p(x) = \sum_{a \in A_n} f(a)\ell_a(x). \tag{4}$$

The mapping $f \mapsto p$ is a projection and hence we sometimes write $p = \pi_{A_n}(f)$. If we regard both $f, p \in C(K)$ then the operator π_{A_n} has operator norm (as is not difficult to see)

$$\|\pi_{A_n}\| = \max_{x \in K} \sum_{a \in A_n} |\ell_a(x)|.$$

This operator norm (sometimes called the Lebesgue constant) gives a bound on how far the interpolant is from the best uniform polynomial approximant to f. To see this, for any $q \in \mathcal{P}_n(K)$, write

$$\begin{aligned} \|f - p\|_K &= \|f - \pi_{A_n}(f)\|_K \\ &= \|f - q - \pi_{A_n}(f - q)\|_K \\ &\leqslant \|f - q\|_K + \|\pi_{A_n}(f - q)\|_K \\ &\leqslant \|f - q\|_K + \|\pi_{A_n}\|_K \|f - q\|_K \\ &= (1 + \|\pi_{A_n}\|)\|f - q\|_K \end{aligned}$$

so that

$$\|f - p\|_K \leqslant (1 + \|\pi_{A_n}\|) \inf_{q \in \mathcal{P}_n(K)} \|f - q\|_K.$$

It follows that the quality of approximation to f provided by the interpolant p is indicated by the size of $\|\pi_{A_n}\|$, the smaller it is the better.

Now, suppose that $F_n \subset K$ is a subset of N_n distinct points for which $A_n = F_n$ maximizes $|\operatorname{vdm}(A_n; B_n)|$. Then by (3), each

$$\|\ell_a\|_K \leqslant 1, \quad a \in F_n \tag{5}$$

and hence the corresponding Lebesgue constants are such that

$$\|\pi_{A_n}\|_K \leqslant N_n.$$

The set F_n is called a set of Fekete points of degree n for K and provides, for any K, a good (often excellent) set of interpolation points.

In one variable, for $K = [-1, 1]$, the Fekete points have been much studied. Fejér [5] showed that F_n consists of $-1, +1$ together with the zeros of $P_n'(x)$, where P_n is the nth Legendre polynomial. Sündermann [8] subsequently showed that the Lebesgue constants are $O(\log(n))$, which is best possible. This confirms that the Fekete points for the interval are indeed excellent interpolation points.

From Fejér's result in particular, it follows that they are asymptotically nearly equally spaced with respect to the arcsin metric,

$$\text{dist}(a, b) = \left|\cos^{-1}(b) - \cos^{-1}(a)\right|. \tag{6}$$

In other words, for each $a \in F_n$,

$$\min_{\substack{b \in F_n \\ b \neq a}} \text{dist}(a, b) \approx \frac{c}{n}$$

for some constant c.

In contrast, as the Fekete points are more dense near the endpoints (just as are the Chebyshev points, for example), in the usual euclidean distance, there are points $a \in F_n$ for which

$$\min_{\substack{b \in F_n \\ b \neq a}} |b - a| \approx \frac{c}{n^2}.$$

More generally, Kövari and Pommerenke [7] have discussed the spacing of complex Fekete points for $K \subset \mathbb{C}$, a continuum.

In several variables, up to now, very little has been known about the spacing of the Fekete points. Dubiner [4] has shown that for general compact sets there is a lower bound

$$\frac{c_1}{n} \leqslant \min_{\substack{b \in F_n \\ b \neq a}} \text{dist}(a, b)$$

for an appropriate analogue of the arcsin metric (6) and $c_1 = \pi/2$ (cf. Theorem 1). We will show that for $K \subset \mathbb{R}^d$, either a sphere, ball or simplex, there is a corresponding upper bound, so that we may conclude there are constants $c_1, c_2 > 0$, depending only on the dimension d, such that

$$\frac{c_1}{n} \leqslant \min_{\substack{b \in F_n \\ b \neq a}} \text{dist}(a, b) \leqslant \frac{c_2}{n}.$$

2. THE BARAN AND DUBINER DISTANCES

The generalizations of the arcsin distance (6) that we will use are the Baran and Dubiner distances studied in [1,2].

First, we recall for a compact set $K \subset \mathbb{C}^d$, the function

$$V_K(z) := \sup\left\{\log\left(|p(z)|^{1/\deg(p)}\right):\ p : \mathbb{C}^d \to \mathbb{C}, \deg(p) \geqslant 1, \|p\|_K \leqslant 1\right\}$$

is known as the Siciak–Zaharjuta extremal function (see the monograph by Klimek [6] for more detail). If $V_K(z)$ is finite, which it is for all $z \in \mathbb{C}^d$ when $K = \overline{\Omega} \subset \mathbb{R}^d$

where Ω is a domain, then for any polynomial p and any point z, from the definition of V_K we have the Bernstein–Walsh inequality

$$|p(z)| \leqslant e^{\deg(p) V_K(z)} \|p\|_K.$$

Definition 1. Suppose that $K = \overline{\Omega}$ where $\Omega \subset \mathbb{R}^d$ is a bounded domain. Then

$$\delta_B(x; y) := \limsup_{t \to 0^+} \frac{V_K(x + ity)}{t},$$

(for $x \in \Omega$ and $y \in \mathbb{R}^N$) defined for compact K for which it is usc, is the *Baran pseudometric for* K and

$$\mathrm{dist}_B(a, b) = \inf_{\gamma} \int_0^1 \delta_B\big(\gamma(t); \gamma'(t)\big)\, dt$$

where the inf is taken over all parametric curves $\gamma : [0, 1] \to K$ with $\gamma(0) = a$ and $\gamma(1) = b$, is the *Baran* distance for K.

We remark, that from the results of [3], δ_B is continuous for $x \in K^o$ if K is an arbitrary convex body. Moreover, in this case, the limsup in the definition of δ_B is actually a limit.

Definition 2. Suppose that $K \subset \mathbb{R}^d$ is compact. Then

$$\mathrm{dist}_D(a, b) := \sup_{\|p\|_K \leqslant 1, \deg p \geqslant 1} \frac{1}{\deg p} \left|\cos^{-1}\big(p(b)\big) - \cos^{-1}\big(p(a)\big)\right|$$

is the *Dubiner* distance on K.

Note that $\mathrm{dist}_B(a, b)$ is only well defined for compact sets which are the closure of a domain, and hence not for a sphere. However, $\mathrm{dist}_D(a, b)$ is well defined for any compact set $K \subset \mathbb{R}^d$, including the sphere. It turns out that, when both are well defined, it is always the case that

$$\mathrm{dist}_D(a, b) \leqslant \mathrm{dist}_B(a, b). \tag{7}$$

For the proof of this and also other properties of these distances we refer the reader to [1,2].

Of importance to us here will be the following general theorem, given by Dubiner [4].

Theorem 1 (Dubiner). *Suppose that $K \subset \mathbb{R}^d$ is compact and that $F_n \subset K$ is a set of Fekete points of degree n. Then, for all $a \in F_n$,*

$$\frac{\pi}{2} \frac{1}{n} \leqslant \min_{\substack{b \in F_n \\ b \neq a}} \mathrm{dist}_D(a, b). \tag{8}$$

Proof. Consider $p = \ell_a$, the Lagrange polynomial of degree n for a. Then by (5), $\|\ell_a\|_K \leqslant 1$ and so $p = \ell_a$ is a candidate in the supremum defining the Dubiner distance. Hence, for any $a \neq b \in F_n$,

$$\begin{aligned}\mathrm{dist}_D(a,b) &\geqslant \frac{1}{n}\left|\cos^{-1}\big(\ell_a(b)\big) - \cos^{-1}\big(\ell_a(a)\big)\right| \\ &= \frac{1}{n}\left|\cos^{-1}(0) - \cos^{-1}(1)\right| \\ &= \frac{1}{n}\frac{\pi}{2}.\end{aligned}$$

□

3. THE SPACING OF FEKETE POINTS ON THE SPHERE

We take $K = S^{d-1} \subset \mathbb{R}^d$ the unit sphere. In this case the Dubiner distance is just the geodesic distance on the sphere (cf. [1]), i.e., for $a, b \in S^{d-1}$,

$$\mathrm{dist}_D(a,b) = \mathrm{dist}(a,b) = \cos^{-1}(a \cdot b).$$

Theorem 2. *There are constants $c_1 = \pi/2$ and $c_2 > 0$, depending only on the dimension d, such that if $F_n \subset S^{d-1}$ is a set of Fekete points of degree n, then for all $a \in F_n$,*

$$\frac{c_1}{n} \leqslant \min_{\substack{b \in F_n \\ b \neq a}} \mathrm{dist}(a,b) \leqslant \frac{c_2}{n}.$$

Proof. The lower bound is given immediately by Theorem 1. To show the upper bound, we will make use of the polynomial provided by the following lemma. We remark that the constants in our estimates below, $c, c_3, c_4, \ldots,$ all depend only on the dimension d. We do not specify their precise values. □

Lemma 1. *There is a constant $c > 0$ such that for all integers $n \geqslant 1$ and points $A \in S^{d-1}$, there exists a spherical polynomial P of degree at most n such that* (a) $P(A) = 1$ *and* (b)

$$|P(x)| \leqslant \frac{c}{n^d}\,\mathrm{dist}(A,x)^{-d}, \quad x \in S^{d-1}. \tag{9}$$

Proof. Let $\phi \in [0, \pi]$ be the angle between $x \in S^{d-1}$ and the point A so that $\cos(\phi) = A \cdot x$. Note that then $\phi = \mathrm{dist}(x, A)$. For $m := \lfloor n/d \rfloor$ set

$$\begin{aligned}Q(x) &= \frac{2}{2m+1}\left\{\frac{1}{2} + \cos(\phi) + \cos(2\phi) + \cdots + \cos(m\phi)\right\} \\ &= \frac{1}{2m+1}\frac{\sin(\frac{2m+1}{2}\phi)}{\sin(\frac{\phi}{2})}.\end{aligned} \tag{10}$$

Note that we may also write

$$Q(x) = \frac{1}{2m+1} U_{2m}\left(\sqrt{\frac{A \cdot x + 1}{2}}\right)$$

where U_k denotes the Chebyshev polynomial of the second kind of degree k.

Then $P(x) := Q(x)^d$ has the desired properties. $\square$

Continuing, fix $a \in F_n$ and let $a^* \in F_n$ be a closest Fekete point to a. Then choose $A \in S^{d-1}$ so that $\operatorname{dist}(a, A) = \frac{1}{2} \operatorname{dist}(a, a^*)$. In particular, $A \notin F_n$.

Then, for all $b \in F_n$, $b \neq a$,

$$\begin{aligned} \operatorname{dist}(b, A) &\geqslant \operatorname{dist}(b, a) - \operatorname{dist}(a, A) \\ &= \operatorname{dist}(b, a) - \frac{1}{2} \operatorname{dist}\bigl(a, a^*\bigr) \\ &= \frac{1}{2} \operatorname{dist}(b, a) + \frac{1}{2}\bigl\{\operatorname{dist}(a, b) - \operatorname{dist}\bigl(a, a^*\bigr)\bigr\} \\ &\geqslant \frac{1}{2} \operatorname{dist}(b, a) \end{aligned} \tag{11}$$

as $\operatorname{dist}(a, b) \geqslant \operatorname{dist}(a, a^*)$ by the definition of a^*.

Now let $P(x)$ be the polynomial of degree n provided by Lemma 1 for the point A. We may write

$$P(x) = \sum_{b \in F_n} P(b) \ell_b(x)$$

so that at $x = A$,

$$1 = P(A) = \sum_{b \in F_n} P(b) \ell_b(A).$$

Taking absolute values, it follows that

$$\begin{aligned} 1 &\leqslant \sum_{b \in F_n} |P(b)| && \text{by (5)} \\ &\leqslant \frac{c}{n^d} \sum_{b \in F_n} \operatorname{dist}(b, A)^{-d} && \text{by (9)} \\ &= \frac{c}{n^d} \left\{ \operatorname{dist}(a, A)^{-d} + \sum_{\substack{b \in F_n \\ b \neq a}} \operatorname{dist}(b, A)^{-d} \right\} \\ &= \frac{c}{n^d} \left\{ 2^d \operatorname{dist}\bigl(a, a^*\bigr)^{-d} + \sum_{\substack{b \in F_n \\ b \neq a}} \operatorname{dist}(b, A)^{-d} \right\} && \text{by the choice of } A \end{aligned} \tag{12}$$

$$\leqslant \frac{c}{n^d}\left\{2^d \operatorname{dist}\bigl(a, a^*\bigr)^{-d} + \sum_{\substack{b\in F_n\\ b\neq a}} 2^d \operatorname{dist}(b, a)^{-d}\right\} \quad \text{by (11)}$$

$$= 2^d \frac{c}{n^d}\left\{\operatorname{dist}\bigl(a, a^*\bigr)^{-d} + \sum_{\substack{b\in F_n\\ b\neq a}} \operatorname{dist}(b, a)^{-d}\right\}.$$

To estimate the sum in (12) we partition S^{d-1} into "strips"

$$S_0 := \left\{b \in S^{d-1} \mid \operatorname{dist}(b, a) \leqslant \frac{1}{2}\operatorname{dist}\bigl(a, a^*\bigr)\right\},$$

$$S_j := \left\{b \in S^{d-1} \,\middle|\, \frac{1}{2}\operatorname{dist}\bigl(a, a^*\bigr) + \frac{j-1}{n} < \operatorname{dist}(a, b) \leqslant \frac{1}{2}\operatorname{dist}\bigl(a, a^*\bigr) + \frac{j}{n}\right\}$$

where j is such that $\frac{1}{2}\operatorname{dist}(a, a^*) + \frac{j-1}{n} < \pi$ (the maximum distance).

It is convenient to denote

$$\lambda := n\left(\frac{1}{2}\operatorname{dist}\bigl(a, a^*\bigr)\right)$$

so that

$$S_j = \left\{b \in S^{d-1} \,\middle|\, \frac{\lambda + j - 1}{n} < \operatorname{dist}(a, b) \leqslant \frac{\lambda + j}{n}\right\}.$$

Note that $S_0 \cap F_n = \{a\}$ as $\operatorname{dist}(a, a^*)$ is minimal. Further, for S_j, we may compute its surface 'area' as

$$\begin{aligned}
V_{d-1}(S_j) &= c_3 \int_{(\lambda+j-1)/n}^{(\lambda+j)/n} \sin^{d-2}(\phi)\, d\phi && \text{for some constant } c_3\\
&= c_3\left\{\frac{\lambda + j}{n} - \frac{\lambda + j - 1}{n}\right\}\sin^{d-2}(\phi_j) && \text{for a } \phi_j \in \left[\frac{\lambda + j - 1}{n}, \frac{\lambda + j}{n}\right]\\
&= \frac{c_3}{n}\sin^{d-2}(\phi_j)\\
&\leqslant \frac{c_4}{n}\phi_j^{d-2} && \text{as } \sin(\theta) \leqslant \theta\\
&\leqslant \frac{c_4}{n}\left(\frac{\lambda + j}{n}\right)^{d-2} && \text{as } \phi_j \leqslant (\lambda + j)/n\\
&= \frac{c_4}{n^{d-1}}(\lambda + j)^{d-2}.
\end{aligned}$$

But, as there is the minimal spacing,

$$\operatorname{dist}\bigl(b, b^*\bigr) \geqslant \frac{c_1}{n},$$

for $b \in F_n$ and $b^* \in F_n$ a closest point to b, there are no other Fekete points in the 'disk' $\{x \in S^{d-1} \mid \operatorname{dist}(x, b) < c_1/n\}$, a set of volume

$$c_5 \int_0^{c_1/n} \sin^{d-2}(\phi)\, d\phi \leqslant \frac{c_6}{n^{d-1}}.$$

Hence there are at most

$$c_7 \frac{(\lambda + j)^{d-2}/n^{d-1}}{1/n^{d-1}} = c_7(\lambda + j)^{d-2} \tag{13}$$

Fekete points in the strip S_j.

It follows that

$$\begin{aligned}
\sum_{\substack{b \in F_n \\ b \neq a}} \operatorname{dist}(a, b)^{-d} &= \sum_{j \geqslant 1} \sum_{b \in F_n \cap S_j} \operatorname{dist}(a, b)^{-d} \\
&\leqslant c_7 \sum_{j \geqslant 1} (\lambda + j)^{d-2} \left(\frac{\lambda + j - 1}{n} \right)^{-d} \\
&= c_7 n^d \sum_{j \geqslant 1} \frac{(\lambda + j)^{d-2}}{(\lambda + j - 1)^d}.
\end{aligned}$$

Now, we may assume that $\lambda \geqslant 1$, for if not, $\lambda < 1$ and hence

$$n\left(\frac{1}{2} \operatorname{dist}\big(a, a^*\big) \right) \leqslant 1$$

and so $\operatorname{dist}(a, a^*) \leqslant 2/n$ and we are done. Making this assumption then, we have

$$\lambda + j \leqslant 2(\lambda + j - 1), \quad j \geqslant 1,$$

so that

$$\begin{aligned}
\sum_{\substack{b \in F_n \\ b \neq a}} \operatorname{dist}(a, b)^{-d} &\leqslant c_7 n^d \sum_{j \geqslant 1} \frac{(\lambda + j)^{d-2}}{(\lambda + j - 1)^d} \\
&\leqslant c_7 n^d \sum_{j \geqslant 1} \frac{2^{d-2}(\lambda + j - 1)^{d-2}}{(\lambda + j - 1)^d} \\
&= c_8 n^d \sum_{j \geqslant 1} \frac{1}{(\lambda + j - 1)^2} \\
&\leqslant c_9 n^d \left\{ \int_0^\infty \frac{1}{(\lambda + x)^2}\, dx + \frac{1}{\lambda^2} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= c_9 n^d \left\{ \frac{1}{\lambda} + \frac{1}{\lambda^2} \right\} \\
&\leqslant c_{10} n^d \frac{1}{\lambda} \quad \text{using again that } \lambda \geqslant 1.
\end{aligned}
$$

Combining this with (12), we have

$$
\begin{aligned}
1 &\leqslant \frac{2^d c}{n^d} \left\{ \frac{n^d}{2^d \lambda^d} + c_{10} n^d \frac{1}{\lambda} \right\} \\
&\leqslant c_{11} \left\{ \frac{1}{\lambda^d} + \frac{1}{\lambda} \right\} \\
&\leqslant c_{12} \frac{1}{\lambda}.
\end{aligned}
$$

Hence $\lambda \leqslant c_{12}$ and we are done.

4. THE SPACING OF FEKETE POINTS ON THE BALL

We now take $K = B^d \subset \mathbb{R}^d$ the unit ball. In this case the Dubiner and Baran distances (cf. [1,2]) are equal and are described as follows. For $a \in B^d$, i.e., $|a| \leqslant 1$, set

$$
\widetilde{a} := \big(a, \sqrt{1 - |a|^2}\big) \in S^d \subset \mathbb{R}^{d+1}.
$$

In other words, $\widetilde{a}$ is a lifted to the circumscribing sphere S^d.

Then, for $a, b \in B^d$, the Dubiner and Baran distances are just the geodesic spherical distance on S^d between $\widetilde{a}$ and $\widetilde{b}$, i.e.,

$$
\begin{aligned}
\mathrm{dist}_D(a, b) &= \mathrm{dist}_B(a, b) \\
&= \cos^{-1}(\widetilde{a} \cdot \widetilde{b}) \\
&= \cos^{-1}\big(a \cdot b + \sqrt{1 - |a|^2}\sqrt{1 - |b|^2}\big).
\end{aligned}
$$

We will refer to either of these as $\mathrm{dist}(a, b)$.

We remark that the surface area measure on S^d pulls back under the mapping $a \mapsto \widetilde{a}$ to a measure on B^d, the ‘surface area’ measure,

$$
d\mu = c \frac{1}{\sqrt{1 - |x|^2}}\, dx \tag{14}
$$

where c is a normalizing constant.

Theorem 3. *There are constants $c_1 = \pi/2$ and $c_2 > 0$, depending only on the dimension d, such that if $F_n \subset B^d$ is a set of Fekete points of degree n, then for all $a \in F_n$,*

$$
\frac{c_1}{n} \leqslant \min_{\substack{b \in F_n \\ b \neq a}} \mathrm{dist}(a, b) \leqslant \frac{c_2}{n}.
$$

Proof. The lower bound is given immediately by Theorem 1. To show the upper bound, we will make use of a polynomial analogous to that provided by Lemma 1. We will first need to establish a technical result.

Lemma 2. *Suppose that* $Q(\phi)$ *is the trigonometric polynomial given by* (10). *Then for* $\phi \in [0, \pi]$,

$$Q(\phi) \geqslant -1/2.$$

Proof. If $Q(\phi) \geqslant 0$ we need proceed no further, and hence we need only consider ϕ for which $Q(\phi) \leqslant 0$. But

$$Q(\phi) = \frac{1}{2m+1} \frac{\sin(\frac{2m+1}{2}\phi)}{\sin(\frac{\phi}{2})}$$

and hence it changes sign at $\phi_k := 2k\pi/(2m+1)$, $k = 1, 2, \ldots, m$. Specifically, $Q(\phi) \leqslant 0$ on the intervals $[2k\pi/(2m+1), 2(k+1)\pi/(2m+1)]$ for *odd* k. On such an interval,

$$\begin{aligned} |Q(\phi)| &\leqslant \left(\frac{1}{2m+1}\right) \frac{1}{\sin(k\pi/(2m+1))} \\ &\leqslant \left(\frac{1}{2m+1}\right) \frac{1}{(2/\pi)k\pi/(2m+1)}, \end{aligned}$$

using the fact that $\sin(\theta) \leqslant \frac{2}{\pi}\theta$, for $\theta \in [0, \pi/2]$. Hence on the interval $[2k\pi/(2m+1), 2(k+1)\pi/(2m+1)]$, $|Q(\phi)| \leqslant 1/(2k) \leqslant 1/2$. □

Lemma 3. *There is a constant* $c > 0$ *such that for all integers* $n \geqslant 1$ *and points* $A \in B^d$, *there exists an algebraic polynomial* P *of degree at most* n *such that* (a) $P(A) = 1$ *and* (b)

$$|P(x)| \leqslant \frac{c}{n^{d+1}} \operatorname{dist}(A, x)^{-(d+1)}, \quad x \in B^d.$$

Proof. For a point $\widetilde{x} \in S^d$ write $\widetilde{x} = (x, z)$ where $x \in B^d$ and $z \in [-1, 1]$. Let $Q(\widetilde{x})$ be the spherical polynomial on S^d given by (10), where ϕ is the angle between $\widetilde{x} \in S^d$ and $\widetilde{A} \in S^d$ and $m = \lfloor n/(d+1) \rfloor$. Note that for $x \in B^d$, $Q(x, z)^{d+1} + Q(x, -z)^{d+1}$ is even in z and hence a function of $z^2 = 1 - |x|^2$, on S^d. Hence

$$P(x) = Q(x, z)^{d+1} + Q(x, -z)^{d+1}$$

is an algebraic polynomial in x. We claim that it has (essentially) the required properties.

First note that by Lemma 2, $P(A) \geqslant 1 - 2^{-(d+1)} > 0$ and hence property (a) follows from a constant re-normalization. To see property (b) just note that

$$\begin{aligned}
\operatorname{dist}_{S^d}\big((x,\pm z),\widetilde{A}\big) &= \cos^{-1}\big(x\cdot A \pm z\sqrt{1-|A|^2}\big)\\
&\geqslant \cos^{-1}\big(x\cdot A + |z|\sqrt{1-|A|^2}\big)\\
&= \cos^{-1}\big(x\cdot A + \sqrt{1-|x|^2}\sqrt{1-|A|^2}\big)\\
&= \operatorname{dist}(x,A).
\end{aligned}$$

Hence both $Q(x,z)^{d+1}$ and $Q(x,-z)^{d+1}$ are bounded by

$$\frac{c}{n^{d+1}}\operatorname{dist}(x,A)^{-(d+1)}.$$

□

The proof of Theorem 3 is now exactly the same as for Theorem 2, except in one dimension higher, using the polynomial $P(x)$ provided by Lemma 3. The volumes of the strips S_j are measured using the measure $d\mu$ of (14) to yield (13). We omit the details. □

5. THE SPACING OF FEKETE POINTS ON THE SIMPLEX

We now take $K = T^d \subset \mathbb{R}^d$ the standard simplex, i.e.,

$$T^d := \left\{ x\in\mathbb{R}^d \;\middle|\; x_i \geqslant 0,\, i=1,\ldots,d,\ \text{ and } \sum_{i=1}^d x_i \leqslant 1\right\}.$$

In this case the Baran distance (cf. [1,2]) is described as follows. For $a\in T^d$, set

$$\widetilde{a} := \left(\sqrt{a_1},\sqrt{a_2},\ldots,\sqrt{a_d},\sqrt{1-\sum_{i=1}^d a_i}\right)\in S^d\subset R^{d+1}.$$

Then, for $a,b\in T^d$,

$$\begin{aligned}
\operatorname{dist}_B(a,b) &= 2\operatorname{dist}_{S^d}(\widetilde{a},\widetilde{b})\\
&= 2\cos^{-1}(\widetilde{a}\cdot\widetilde{b}).
\end{aligned}$$

We remark that the surface area measure on S^d pulls back under the mapping $a\mapsto\widetilde{a}$ to a measure on T^d,

$$d\mu = c\frac{1}{\sqrt{x_1x_2\cdots x_d(1-\sum_{i=1}^d x_i)}}\,dx$$

where c is a normalizing constant.

A closed form equation for the Dubiner distance is not known, but one will not be needed here.

Theorem 4. *There are constants $c_1 = \pi/2$ and $c_2 > 0$, depending only on the dimension d, such that if $F_n \subset T^d$ is a set of Fekete points of degree n, then for all $a \in F_n$,*

$$\frac{c_1}{n} \leqslant \min_{\substack{b \in F_n \\ b \neq a}} \mathrm{dist}_D(a, b) \leqslant \min_{\substack{b \in F_n \\ b \neq a}} \mathrm{dist}_B(a, b) \leqslant \frac{c_2}{n}.$$

Proof. The lower bound is given immediately by Theorem 1. To show the upper bound, we will again make use of a polynomial analogous to that provided by Lemmas 1 and 3.

Lemma 4. *There is a constant $c > 0$ such that for all integers $n \geqslant 1$ and points $A \in T^d$, there exists an algebraic polynomial P of degree at most n such that* (a) $P(A) = 1$ *and* (b)

$$|P(x)| \leqslant \frac{c}{n^{d+1}} \mathrm{dist}_B(A, x)^{-(d+1)}, \quad x \in T^d.$$

Proof. We let $\widetilde{x}$ denote a point in S^d. Then, let $Q(\widetilde{x})$ be the spherical polynomial on S^d given by (10), where ϕ is the angle between $\widetilde{x} \in S^d$ and $\widetilde{A} \in S^d$ and $m = \lfloor 2n/(d+1) \rfloor$.

Now, let $\mathcal{M}$ denote the set of $(d+1) \times (d+1)$ diagonal matrices with ± 1 on the diagonal. There are $\#\mathcal{M} = 2^{d+1}$ elements in $\mathcal{M}$.

Then, set

$$\widetilde{P}(\widetilde{x}) := \sum_{M \in \mathcal{M}} Q(M\widetilde{x})^{d+1}.$$

$\widetilde{P}(\widetilde{x})$ is a polynomial of degree at most $2n$ in $\widetilde{x}$ that is symmetric under each of the mappings $\widetilde{x}_i \mapsto -\widetilde{x}_i$. Hence it is actually a polynomial in the $\widetilde{x}_i^2$. Then, as $\widetilde{x}_i = \sqrt{x_i}$ $1 \leqslant i \leqslant d$ and $\widetilde{x}_{d+1} = \sqrt{1 - \sum_{i=1}^d x_i}$,

$$P(x) := \widetilde{P}(\widetilde{x})$$

is an algebraic polynomial of degree at most n in x. We claim that $P(x)$ (essentially) satisfies the desired properties.

First note that

$$\begin{aligned} P(A) &= \sum_{M \in \mathcal{M}} Q(M\widetilde{A})^{d+1} \\ &= Q(\widetilde{A})^{d+1} + \sum_{\substack{M \in \mathcal{M} \\ M \neq I}} Q(M\widetilde{A})^{d+1} \\ &= 1 + \sum_{\substack{M \in \mathcal{M} \\ M \neq I}} Q(M\widetilde{A})^{d+1} \end{aligned}$$

$$
\begin{aligned}
&\geqslant 1 - \sum_{\substack{M \in \mathcal{M} \\ M \neq I}} (1/2)^{d+1} \quad \text{by Lemma 2} \\
&= 1 - \big(2^{d+1} - 1\big) 2^{-(d+1)} \\
&= 2^{-(d+1)} > 0.
\end{aligned}
$$

Thus property (a) is attained by a renormalization.

To see property (b), note that for each $M \in \mathcal{M}$ and $\widetilde{x} := (\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_d}, \sqrt{1 - \sum_{i=1}^{d} x_i})$

$$
(M\widetilde{x}) \cdot \widetilde{A} \leqslant \widetilde{x} \cdot \widetilde{A}
$$

so that, as $\cos^{-1}$ is a decreasing function,

$$
\operatorname{dist}_{S^d}(M\widetilde{x}, \widetilde{A}) \geqslant \operatorname{dist}_{S^d}(\widetilde{x}, \widetilde{A}).
$$

It follows that for all $M \in \mathcal{M}$,

$$
Q(M\widetilde{x})^{d+1} \leqslant \frac{c}{n^{d+1}} \operatorname{dist}_B(x, A)^{-(d+1)}
$$

and hence $P(x)$ satisfies (b). □

The proof of Theorem 4 is now exactly the same as for Theorems 2 and 3, using the polynomial $P(x)$ provided by Lemma 4. The volumes of the strips S_j are measured using the measure $d\mu$ of (15) to yield (13). We again omit the details. □

REFERENCES

[1] Bos L., Levenberg N., Waldron S. – Metrics associated to multivariate polynomial inequalities, in: Advances in Constructive Approximation, Neamtu M., Saff E.B. (Eds.), Nashboro Press, Nashville, 2004, pp. 133–147.

[2] Bos L., Levenberg N., Waldron S. – Pseudometrics, distances and multivariate polynomial inequalities, to appear.

[3] Burns D., Levenberg N., Ma'u S., Révész Sz. – Monge–Ampère measures for convex bodies and Bernstein–Markov type inequalities, submitted.

[4] Dubiner M. – The theory of multidimensional polynomial approximation, J. Anal. Math. **67** (1995) 39–116.

[5] Fejér L. – Bestimmung der jenigen Abszissen eines Intervalles, für welche die Quadratsumme der Grundfunktionen der Lagrangeschen Interpolation im Intervalle ein Möglichst kleines Maximum Besitzt, Annali della Scuola Normale Superiore di Pisa – Classe di Scienze, Ser. 2 **1** (1932) 263–276.

[6] Klimek M. – Pluripotential Theory, Clarendon Press, Oxford, 1991.

[7] Kövari T., Pommerenke Ch. – On the distribution of Fekete points, Mathematika **15** (1968) 70–75.

[8] Sündermann B. – Lebesgue constants in Lagrangian interpolation at the Fekete points, Mitt. Math. Ges. Hamb. **11** (1983) 204–211.

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