TECHNION - ISRAEL INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATHEMATICS

Schmidt's Inequality

Shayne Waldron* (*waldron@math.auckland.ac.nz*)

Technical Report
January 1997

ABSTRACT

The main result is the computation of the best constant in the Wirtinger–Sobolev inequality

$$\|f\|_p \leq C_{p,q,\theta}\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}}\|Df\|_q,$$

where

$$f(\theta)=0,$$

and θ is some point in $[a,b]$, or, equivalently, the determination of the norm of the (bounded) linear map

$$A : L_q[a,b] \to L_p[a,b]$$

given by

$$Af(x) := \int_\theta^x f(t)\,dt.$$

This and other results are seen to be closely related to an inequality of Schmidt 1940.

The method of proof is elementary, and should be the main point of interest for most readers since it clearly illustrates a technique that can be applied to other situations. These include the generalisations of Hardy's inequality where $\theta = a$ and $\|\cdot\|_p$, $\|\cdot\|_q$ are replaced by weighted L_p, L_q norms, and higher order Wirtinger–Sobolev inequalities involving boundary conditions at a single point.

Key Words: Schmidt's inequality, Hardy–type inequalities, Wirtinger–Sobolev inequalities, Poincaré inequalities, Hölder's inequality, isoperimetric calculus of variations problems, n–widths

AMS (MOS) Subject Classifications: primary 41A44, 41A80, 47A30, secondary 34B10, 34L30

$*$ Supported by the Israel Institute of Technology

1. Introduction

There has recently been considerable progress in the problem of estimating the best constant C in the inequality

$$\|D^j(f - H_\Theta f)\|_p \le C\,(b-a)^{n-j+\frac{1}{p}-\frac{1}{q}}\|D^n f\|_q, \qquad \forall f \in W_q^n, \tag{1.1}$$

where $H_\Theta f$ is the Hermite interpolant to f at the some multiset Θ of n points in $[a,b]$, and $0 \le j < n$. In Shadrin [S95], the best constant was determined for $p = q = \infty$, $0 \le j < n$, and all Θ. The remaining estimates in the extensive literature on this problem were extended and put within a unified framework based on a single 'basic estimate' in Waldron [W96]. Inequalities of the form (1.1) belong to the class of Wirtinger(–Sobolev) inequalities (also called Poincaré inequalities), see, e.g., Fink, Mitrinović and Pečarić [FMP91:p66].

Towards a better understanding of what, if any, improvements to these estimates might be possible (for $p, q \ne \infty$), the best constant in (1.1) is computed in the simplest case, when $n = 1$ ($j = 0$), for $1 \le p, q \le \infty$. Here $\Theta = \{\theta\}$, a single point in $[a,b]$, and

$$H_\Theta f = f(\theta), \tag{1.2}$$

the constant polynomial which matches f at θ.

Since

$$f(x) - H_\Theta f(x) = f(x) - f(\theta) = \int_\theta^x Df(t)\,dt, \tag{1.3}$$

finding the best constant in (1.1) is equivalent to computing the norm of the linear map

$$A : L_q[a,b] \to L_p[a,b]$$

given by

$$Af(x) := \int_\theta^x f(t)\,dt,$$

and since

$$Df = D(f - f(\theta)) = D(f - H_\Theta f),$$

it is also equivalent to finding the best constant C in the inequality: for $f \in W_q^1$ with $f(\theta) = 0$,

$$\|f\|_p \le C\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}}\|Df\|_q. \tag{1.4}$$

It is the last of these equivalencies which appears most commonly, and we will solve the problem in these terms. The solution is given in Theorem 4.1.

The rest of the paper is set out as follows. In Section 2, the (standard) variational approach to finding the best constant in (1.4) is outlined. In Section 3, the 'elementary argument' which allows the problem to be split into two problems with boundary conditions of the form $f(a) = 0$ (equivalently $f(b) = 0$), and thereby reduced to a 'maximisation problem' of one variable, is given. In Section 4, the 'maximisation problem' is solved and the best constant and corresponding extremal functions (when they exist) are computed.

In Section 5, a number of related results concerning extremal problems and n-widths which can be obtained from an inequality of Schmidt 1940 by using simple geometric arguments (such as those in this paper) are discussed.

2. The variational approach

Let $W_p^1 := W_p^1[a,b]$ be the *Sobolev space* of functions f which are absolutely continuous on $[a,b]$, with $D^1 f \in L_p := L_p[a,b]$. To solve isoperimetric extremal problems such as

$$\sup_{f\neq 0}\left\{\frac{\|f\|_p}{\|Df\|_q} : f(\theta)=0,\ f\in W_q^1\right\} = \sup\left\{\|f\|_p : f(\theta)=0,\ \|Df\|_q=1,\ f\in W_q^1\right\}, \quad (2.1)$$

the standard approach is to use the calculus of variations.

For $\theta = a,b$ the Euler–Lagrange equation for (2.1) has been solved. The following result is essentially due to Schmidt [Sc40:(20),p306].

Lemma 2.2. *Let $0<p\le\infty$, $1\le q\le\infty$. Then, for all $f\in W_q^1[a,b]$ satisfying*

$$f(a)=0, \qquad (\textit{equivalently } f(b)=0) \quad (2.3)$$

there is the sharp inequality

$$\|f\|_p \le C(p,q)\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}}\|Df\|_q, \quad (2.4)$$

where $0<C(p,q)\le 1$ is defined by

$$C(p,q) := \frac{(\frac{1}{p}+\frac{1}{q'})^{-\frac{1}{p}-\frac{1}{q'}}}{(\frac{1}{p})^{-\frac{1}{p}}(\frac{1}{q'})^{-\frac{1}{q'}}}\,\frac{\Gamma(1+\frac{1}{p}+\frac{1}{q'})}{\Gamma(1+\frac{1}{p})\Gamma(1+\frac{1}{q'})}, \quad (2.5)$$

with q' the conjugate exponent of q, and $1/\infty$ is to be interpreted (in the usual way) as 0.

For $\theta \neq a,b$ the Euler–Lagrange equation for (2.1) splits into a pair of equations, similar to that for the case $\theta=a$, which are connected by a common parameter. In principle, this pair of equations can be solved by using the solution for when $\theta=a$. Instead, we perform effectively this argument in terms of the inequalities (1.4). This is our 'elementary argument'. It provides the extremals in the cases when they exist, by using the following.

For all values of p,q in Lemma 2.2, except $q=1$, $p\neq\infty$, extremals exist for (2.4), and are given by the scalar multiples of $E_{p,q}\left((\cdot-a)/(b-a)\right)$, where

$$E_{p,q} : [0,1]\to[0,1]$$

is defined as the (unique) solution of the initial value problem

$$Df = I_{p,q}(1-f^p)^{\frac{1}{q}}, \qquad f(0)=0, \quad (2.6)$$

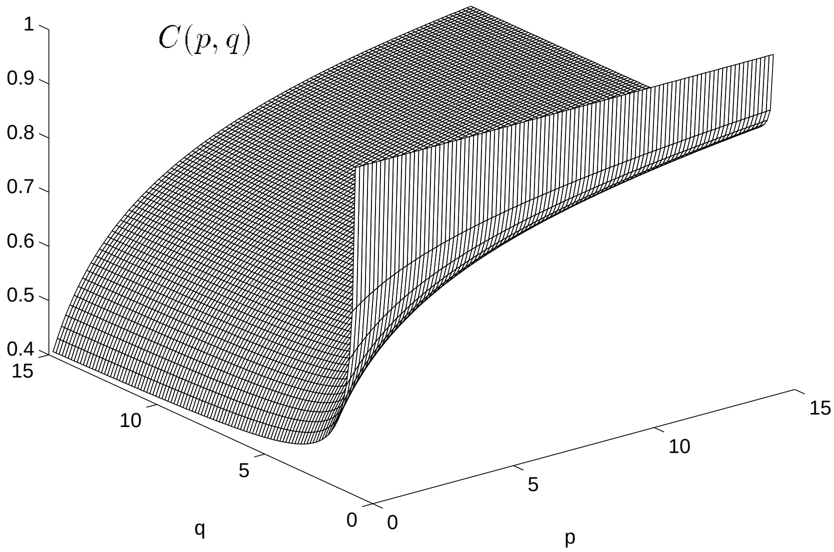

Fig. 2.1. The graph of $(p, q) \mapsto C(p, q)$ over $(0, 15] \times [1, 15]$. Note $C(p, 1) = 1$.

where

$$I_{p,q} := \int_0^1 \frac{d\xi}{(1-\xi^p)^{1/q}} = \frac{\Gamma(\frac{1}{p})\Gamma(\frac{1}{q'})}{p\Gamma(\frac{1}{p}+\frac{1}{q'})} = \frac{\Gamma(\frac{1}{p}+1)\Gamma(\frac{1}{q'})}{\Gamma(\frac{1}{p}+\frac{1}{q'})}, \tag{2.7}$$

with $I_{p,q}$ interpreted as 1 when $p = \infty$ or $q = \infty$. From (2.6), it is easily seen that $E_{p,q}$ is strictly increasing, concave, and satisfies

$$E_{p,q}(0) = 0, \quad E_{p,q}(1) = 1, \qquad DE_{p,q}(0) = I_{p,q}, \quad DE_{p,q}(1) = 0,$$

with

$$E_{2,2}(x) = \sin(\frac{\pi}{2}x).$$

The functions $E_{p,q}$ can be expressed as the p–th power of the inverse of an *incomplete Gamma function*. It is claimed incorrectly in Fink [F74:Lemma 2] that extremals for (2.4) exist unless $(p, q) = (1, 1)$ or (∞, ∞). The argument given there mistakenly concludes that since $C(p, q) = C(q', p')$, if extremals exist for the choice of norms (p, q), then they must exist for the choice (q', p'). For fixed p, $E_{p,q} \to \chi_{(0,1]}$ pointwise as $q \to 1^+$.

The numerical solution of (2.6) provides no obstacles. In our case, the graphs of $E_{p,q}$ appearing in this paper were done using `MATLAB` to compute the values of $E_{p,q}$ (at equally spaced points) by the *Runge-Kutta method of order* 4.

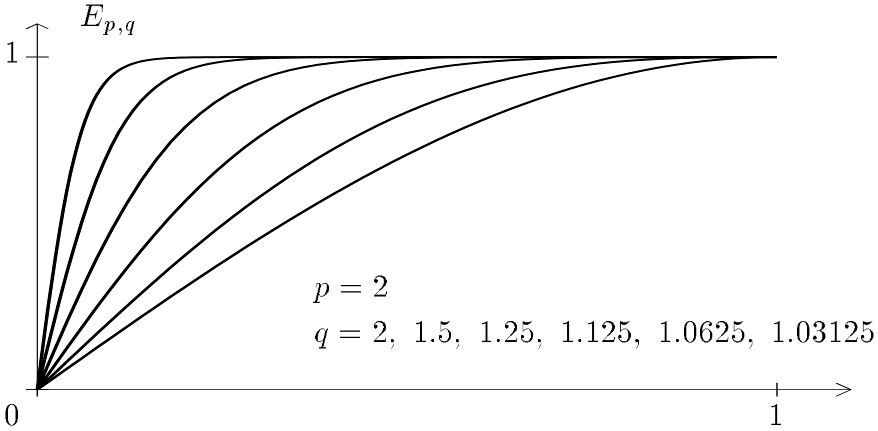

Fig. 2.2. The asymptotic behaviour of $E_{p,q}$ as $q \to 1^+$.

3. The elementary argument

The key to solution of (2.1) for $a < \theta < b$ presented below, is the observation that functions $f \in W_q^1[a,b]$ with $f(\theta) = 0$ are of the form

$$f(x) = \begin{cases} g(x), & a \le x \le \theta; \\ h(x), & \theta \le x \le b, \end{cases} \tag{3.1}$$

where

$$g \in W_q^1[a,\theta], \text{ with } g(\theta) = 0, \qquad h \in W_q^1[\theta,b], \text{ with } h(\theta) = 0,$$

together with the fact that

$$\int_a^b |f|^p = \int_a^\theta |g|^p + \int_\theta^b |h|^p, \qquad 0 < p < \infty,$$

$$\int_a^b |Df|^q = \int_a^\theta |Dg|^q + \int_\theta^b |Dh|^q, \qquad 1 \le q < \infty.$$

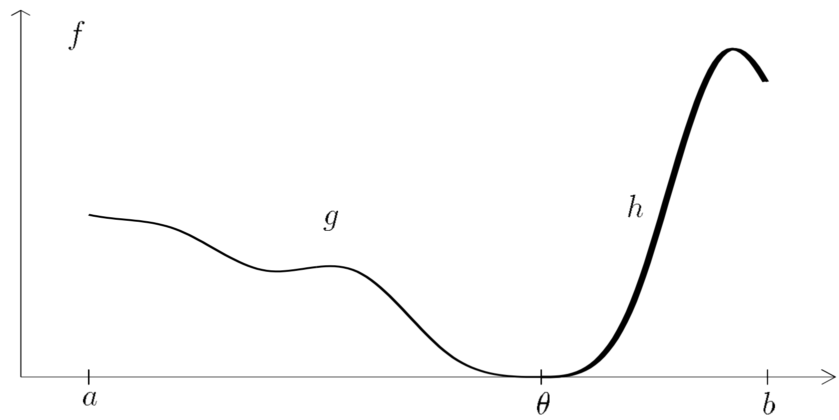

Fig. 3.1. The splitting of f into g and h (thicker).

It will be convenient to have (2.4) in the form

$$\sup_{\substack{f(a)=0 \\ \int_a^b |Df|^q=\alpha}} \int_a^b |f|^p = \alpha^{\frac{p}{q}} C(p,q)^p (b-a)^{p(1+\frac{1}{p}-\frac{1}{q})}, \qquad 0<p<\infty,\ 1\le q<\infty, \tag{3.2}$$

where $\alpha > 0$, (and the supremum is over functions in $W_q^1[a,b]$). It is to be understood that (3.2) also holds when the condition $f(a)=0$ is replaced by $f(b)=0$.

For simplicity, assume without loss of generality that $[a,b]=[0,1]$. Let $0\le\theta\le 1$, and $0<p<\infty$, $1\le q<\infty$. Then, by using the splitting (3.1) and (3.2), we compute that the solution of (2.1) satisfies

$$\begin{aligned}
&\sup\left\{\|f\|_p : f(\theta)=0,\ \|Df\|_q=1,\ f\in W_q^1\right\} \\
&= \sup_{0\le A\le 1}\left\{\Big(\int_0^\theta |g|^p + \int_\theta^1 |h|^p\Big)^{\frac{1}{p}} : g(\theta)=h(\theta)=0,\ \int_0^\theta |Dg|^q = A,\ \int_\theta^1 |Dh|^q = 1-A\right\} \\
&= \sup_{0\le A\le 1}\Bigg(\sup_{\substack{g(\theta)=0 \\ \int_0^\theta |Dg|^q=A}} \int_0^\theta |g|^p + \sup_{\substack{h(\theta)=0 \\ \int_\theta^1 |Dh|^q=1-A}} \int_\theta^1 |h|^p\Bigg)^{1/p} \\
&= C(p,q) \max_{0\le A\le 1}\Big(A^{\frac{p}{q}}\theta^{p(1+\frac{1}{p}-\frac{1}{q})} + (1-A)^{\frac{p}{q}}(1-\theta)^{p(1+\frac{1}{p}-\frac{1}{q})}\Big)^{1/p}.
\end{aligned} \tag{3.3}$$

Thus, the problem (2.1) has been reduced to the maximisation problem of 1 variable of finding

$$M(p,q,\theta)^p := \max_{0\le A\le 1} f(A), \tag{3.4}$$

where

$$f := f_{p,q,\theta} : A \mapsto A^{\frac{p}{q}}\theta^{p(1+\frac{1}{p}-\frac{1}{q})} + (1-A)^{\frac{p}{q}}(1-\theta)^{p(1+\frac{1}{p}-\frac{1}{q})}. \tag{3.5}$$

This maximum is found in the next section.

Application to Hardy–type inequalities

The careful reader will notice that the argument just outlined also applies to a variety of similar situations – some of interest.

One such example is when $\|\cdot\|_p$, $\|\cdot\|_q$ are replaced by weighted L_p, L_q norms $\|\cdot\|_p^*$, $\|\cdot\|_q^*$ respectively. The corresponding inequalities

$$\|f\|_p^* \le C\,\|Df\|_q^*, \qquad \forall f\in W_q^1, \tag{3.6}$$

where $f(a)=0$, analogous to (2.4), are called *Hardy–type* inequalities. The original Hardy's inequality is the case $p=q>1$, where

$$\|f\|_p^* := \Big(\int_0^\infty \Big|\frac{f(x)}{x}\Big|^p\,dx\Big)^{1/p}, \qquad \|f\|_q^* := \|f\|_{L_p[0,\infty)}, \qquad 1<p<\infty,$$

with the condition $f(0) = 0$. Here the best constant is $C = p/(p-1)$. Often Hardy's inequality is stated with f in the form

$$f(x) = \int_0^x g(t)\,dt.$$

There is considerable interest in Hardy–type inequalities, see, e.g., the monograph of Opic and Kufner [OK90]. The author has made no attempt to translate the various conditions for the existence of an inequality of the form (3.6) and estimates for the best constant to when the condition $f(a) = 0$ is replaced by $f(\theta) = 0$ with θ some point inside the interval of interest (which has left endpoint a).

Another situation of interest where the argument applies is higher order Wirtinger inequalities

$$\|f\|_p \le C\,(b-a)^{n+\frac{1}{p}-\frac{1}{q}}\|D^n f\|_q, \qquad \forall f \in W_q^n,$$

where f satisfies boundary conditions at a single point θ, $a < \theta < b$, to which could be added boundardy conditions at the endpoints (same conditions at both endpoints). A particular case of note is when f vanishes to order n at θ. For $\theta = a$, this extremal problem has recently been investigated by Buslaev [B95].

4. The best constant

In this section, the solution of (2.1) is completed by finding the points A^* where the maximum (3.4) is attained, and by treating the cases $p = \infty$, $q = \infty$ using 'continuity' arguments. This leads to the main result, which is the following.

Theorem 4.1. *Let $0 < p \le \infty$, $1 \le q \le \infty$, and $a \le \theta \le b$. Then, for all $f \in W_q^1[a,b]$ satisfying*

$$f(\theta) = 0,$$

there is the sharp inequality

$$\|f\|_p \le W\Big(\frac{1}{p} - \frac{1}{q}, \frac{\theta - a}{b-a}\Big) C(p,q)\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}}\|Df\|_q, \tag{4.2}$$

where $1/2 \le W \le 1$ is defined by

$$W(t,\theta) := \begin{cases} (1/2 + |1/2-\theta|)^{1+t}, & -1 \le t \le 0; \\ (\theta^{1+\frac{1}{t}} + (1-\theta)^{1+\frac{1}{t}})^t, & 0 < t < \infty, \end{cases} \tag{4.3}$$

and $0 < C \le 1$ is defined by (2.5).

The inequality (4.2) will be referred to as **Schmidt's inequality** (see Section 5). Inequalities of this type are used in the spectral analysis of certain ordinary differential equations (see, e.g., Brown, Hinton and Schwabik [BHS96]).

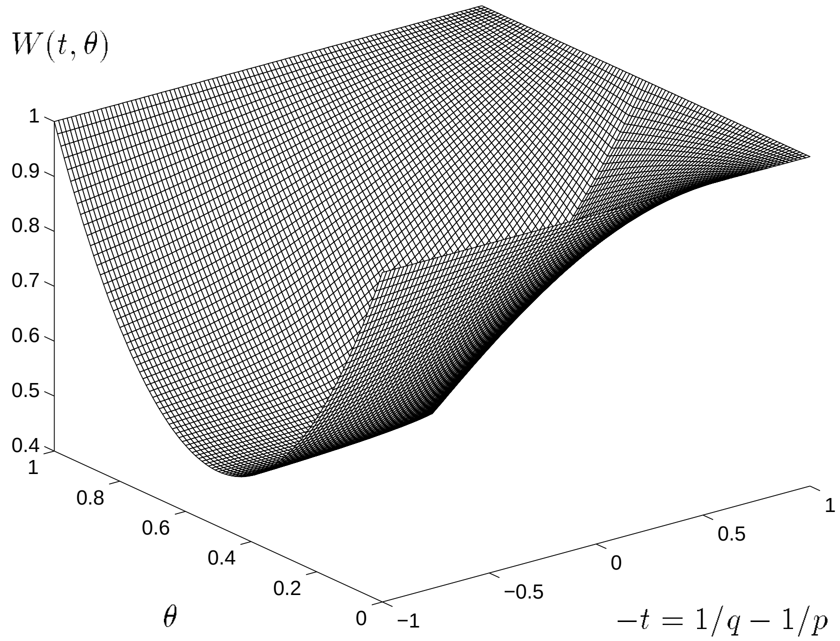

Fig. 4.1. Graph of $(t,\theta) \mapsto W(t,\theta)$ for $-1 \le t \le 1$. The range of $-t$ corresponds to the values $1 \le p, q \le \infty$.

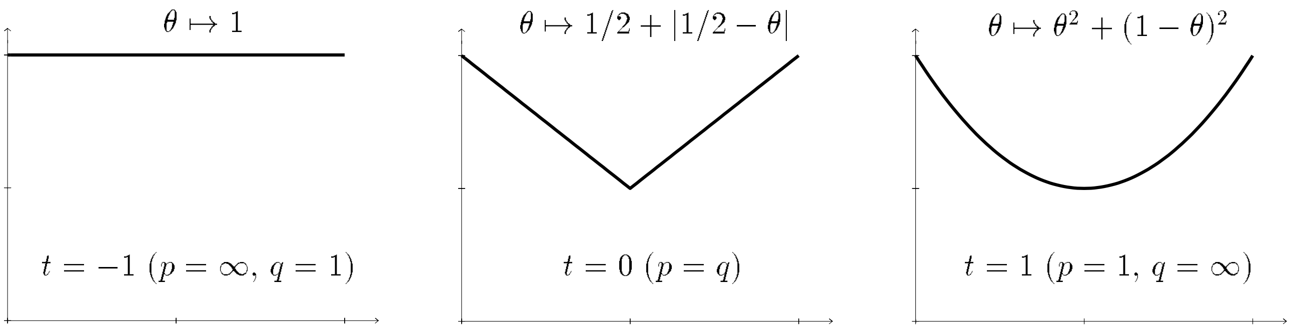

Fig. 4.2. The graph of $\theta \mapsto W(t,\theta)$ for the values $t = -1, 0, 1$.

To prove Theorem 4.1, we require the maximum over $0 \le A \le 1$ of

$$f := f_{p,q,\theta} : A \mapsto A^{\frac{p}{q}} \theta^{p(1+\frac{1}{p}-\frac{1}{q})} + (1-A)^{\frac{p}{q}} (1-\theta)^{p(1+\frac{1}{p}-\frac{1}{q})}, \tag{4.4}$$

where $0 < \theta < 1$. Since the second derivative of f is

$$D^2 f(A) = \frac{p}{q}\left(\frac{p}{q} - 1\right) \left\{ A^{\frac{p}{q}-2} \theta^{p(1+\frac{1}{p}-\frac{1}{q})} + (1-A)^{\frac{p}{q}-2} (1-\theta)^{p(1+\frac{1}{p}-\frac{1}{q})} \right\}, \tag{4.5}$$

where the term inside the $\{\ \}$ is positive, f is either convex, linear, or concave, depending on the values of p, q.

To describe the extremal functions we need the following. Recall $E_{p,q}$ is defined by (2.6). For $0 \le \theta < 1$, let

$$E^{\theta+}_{p,q}(x) := \begin{cases} 0, & 0 \le x \le \theta; \\ E_{p,q}\left(\frac{x-\theta}{1-\theta}\right), & \theta \le x \le 1, \end{cases} \tag{4.6}$$

which is continuous and supported on $[\theta, 1]$, and, for $0 < \theta \leq 1$, let

$$E_{p,q}^{\theta-}(x) := \begin{cases} E_{p,q}\left(\frac{\theta-x}{\theta}\right), & 0 \leq x \leq \theta; \\ 0, & \theta \leq x \leq 1, \end{cases} \tag{4.7}$$

which is supported on $[0, \theta]$.

We now complete the proof of Theorem 4.1, and give the extremals, by finding the maximum of (4.4) for the various values of p, q.

The case $1 \leq q \leq p < \infty$

By (4.5), f is convex when $p > q$, and linear when $p = q$, and so it attains its maximum at an endpoint given by

$$A^* = \begin{cases} 0, & 0 \leq \theta < 1/2 \\ 0, 1, & \theta = 1/2,\ p > q \\ [0, 1], & \theta = 1/2,\ p = q \\ 1, & 1/2 < \theta \leq 1. \end{cases}$$

Thus, since

$$\max\{\theta, 1-\theta\} = 1/2 + |1/2 - \theta|,$$

we obtain

$$M(p, q, \theta) = (1/2 + |1/2 - \theta|)^{1+\frac{1}{p}-\frac{1}{q}}, \qquad 1 \leq q \leq p < \infty. \tag{4.8}$$

For $0 \leq \theta < 1/2$, $q \neq 1$, the corresponding extremal function is $E_{p,q}^{\theta+}$, and this is the unique extremal upto a multiplication by a constant. Similarly, for $1/2 < \theta \leq 1$, the extremal function is $E_{p,q}^{\theta-}$. For $\theta = 1/2$, $1 < q < p$, there are two extremal functions $E_{p,q}^{1/2+}$ and $E_{p,q}^{1/2-}$ (corresponding to $A^* = 0, 1$ respectively). For $\theta = 1/2$, $p = q > 1$, any (nontrivial) linear combination of $E_{p,q}^{1/2+}$ and $E_{p,q}^{1/2-}$ is an extremal.

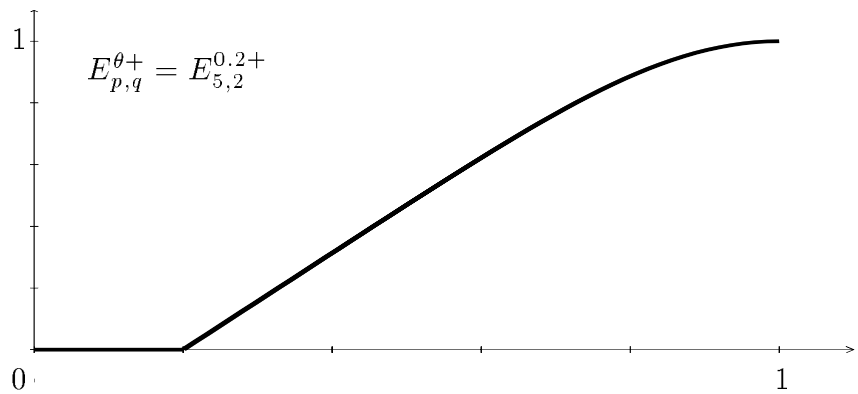

Fig. 4.3. Behaviour of the extremal function $E_{p,q}^{\theta+}$ for $0 < \theta < 1/2$ and $p > q > 1$.

The case $0 < p < q < \infty$, $1 \le q < \infty$

By (4.5), if $p < q$, then f is concave, and so we need to compute any local maxima of f. Since the first derivative of f is

$$Df(A) = \frac{p}{q} A^{\frac{p}{q}-1} \theta^{p(1+\frac{1}{p}-\frac{1}{q})} - \frac{p}{q}(1-A)^{\frac{p}{q}-1}(1-\theta)^{p(1+\frac{1}{p}-\frac{1}{q})}, \tag{4.9}$$

f has a stationary point when

$$A^{-\alpha}\theta^{\beta} = (1-A)^{-\alpha}(1-\theta)^{\beta},$$

where

$$\alpha := 1 - p/q > 0, \qquad \beta := p(1 + 1/p - 1/q) > 0.$$

This has one solution (for $0 < \theta < 1$)

$$A^* = \frac{(1-\theta)^{-\beta/\alpha}}{\theta^{-\beta/\alpha} + (1-\theta)^{-\beta/\alpha}} \in (0,1).$$

Thus, f has a maximum at A^* given by

$$f(A^*) = \frac{(1-\theta)^{-\beta(1-\alpha)/\alpha}\theta^{\beta} + \theta^{-\beta(1-\alpha)/\alpha}(1-\theta)^{\beta}}{(\theta^{-\beta/\alpha} + (1-\theta)^{-\beta/\alpha})^{1-\alpha}} = \left(\theta^{\beta/\alpha} + (1-\theta)^{\beta/\alpha}\right)^{\alpha},$$

and we obtain

$$M(p,q,\theta) = \left(\theta^{1+1/(\frac{1}{p}-\frac{1}{q})} + (1-\theta)^{1+1/(\frac{1}{p}-\frac{1}{q})}\right)^{\frac{1}{p}-\frac{1}{q}}, \qquad p < q. \tag{4.10}$$

Let $f = g + h$ given by

$$g := G\, E_{p,q}^{\theta-},\ G > 0, \qquad h := H\, E_{p,q}^{\theta+},\ H > 0,$$

be the extremal function which attains the supremum in (3.3). Then

$$\int_0^{\theta} |Dg|^q = G^q \theta^{1-q} \|E_{p,q}\|_q^q = A^*, \qquad \int_{\theta}^{1} |Dh|^q = H^q (1-\theta)^{1-q} \|E_{p,q}\|_q^q = 1 - A^*,$$

so that

$$\frac{G}{H} = \left(\frac{A^*}{1-A^*}\frac{(1-\theta)^{1-q}}{\theta^{1-q}}\right)^{1/q} = \left(\frac{\theta}{1-\theta}\right)^{1+\frac{p}{q-p}}.$$

Thus, the extremal functions are (scalar multiples of)

$$\theta^{1+\frac{p}{q-p}} E_{p,q}^{\theta-} \pm (1-\theta)^{1+\frac{p}{q-p}} E_{p,q}^{\theta+}. \tag{4.11}$$

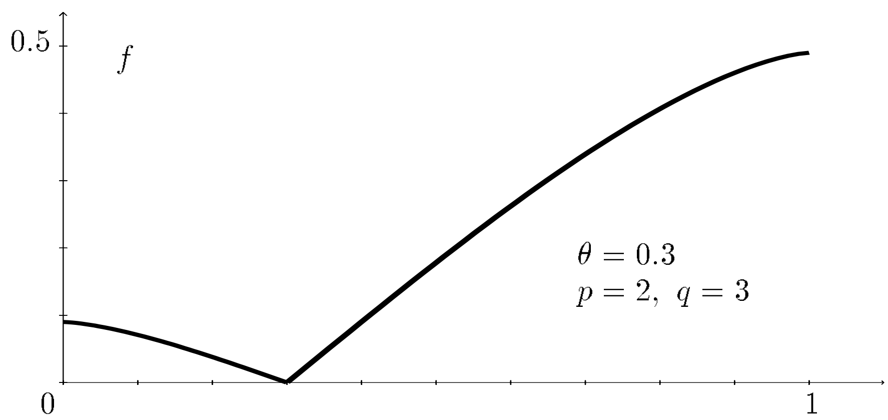

Fig. 4.4. Behaviour of the extremal function $f := \theta^{1+\frac{p}{q-p}} E_{p,q}^{\theta-} + (1-\theta)^{1+\frac{p}{q-p}} E_{p,q}^{\theta+}$ for $0 < \theta < 1/2$ and $p < q$.

The cases $p = \infty$, $q = \infty$

From (4.8), (4.10) we see that for $0 < p < \infty$, $1 \le q < \infty$ the maximum $M(p, q, \theta)$ depends only on $1/p - 1/q$ and θ, i.e.,

$$M(p, q, \theta) = W(\frac{1}{p} - \frac{1}{q}, \theta),$$

where $W(t, \theta)$ is defined by (4.3). It is easily seen from (2.6) that the extremal functions do not depend only on $1/p - 1/q$ and θ. Similary, $C(p, q)$ is not a function of $1/p - 1/q$.

Since the solution of (2.1) is a bounded and continuous function of $0 < p < \infty$, $1 \le q < \infty$, it would be expected the remaining values ($p = \infty$ and $q = \infty$) could be obtained by taking the appropriate limits. The argument is as follows.

The case $p = \infty$, $1 \le q < \infty$. For $f \in W_q^1$ with $f(\theta) = 0$, there is the sharp inequality

$$\|f\|_{p^*} \le W(\frac{1}{p^*} - \frac{1}{q}, \theta) C(p^*, q) \, \|Df\|_q, \tag{4.12}$$

where $0 < p^* < \infty$ is fixed. Since $f \in W_q^1 \subset L_{p^*}$, for all $0 < p^* \le \infty$, the limit of (4.12) as $p^* \to \infty$ can be taken to obtain

$$\|f\|_\infty \le W(\frac{1}{\infty} - \frac{1}{q}, \theta) C(\infty, q) \, \|Df\|_q,$$

which is sharp, since for each $p^* < \infty$ an f with $\|Df\|_q = 1$ which gives as close to equality in (4.12) as desired can be chosen.

The case $1 \le p \le \infty$, $q = \infty$. For $f \in W_\infty^1$ with $f(\theta) = 0$, there is the sharp inequality

$$\|f\|_p \le W(\frac{1}{p} - \frac{1}{q^*}, \theta) C(p, q^*) \, \|Df\|_{q^*}, \tag{4.13}$$

where $1 \le q^* < \infty$. As before, the limit as $q^* \to \infty$ can be taken to obtain the sharp inequality

$$\|f\|_p \le W(\frac{1}{p} - \frac{1}{\infty}, \theta) C(p,\infty) \|Df\|_\infty.$$

This completes the proof of Theorem 4.1.

Remarks

With a mind to identifying phenomenon that might also hold for other Wirtinger inequalities of the form (1.1), it is of interest to know when the extremals for (2.1) are (polynomial) splines. It can be shown that this occurs only when

$$\begin{aligned} &p = 1,\ q = 2 \qquad \text{(quadratic splines)} \\ &p = \infty, \quad \text{or} \quad q = \infty \qquad \text{(linear splines)}. \end{aligned}$$

The only case where the best constant in (4.2) has been investigated for $\theta \ne a, b$ is in Tikhomirov [T76:§2.5.2], where

$$\theta = \frac{a+b}{2},$$

the midpoint of the interval, and $1 < p < \infty$, $1 \le q < \infty$. Tikhomirov used this best constant in his calculation of the n-width $s_n(B_q^1, L_p)$, which is given in Theorem 5.9.

For simplicity, suppose that $[a,b] = [0,1]$ and $\theta = 1/2$. For this choice of θ, we have from (4.3), that

$$W(\frac{1}{p} - \frac{1}{q}, \frac{1}{2}) = \begin{cases} \frac{1}{2}, & p < q \\ \left(\frac{1}{2}\right)^{1+\frac{1}{p}-\frac{1}{q}}, & p \ge q, \end{cases} \tag{4.14}$$

for $0 < p \le \infty$, $1 \le q \le \infty$. For $1 < p \le q < \infty$, this agrees with the result of Tikhomirov [T76:p127]. But, for $1 \le q < p < \infty$, Tikhomirov claims the best constant (4.2) is

$$\frac{1}{2} C(p,q), \tag{4.15}$$

rather than the larger constant

$$\left(\frac{1}{2}\right)^{1+\frac{1}{p}-\frac{1}{q}} C(p,q),$$

given by (4.14). Tikhomirov's constant (4.15) would be correct, if it could be assumed that the extremal function was symmetric about $\theta = (a+b)/2$ (as it is in the case $p \le q$). However, as we have seen, for $1 < q < p \le \infty$ the extremal function is only supported on half of the interval $[0,1]$. Indeed, we compute that the best constant in (4.2) must be at least as large as

$$\frac{\|E_{p,q}^{1/2+}\|_p}{\|DE_{p,q}^{1/2+}\|_q} = \frac{(1/2)^{1/p}\|E_{p,q}\|_p}{(1/2)^{1/q-1}\|DE_{p,q}\|_q} = \left(\frac{1}{2}\right)^{1+\frac{1}{p}-\frac{1}{q}} C(p,q), \qquad 1 < q < p \le \infty,$$

where $1/2 < (1/2)^{1+\frac{1}{p}-\frac{1}{q}} < 1$, and this is the best constant.

5. Schmidt's inequality

There are several inequalities of the form

$$\|f\|_p \le C\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}}\|Df\|_q$$

where f belongs to some class of functions, with the best constant and extremals related to $C(p,q)$ and $E_{p,q}$ respectively, which are closely related to the following result of Schmidt.

Schmidt's inequality([Sc40:(4),p302]) 5.1. *Let $0 < p \le \infty$, $1 \le q \le \infty$. Then, for all $f \in W_q^1[a,b]$ satisfying*

$$f(a) = f(b), \qquad \max_{t\in[a,b]} f(t) + \min_{t\in[a,b]} f(t) = 0, \tag{5.2}$$

there is the sharp inequality

$$\|f\|_p \le \frac{1}{4}C(p,q)\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}}\|Df\|_q, \tag{5.3}$$

where $C(p,q)$ is defined by (2.5).

Schmidt's proof of 5.1 does not use the calculus of variations, but instead uses Hölder's inequality in a very clever way. Schmidt indicates that this argument can be modified to obtain other inequalities. The first of these is the following.

Theorem([Sc40:(13),p304]) 5.4. *Let $0 < p \le \infty$, $1 \le q \le \infty$. Then, for all $f \in W_q^1[a,b]$ satisfying*

$$f(a) = f(b) = 0, \tag{5.5}$$

there is the sharp inequality

$$\|f\|_p \le \frac{1}{2}C(p,q)\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}}\|Df\|_q, \tag{5.6}$$

where $C(p,q)$ is defined by (2.5).

This result is stated for $1 \le p,q \le \infty$ with the value of $C(p,q)$ given incorrectly (due to a slight error in its proof) in Fink [F74:p408] (see the constant $D(1,1,p,q)$), and is stated correctly for $1 \le p < \infty$, $1 < q < \infty$ (together with the extremal functions) but without proof in Talenti [Ta76:p357] (there it is mentioned as a 1-dimensional analogue of a Sobolev inequality for functions in $W_q^1(\mathbb{R}^m)$). Neither author makes reference to Schmidt [Sc40].

In Schmidt's statement of Theorem 5.4, the condition (5.5) is given as

$$f(a) = f(b), \qquad f \text{ has (at least) one zero on } [a,b].$$

It is not difficult to see that a variation of the 'elementary argument' where f is split by (3.1) into functions g and h with

$$g(a) = g(\theta) = h(\theta) = h(b) = 0$$

can be used, together with Theorem 5.4, to compute the best constant in the inequality

$$\|f\|_p \leq \frac{1}{2} C(p,q)\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}} \|Df\|_q, \tag{5.7}$$

where $f \in W_q^1$ satisfies

$$f(a) = f(\theta) = f(b).$$

More generally, by using similar variations of the 'elementary argument' and induction, it is (in principle) possible to compute the best constant in (5.7) where $f \in W_q^1$ satisfies

$$f(\theta_1) = f(\theta_2) = \cdots = f(\theta_n) = 0,$$

for some choice

$$a \leq \theta_1 < \theta_2 < \cdots < \theta_n \leq b,$$

together with the extremal functions, which exist except when $q = 1$, $p \neq \infty$, and can be constructed from $E_{p,q}$.

A third result mentioned by Schmidt [Sc40:(20),p306] is the inequality that: for $f \in W_q^1$ with $f(a) = 0$,

$$\|f\|_p \leq C(p,q)\,(b-a)^{1+\frac{1}{p}-\frac{1}{q}} \|Df\|_q, \tag{5.8}$$

which was stated earlier as Lemma 2.2.

The special case of this result when

$$p = q = 2k \quad \text{an even integer}$$

was given by Hardy and Littlewood [HL32:Th.5] (see also [HLP34:256,p182]) using what they describe as a proof of 'type C' (depends essentially on the calculus of variations). They refer to the extremal functions $E_{2k,2k}$ (for $[a,b] = [0,1]$) as *hyperelliptic curves* and compute

$$C(2k,2k) = \left(\frac{1}{2k-1}\right)^{1/2k} \left(\frac{2k}{\pi} \sin\frac{\pi}{2k}\right).$$

This example motivated Schmidt to give his proof of 'type A' (strictly elementary). To quote Hardy and Littlewood [HL32] (and [HLP34]): (their) "proof is of 'type C' and (in view of the difficulty of calculating the slope-function) is might be difficult to construct a much more elementary proof" (either of 'type B' or simpler still of 'type A'). There is further discussion of the results of Schmidt in Levin and Stechkin's supplement [LS48] to the Russian edition of Hardy, Littlewood and Pólya's book on inequalities [HLP48].

For $1 \leq p,q \leq \infty$ Lemma 2.2 is given in [F74:p407]. There $C(p,q)$ is denoted by $C(1,1,p,q)$ and it is given incorrectly (due to a slight error in the proof). Fink was unaware of the earlier result of Schmidt.

The statement of Lemma 2.2 given by Schmidt is infact slightly stronger. It asserts the sharp inequality in (5.8) where the condition (2.3) is replaced by

$$f \text{ has (at least) one zero on } [a,b]$$

(and the discussion of the extremals shows the sharpness occurs only when the zero is at a or b). A quantative form of this result is given by our Theorem 4.1.

The constant $C(p,q)$ also naturally occurs in the computation of the n-widths of the set

$$B_q^1 := \{f \in W_q^1 : \|Df\|_q \leq 1\}$$

in L_p, since for $1 \leq p \leq q \leq \infty$ an optimal linear operator of rank n is given by Lagrange interpolation by piecewise constants (cf (1.2)).

Theorem ([T70]) 5.9. *Let $[a,b]=[0,1]$. Then, for $1 \leq p \leq q \leq \infty$*

$$s_n(B_q^1, L_p) = \frac{1}{2} C(p,q) \frac{1}{n}, \qquad n = 1,2,3,\ldots, \tag{5.10}$$

where $s_n(B_q^1, L_p)$ denotes the common value of the Kolmogorov, linear and Gel'fand n-widths. The space F_n consisting of step functions with break points

$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$$

is an optimal n-dimensional subspace, and the operator L_n of Lagrange interpolation from F_n at the points

$$\frac{1}{2n}, \frac{3}{2n}, \frac{5}{2n} \cdots \frac{2n-1}{2n} \qquad \textit{(the midpoints of each step)}$$

is an optimal linear operator of rank n.

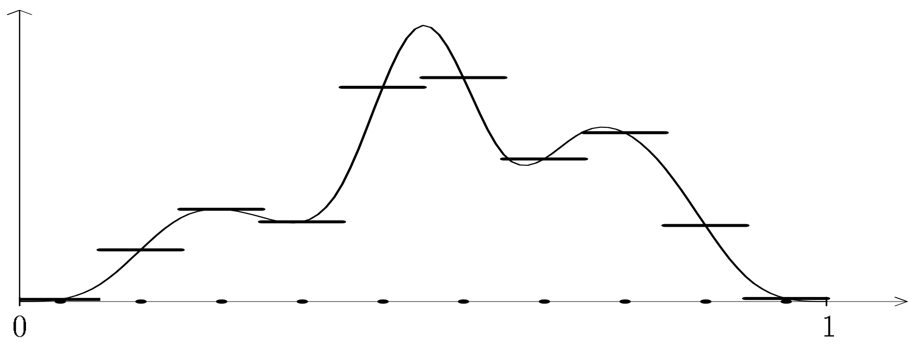

Fig. 5.4. Example of the Lagrange interpolant $L_n f$ to f from the space of step functions F_n (n=10).

Acknowledgement

I would like to thank Evsei Dyn'kin for many helpful conversations related to this paper. This work was supported by the *Israel Council for Higher Education* through its Postdoctoral Fellowship Scheme.

References

[BHS96] R. C. Brown, D. B. Hinton and S. Schwabik, Applications of a one-dimensional Sobolev inequality to eigenvalue problems, *Differential and integral Equations* **9**(1996), 481–498.

[B95] A. P. Buslaev, Some properties of the spectrum of nonlinear equations of Sturm-Liouville type, *Russian Acad. Sci. Sb. Math.* **80(1)**(1995), 1–14.

[F74] M. A. Fink, Conjugate inequalities for functions and their derivative, *SIAM J. Math. Anal.* **5(3)**(1974), 399–411.

[FMP91] A. M. Fink, D. S. Mitrinović and J. E. Pečarić, Inequalities involving functions and their integrals and derivatives, Kluwer Academic Publishers, Dordrecht, 1991.

[HL32] G. H. Hardy and J. E. Littlewood, Some integral inequalities connected with the calculus of variations, *Quart. J. Math. Oxford Ser.* **3**(1932), 241–252. val

[HLP34] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge University Press, Cambridge, 1934.

[HLP48] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities [Russian edition], Foriegn Literature, Moscow, 1948.

[LS48] V. E. Levin and S. B. Stechkin, Supplement to the Russian edition of Hardy, Littlewood and Polya, In: Inequalities (Russian edition) (HLP, Eds), pp. 361–441, Foreign Literature, Moscow, 1948.

[OK90] B. Opic and A. Kufner, Hardy-type inequalities, Longman, Harlow, 1990.

[Sc40] E. Schmidt, Über die Ungleichung, welche die Integrale über eine Potenz einer Funktion und über eine andere Potenz ihrer Ableitung verbindet, *Math. Ann.* **117**(1940), 301–326.

[S95] A. Yu. Shadrin, Error bounds for Lagrange interpolation, *J. Approx. Theory* **80**(1995), 25–49.

[Ta76] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.* **110**(1976), 353–372.

[T70] V. M. Tikhomirov, Some problems in approximation theory, dissertation, Moscow State University, 1970. (In Russian)

[T76] V. M. Tikhomirov, Some questions of Approximation Theory, Moscow State University, Moscow, 1976. (In Russian)

[W96] S. Waldron, L_p-error bounds for Hermite interpolation and the associated Wirtinger inequalities, *Constr. Approx.* **xx**(1996), xx-xx.