Extremising the L_p-norm of a monic polynomial with roots in a given interval and Hermite interpolation

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Abstract:

Let Θ be a multiset of n points in $[a,b]$, and

$$\omega_\Theta := \prod_{\theta\in\Theta}(\cdot - \theta).$$

In this paper we investigate the extrema of $\Theta \mapsto \|\omega_\Theta\|_p$. Consequences of the results we obtain include: L_p-bounds for Hermite interpolation, error estimates for Gauss quadrature formulæ with multiple nodes, and certain quantitative statements about good and best approximation by polynomials of fixed degree.

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1. Introduction

Let Θ be a multiset of n points in $[a,b]$, and

$$\omega_\Theta := \prod_{\theta\in\Theta}(\cdot-\theta)\in\Pi_n.$$

In this paper we discuss the size of $\|\omega_\Theta\|_p$ as a function of Θ. This constant $\|\omega_\Theta\|_p$ arises naturally in error bounds for Hermite interpolation. For example, if $H_\Theta f\in\Pi_{<n}$ is the Hermite interpolant to f at the points Θ (counting multiplicities), then

$$\|f-H_\Theta f\|_p \le \frac{\|\omega_\Theta\|_p}{n!}\|D^n f\|_\infty,\quad \forall f\in W_\infty^n, \tag{1.1}$$

with equality iff $f\in\Pi_n$.

In Section 2, we show that if some of the points in Θ are prescribed, then $\|\omega_\Theta\|_p$ is maximised by an appropriate choice of the remaining points from $\{a,b\}$. As an application, we provide L_p-error bounds for Hermite interpolation, in cases where some of the points in Θ are known to be from $\{a,b\}$.

In Section 3, we show that $\|\omega_\Theta\|_p$ is minimised for a certain choice of Θ, consisting of n distinct points in (a,b). These points are precisely the roots of the error in the unique best L_p-approximation from $\Pi_{<n}$ to any polynomial of (exact) degree n. This result is closely related to Gauss quadrature formulæ with multiple nodes (via s-orthogonal polynomials), for which we are able to give error bounds. Other applications in this section include error bounds for best L_p-approximation by polynomials of fixed degree.

2. Maximising $\|\omega_\Theta\|_p$

Throughout, Θ will be used for a multiset of n points from $[a,b]$. Our functions will be defined on the closed interval $[a,b]$, $b-a>0$. Thus $\|\cdot\|_p := \|\cdot\|_{L_p[a,b]}$, and $W_p^n := W_p^n[a,b]$ the **Sobolev** space of functions f with $D^{n-1}f$ absolutely continuous on $[a,b]$ and $D^n f\in L_p := L_p[a,b]$. The space of polynomials of degree $\le n$ will be denoted by Π_n.

(2.1) Theorem. *Let Θ' be a fixed multiset of $\le n$ points from $[a,b]$. The maximum of*

$$\{\|\omega_\Theta\|_p : \Theta\supset\Theta'\}$$

is attained when $\Theta \setminus \Theta'$ *is in* $\{a, b\}$.

Proof. Let $\mathcal{C}$ be the convex hull of the compact set

$$\mathcal{W} := \{\omega_\Theta : \Theta \supset \Theta'\} \subset \Pi_n.$$

Since $\mathcal{C} \to \mathbb{R} : f \mapsto \|f\|_p$ is a continuous convex function, it attains its maximum at an extreme point of $\mathcal{C}$. Since each point in $\mathcal{C} \setminus \mathcal{W}$ can be written as a (nontrivial) convex combination of two points in $\mathcal{C}$, the extreme points of $\mathcal{C}$ are contained in $\mathcal{W}$.

Suppose $\omega_\Theta \in \mathcal{W}$ is an extreme point of $\mathcal{C}$, with $\{\xi, \Theta'\} \subset \Theta$, for some $\xi \in (a, b)$. Then for small ε

$$\omega_\Theta = \frac{1}{2}(\cdot - (\xi - \varepsilon))\,\omega_{\Theta\setminus\xi} + \frac{1}{2}(\cdot - (\xi + \varepsilon))\,\omega_{\Theta\setminus\xi},$$

a convex combination of points in $\mathcal{W}$, contradicting the fact ω_Θ is an extreme point of $\mathcal{C}$. Thus the extreme points of $\mathcal{C}$ are given by ω_Θ, where Θ consists of Θ' together with points from $\{a, b\}$. □

We now use this result to find the maximum of $\Theta \mapsto \|\omega_\Theta\|_p$ over $\mathcal{A}_n(i, j)$, which is, by definition, the set of those Θ containing one endpoint at least i times and the other at least j times, where $i + j \le n$. Notice that $\mathcal{A}_n(i, j)$ is symmetric in i, j, that $\mathcal{A}_n(0, 0)$ consists of all Θ, and that $\mathcal{A}_n(m, n - m)$ has at most two elements.

Let B be the **beta function**

$$B(x, y) := \int_0^1 t^{x-1}(1 - t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \quad \forall x, y > 0.$$

Recall that B is symmetric, and satisfies: $0 < B(x, y) \le \min\{1, 1/\max\{x, y\}\}, \quad \forall x, y > 0.$

(2.2) Corollary. *Let* $m := \min\{i, j\}$, *and* $0^0 := 1$. *Then*

$$\max_{\Theta\in\mathcal{A}_n(i,j)} \|\omega_\Theta\|_p = (b - a)^{n+\frac{1}{p}} \begin{cases} B(pm + 1, p(n - m) + 1)^{\frac{1}{p}}, & 1 \le p < \infty \\ m^m(n - m)^{n-m}/n^n, & p = \infty, \end{cases}$$

with the maximum achieved iff $\Theta \in \mathcal{A}_n(m, n - m)$.

Proof. By Theorem (2.1), the maximum occurs when all the points in Θ are from $\{a, b\}$. For $\Theta \in \mathcal{A}_n(k, n - k)$, we compute

$$\|\omega_\Theta\|_p = (b - a)^{n+\frac{1}{p}} \begin{cases} B(pk + 1, p(n - k) + 1)^{\frac{1}{p}}, & 1 \le p < \infty \\ k^k(n - k)^{n-k}/n^n, & p = \infty, \end{cases}$$

and then observe that the maximum of $\|\omega_\Theta\|_p$ over $m \le k \le n - \max\{i, j\}$ occurs when $k = m$. □

This improves upon the weaker result of Agarwal [Ag91], that

$$\max_{\Theta\in\mathcal{A}_n(i,j)} \|\omega_\Theta\|_p \le (b - a)^{n+\frac{1}{p}}(2B_{1/2}(pm + 1, p(n - m) + 1))^{\frac{1}{p}}, \quad 1 \le p < \infty. \tag{2.3}$$

Here $B_{1/2}$ is the **incomplete beta function**

$$B_{1/2}(x,y) := \int_0^{1/2} t^{x-1}(1-t)^{y-1}\,dt, \quad \forall x,y>0.$$

We observe that $B_{1/2}$ is not symmetric, and satisfies $B(x,y) \le 2B_{1/2}(x,y)$, $\forall\, 1 \le x \le y$, with strict inequality unless $x=y$. Thus Corollary (2.2) gives better bounds than (2.3) whenever $m \ne n-m$, and the same bounds otherwise.

L_p-Error bounds for Hermite interpolation

Let $1 \le p,q \le \infty$, and $H_\Theta f \in \Pi_{<n}$ be the Hermite interpolant to f at Θ (counting multiplicities). Recently, see Waldron [Wa94], the author has shown that:

$$\|f - H_\Theta f\|_p \le \mathrm{const}_{n,p,q,\Theta}(b-a)^{n+\frac{1}{p}-\frac{1}{q}}\|D^n f\|_q, \quad \forall f \in W_q^n, \tag{2.4}$$

where

$$\mathrm{const}_{n,p,q,\Theta} := \frac{n^{\frac{1}{q}}}{n!}\left\| x \mapsto \frac{\omega_\Theta(x)}{(\mathrm{diam}\{x,\Theta\})^{1/q}}\right\|_p (b-a)^{-(n+\frac{1}{p}-\frac{1}{q})}.$$

Here **diam** denotes the diameter of a (multi)set of points. Using Corollary (2.2), we may estimate the constants $\mathrm{const}_{n,p,q,\Theta}$.

(2.5) Hermite error bounds. *Let $\Theta \in \mathcal{A}_n(i,j)$, with $m := \min\{i,j\} > 0$. Then*

$$\mathrm{const}_{n,p,q,\Theta} \le \frac{n^{\frac{1}{q}}}{n!}\begin{cases} B(pm+1, p(n-m)+1)^{\frac{1}{p}}, & 1 \le p < \infty \\ m^m(n-m)^{n-m}/n^n, & p=\infty. \end{cases}$$

Proof. Since $m>0$, $\mathrm{diam}\{x,\Theta\} = b-a$, and we obtain

$$\mathrm{const}_{n,p,q,\Theta} = \frac{n^{\frac{1}{q}}}{n!}\|\omega_\Theta\|_p (b-a)^{-(n+\frac{1}{p})}.$$

To this, apply Corollary (2.2). □

This improves upon the bounds in [Ag91], which involve $B_{1/2}$. In the case $m=0$, the above argument can be modified, by observing that

$$\left\| x \mapsto \frac{\omega_\Theta(x)}{(\mathrm{diam}\{x,\Theta\})^{1/q}}\right\|_p \le \left(\|\omega_\Theta\|_{p(1-\frac{1}{nq})}\right)^{1-\frac{1}{nq}}. \tag{2.6}$$

For a full discussion, including the cases of equality in (2.4), and mention of some related inequalities of Brink [Br72], see [Wa94].

Application to the solution of ordinary differential equations

The Hermite error bounds (2.5) can be applied to the analysis of the boundary value problem: $D^n f = g$, with Hermite multipoint conditions given by $H_\Theta f = 0$. See, e.g., Agarwal and Wong [AW93].

3. Minimising $\|\omega_\Theta\|_p$

To show that $\Theta \mapsto \|\omega_\Theta\|_p$ has a unique minimum, we use the following well-known result, see, e.g., [DL93:Ch.3,§5,§10].

(3.1) Theorem. *If $P \subset C[a,b]$ is an n-dimensional Haar space, then g^*, the unique best L_p-approximation to $f \in C[a,b]$ from P, interpolates f at n distinct points in (a,b).*

For $1 \le p < \infty$, by the characterisation theorem for best L_p-approximation (see, e.g., [DL93:p83]) g^* is uniquely determined by

$$\int_a^b |f - g^*|^{p-1} \operatorname{sign}(f - g^*)\, g = 0, \quad \forall g \in P, \tag{3.2}$$

where **sign** denotes the signum function.

For a more detailed analysis, dealing with the interlacing of the zeros of errors in best L_p-approximations, see Pinkus and Ziegler [PZ76].

Taking $P = \Pi_{<n}$, and $f = (\cdot)^n$, we obtain:

(3.3) Corollary. *There is a unique Θ which minimises $\|\omega_\Theta\|_p$. This Θ consists of n distinct points in (a,b), which are the roots of $M_{n,p} \in \Pi_n$, which is, by definition, the error in the unique best L_p-approximation to $(\cdot)^n$ from $\Pi_{<n}$. We have*

$$\frac{1}{4^n}(b-a)^{n+\frac{1}{p}} \le \min_\Theta \|\omega_\Theta\|_p = \|M_{n,p}\|_p \le \frac{2}{4^n}(b-a)^{n+\frac{1}{p}},$$

with equality only when $p = 1, \infty$, respectively. In addition

$$\min_\Theta \|\omega_\Theta\|_2 = \|M_{n,2}\|_2 = \frac{(n!)^2}{(2n)!\sqrt{2n+1}}(b-a)^{n+\frac{1}{2}}.$$

Proof. Taking $P = \Pi_{<n}$, and $f = (\cdot)^n$, in Theorem (3.1), we see that $M_{n,p}$, the error in best approximation, is of the form $M_{n,p} = \omega_\Theta$, for a certain Θ consisting of distinct points in (a,b). Thus, this choice of Θ uniquely minimises $\|\omega_\Theta\|_p$ (even if Θ is not restricted to lie within $[a,b]$).

From Hölder's inequality, it follows that

$$p \mapsto C_p := \|M_{n,p}\|_p (b-a)^{-(n+\frac{1}{p})} = \min_\Theta \|\omega_\Theta\|_p (b-a)^{-(n+\frac{1}{p})}$$

is strictly increasing.

For $p = 1$, $M_{n,p}$ is, up to an affine change of variables equal to U_n, the **Chebyshev polynomial of the second kind**, and we calculate

$$C_1 = \|2^{-n} U_n\|_1 \, 2^{-(n+\frac{1}{1})} = \frac{1}{4^n}.$$

Similarly $M_{n,2}$, $M_{n,\infty}$ are P_n, T_n. i.e., the **Legendre**, **Chebyshev polynomials**, respectively, and

$$C_2 = \|\frac{2^n (n!)^2}{(2n)!} P_n\|_2 \, 2^{-(n+\frac{1}{2})} = \frac{(n!)^2}{(2n)!\sqrt{2n+1}},$$

$$C_\infty = \|2^{-(n-1)} T_n\|_\infty \, 2^{-(n+\frac{1}{\infty})} = \frac{2}{4^n}.$$

The facts about U_n, P_n, T_n that we have used above can be found in any standard book on orthogonal polynomials. □

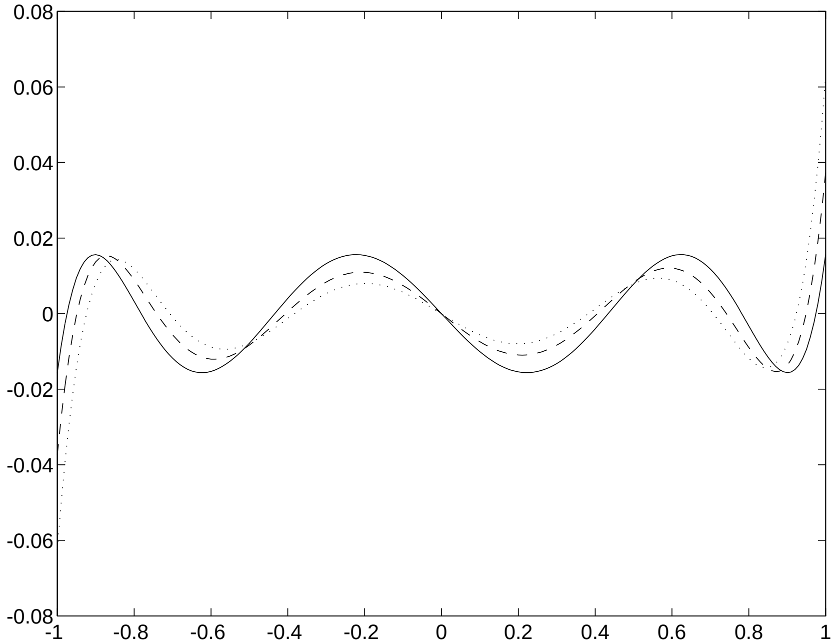

Fig 3.1 Graphs of the polynomials $M_{7,1}$ (dotted), $M_{7,2}$ (dashed), and $M_{7,\infty}$ (line)

Corollary (3.3) is a collection of classical results from the theory of orthogonal polynomials, see, e.g., Szegö [Sz59:p41]. One generalisation of it, of interest to approximation theorists, is Fejér's convex hull theorem, see Davis [Da75:p244].

As mentioned in the proof, when $p = 1, 2, \infty$, the $M_{n,p}$ are well known orthogonal polynomials. For other values of p, no recurrence relations are known for $M_{n,p}$. By (3.2), for $1 \le p < \infty$, $M_{n,p}$ is the unique $m \in \Pi_n$ with leading term $(\cdot)^n$ and

$$\int_a^b |m|^{p-1} \mathrm{sign}(m)\, g = 0, \quad \forall g \in \Pi_{<n}. \tag{3.4}$$

It is possible to view (3.4) as a nonlinear system of equations in the roots of $M_{n,p}$ (with a unique solution), and solve it numerically. For two different iterative schemes, together with sample results, see Burgoyne [Bu67], and Vincenti [Vi86].

Good approximation by polynomials

Combining Corollaries (2.2) and (3.3), we obtain:

$$\frac{1}{2} \frac{4^n}{(np+1)^{1/p}} \le \frac{\max_\Theta \|\omega_\Theta\|_p}{\min_\Theta \|\omega_\Theta\|_p} \le \frac{4^n}{(np+1)^{1/p}},$$

where $(np+1)^{1/p} := 1$, when $p = \infty$. Thus, a good choice of Θ can greatly improve the size of the constant $\|\omega_\Theta\|_p$ occurring in (1.1), over that for a poor choice.

For example, with Θ_{Eq} consisting of points with equal spacing $h := (b-a)/(n-1)$, and Θ_{Ch} the Chebyshev points, Isaacson and Keller [IK66:p267] provide the estimate:

$$\frac{\|\omega_{\Theta_{\mathrm{Eq}}}\|_\infty}{\|\omega_{\Theta_{\mathrm{Ch}}}\|_\infty} > \frac{\sqrt{2}}{n-1}\left(\frac{4}{e}\right)^{n-1},$$

for large n, in support of doing Lagrange interpolation at the Chebyshev points.

Best approximation by polynomials

By Theorem (3.1), the unique best L_p-approximation to $f \in C[a,b]$ from $\Pi_{<n}$ is obtained by Lagrange interpolation at n points in (a,b). Thus, in view of (1.1), we expect some relation between $\min_\Theta \|\omega_\Theta\|_p$, and the error

$$E_{n,p}(f) := \inf_{g\in\Pi_{<n}} \|f-g\|_p$$

in best L_p-approximation. The main result in this direction, which is due to Phillips, is the following.

(3.5) Theorem ([Ph70]). *If $f \in C^n[a,b]$, then $\exists\, \xi \in [a,b]$, such that*

$$E_{n,p}(f) = \frac{\|M_{n,p}\|_p}{n!}|D^n f(\xi)| \le \frac{\|M_{n,p}\|_p}{n!}\|D^n f\|_\infty,$$

with equality iff $f \in \Pi_n$.

Along the same lines, Fink [Fi77], defines $B(n,p,q)$ as the smallest constant such that

$$E_{n,p}(f) \le B(n,p,q)(b-a)^{n+\frac{1}{p}-\frac{1}{q}}\|D^n f\|_q, \quad \forall f \in W_q^n,$$

and gives some equivalent definitions.

Since best approximations are given by Lagrange interpolation, we might hope to estimate $B(n,p,q)$ by interpolating f at some Θ, as does Phillips in Theorem (3.5), where he shows:

$$B(n,p,\infty) = \frac{\|M_{n,p}\|_p}{n!}(b-a)^{-(n+\frac{1}{p})}. \tag{3.6}$$

Pursuing this idea, we are able to estimate $B(n,p,q)$ to within a factor of $8n$.

(3.7) Estimate for Fink's constant.

$$\frac{1}{n!}\frac{1}{4^n} \le B(n,p,q) \le \frac{n^{\frac{1}{q}}}{n!}\left(\frac{2}{4^n}\right)^{1-\frac{1}{nq}} \le 8n\frac{1}{n!}\frac{1}{4^n}.$$

Proof. Let $b-a=1$. First the lower bound. Since $M_{n,p}$ is the error in approximating $f=(\cdot)^n$, which has $D^n f = n!$, we must have

$$B(n,p,q) \ge \frac{\|M_{n,p}\|_p}{n!} \ge \frac{1}{n!}\frac{1}{4^n}.$$

By (2.4) and (2.6):

$$B(n,p,q) \leq \frac{n^{\frac{1}{q}}}{n!} \| \omega_\Theta^{1-\frac{1}{nq}} \|_p = \frac{n^{\frac{1}{q}}}{n!} \left(\|\omega_\Theta\|_{p(1-\frac{1}{nq})} \right)^{1-\frac{1}{nq}}. \tag{3.8}$$

Choosing $\omega_\Theta = M_{n,p(1-1/nq)}$, then applying Corollary (3.3) to (3.8), we obtain the upper bound. □

Gauss quadrature formulæ with multiple nodes

The polynomials $M_{n,p}$ have the following interesting connection with quadrature, see Turan [Tu50], also Ghizzetti and Ossicini [GO70:p74].

If $p = 2s+2$, $s = 0,1,2,\ldots$, then (3.2) reduces to

$$\int_a^b m^{2s+1} g = 0, \quad \forall g \in \Pi_{<n}.$$

The corresponding m $(= M_{n,2s+2})$ is called s-**orthogonal** (with weight dx).

There is a quadrature formula of the form

$$Q(f) := \sum_{i=0}^{2s} \sum_{v\in\Theta} w(i,v) D^i f(v), \tag{3.9}$$

for the integral $I(f) := \int_a^b f$, of precision $(2s+2)n-1$, iff Θ is the zeros of $M_{n,2s+2}$. In keeping with the special case $s=0$, such a Q is referred to as a **Gauss formulæ with multiple nodes**, or simply as a s-**Gauss** formula, and $M_{n,2s+2}$ is called a **Legendre** s-**polynomial**.

The s-Gauss formulæ are **interpolatory**, i.e. $Q(f) = I(H_{\Theta^*} f)$, where Θ^* is any set of $\leq n(2s+2)$ points, which contains each zero of $M_{n,2s+2}$ with multiplicity at least $2s+1$. This allows us to estimate the error for these formulæ.

(3.10) Error bound for s-Gauss formulæ. *Let Θ be the zeros of $M_{n,2s+2}$. Then*

$$|I(f) - Q(f)| \leq \frac{1}{(n(2s+2))!} \left(\|\omega_\Theta\|_{2s+2} \right)^{2s+2} \|D^{n(2s+2)} f\|_\infty, \quad \forall f \in W_\infty^{n(2s+2)},$$

with equality for all $f \in \Pi_{n(2s+2)}$. In addition

$$\left(\|\omega_\Theta\|_{2s+2} \right)^{2s+2} < \left(\frac{2}{4^n} \right)^{2s+2} (b-a)^{n(2s+2)+1},$$

which differs from equality by a factor of $< 2^{2s+2}$.

Proof. With Θ^* as above, by (1.1)

$$|I(f) - Q(f)| = |I(f - H_{\Theta^*} f)| \leq \|f - H_{\Theta^*} f\|_1 \leq \frac{1}{(n(2s+2))!} \|\omega_{\Theta^*}\|_1 \|D^{n(2s+2)} f\|_\infty. \tag{3.11}$$

Let Θ^* consist of the points Θ, each with multiplicity $2s+2$. For this choice,

$$\|\omega_{\Theta^*}\|_1 = \left(\|\omega_\Theta\|_{2s+2}\right)^{2s+2}.$$

Further, if $f \in \Pi_{n(2s+2)}$, then $f - H_{\Theta^*}f$ is a scalar multiple of ω_Θ^{2s+2}, which is nonnegative, and so equality holds in (3.11). Finally by Corollary (3.3)

$$\left(\|\omega_\Theta\|_{2s+2}\right)^{2s+2} < \left(\frac{2}{4^n}(b-a)^{n+\frac{1}{2s+2}}\right)^{2s+2} = \left(\frac{2}{4^n}\right)^{2s+2}(b-a)^{n(2s+2)+1},$$

which differs from equality by a factor of $< 2^{2s+2}$. □

Only when $s = 0$ is this result known; see, e.g., Davis and Rabinowitz [DR75:p98]. In this case $\|\omega_\Theta\|_2$ is the L_2-norm of a Legendre polynomial, and can be computed exactly. For a full account of s-Gauss formulæ, including other error estimates, see the survey article of Gautschi [Ga81].

By using (2.4) and (2.6), it is possible to run through the above argument, to get error bounds for s-Gauss formulæ in terms of $\|D^m f\|_q$, where $n(2s+1) \le m \le n(2s+2)$.

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