EXTREMAL GROWTH OF POLYNOMIALS

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ABSTRACT. We give an exposition of some simple but applicable cases of worst-case growth of a polynomial in terms of its uniform norm on a given compact set $K \subset \mathbb{C}^d$. Included is a direct verification of the formula for the pluripotential extremal function for a real simplex. Throughout we attempt to make the exposition as accessible to a general (analytic) audience as possible, avoiding wherever possible the finer details of Pluripotential Theory.

1. INTRODUCTION

In this expository article we discuss the following basic problem. Suppose that $K \subset \mathbb{C}^d$ is compact and that $p \in \mathbb{C}[\mathbf{z}]$, $\mathbf{z} \in \mathbb{C}^d$, is a (complex) polynomial such that $\|p\|_K := \max_{\mathbf{z}\in K} |p(\mathbf{z})| \le 1$. Then

how big can $|p(\mathbf{z}_0)|$ *be for* $\mathbf{z}_0 \in \mathbb{C}^d \backslash K$*?*

The best answer to this question is given by the celebrated Siciak-Zaharjuta extremal function of Pluripotential Theory which may be defined as

(1)
$$V_K(\mathbf{z}) = \sup\left\{\frac{1}{\deg(p)}\log|p(\mathbf{z})| \,:\, p(\mathbf{z}) \text{ is a non-constant polynomial, } \|p\|_K \le 1\right\}.$$

Consequently, for any $p \in \mathbb{C}[\mathbf{z}]$ and $\mathbf{z}_0 \in \mathbb{C}^d\backslash K$

$$|p(\mathbf{z}_0)| \le \|p\|_K \, \exp(\deg(p) V_K(\mathbf{z}_0)), \tag{2}$$

and the extremal function indeed gives the worst-case growth of a polynomial at a point *outside* K relative to its maximum norm on K.

In dimension $d = 1$ the (uppersemicontinuous regularization of the) extremal function is identical to the exterior Green's function for K with pole at infinity and thus generalizes this important function to several variables. We mention three of its important applications.

1. Siciak's Generalization of the Bernstein-Walsh Approximation Theorem.

Theorem 1.1. *(Siciak [16]) Let $K \subset \mathbb{C}^d$ be compact with V_K continuous on $\mathbb{C}^d$. Let $R > 1$, and let $\Omega_R := \{z : V_K(z) < \log R\}$. Let f be continuous on K. Then for the best approximation errors*

$$D_n(f, K) := \inf\{\|f - p\|_K \,:\, p \text{ is a polynomial of degree at most } n\},$$

$$\limsup_{n\to\infty} D_n(f, K)^{1/n} \le 1/R$$

if and only if f is the restriction to K of a function holomorphic in Ω_R.

Key words and phrases. multivariate polynomials, extremal function, pluripotential theory.

2. Baran's Generalization of Bernstein-Markov Polynomial Inequalities.

Theorem 1.2. *(Baran [2]) Let K be a compact set in $\mathbb{R}^d$ with nonempty interior. Then for every $x \in \mathrm{int}(K)$ we have the following inequality for a real polynomial p*

$$|D_j p(x)| \le (\deg P) D_j^+ V_K(x)(\|p\|_K^2 - p^2(x))^{1/2} \text{ for } j = 1, \cdots, d.$$

Here $D_j p$ is the partial derivative with respect to x_j and

$$D_j^+ V_K(x) := \liminf_{\epsilon \to 0^+} \frac{1}{\epsilon} V_K(x + i\epsilon e_j)$$

with $e_j \in \mathbb{R}^d$, the j-th elementary coordinate vector.

3. The Asymptotics of Fekete Points.

For a given compact $K \subset \mathbb{C}^d$ the polynomials of degree at most n, when restricted to K, form a vector space of dimension N_n, say. Suppose that $\{P_1, P_2, \cdots, P_{N_n}\}$ is a basis for this space. Then the associated Vandermonde determinant is defined by

$$\mathrm{vdm}(x_1, \cdots, x_{N_n}) := \det([P_i(x_j)]_{1 \le i,j \le N_n}), \ x_1, x_2, \cdots, x_{N_n} \in \mathbb{R}^d.$$

A set of points $\{z_1, \cdots, z_{N_n}\} \subset K$ for which

$$|\mathrm{vdm}(z_1, \cdots, z_{N_n})| = \max_{x_1, \cdots, x_{N_n} \in K} |\mathrm{vdm}(x_1, \cdots, x_{N_n})|$$

is said to be a set of *Fekete* points of degree n for K. Such sets of points are nearly optimal for polynomial interpolation (cf. [6]) and, in particular, are used in the Spectral Element Method for the numerical solution of PDEs. Recently it has been shown that their asymptotics are given by the so-called equilibrium measure of K given by the (non-linear) complex Monge-Ampere operator (cf. [11]) applied to V_K.

Theorem 1.3. *(Berman, Boucksom and Nystrom [5]) Suppose that $K \subset \mathbb{C}^d$ is compact and such that V_K is continuous on $\mathbb{C}^d$. For each $n \in \mathbb{N}$ let $\{z_1^{(n)}, \ldots, z_{N_n}^{(n)}\}$ be a set of Fekete points of degree n for K. Let μ_K be the pluripotential equilibrium measure of K. Then the sequence of equally weighted discrete probability measures based on Fekete points*

$$\lim_{n \to \infty} \frac{1}{N_n} \sum_{k=1}^{N_n} \delta_{z_k^{(n)}} = \mu_K$$

in the weak- sense.*

Here, for $z \in \mathbb{C}^d$, δ_z denotes the Dirac measure supported at z.

We hope that we have persuaded the reader of the value of the extremal function. Its general theory is an intimate part of Pluripotential Theory which an interested reader may pursue in Klimek's excellent monograph [11].

However, in some cases the answer to our basic question may be provided using entirely elementary means. It is on these special cases that we focus on in this article, with a special emphasis on when $K \subset \mathbb{R}^d$ is a *real* simplex. Throughout, the Chebyshev polynomials (of the first kind) will play an important role. They may be defined by either

1. $T_n(x) = \cos(n\theta)$, $x = \cos(\theta) \in [-1, 1]$
2. $T_n(z) = \frac{1}{2}\left(\left(z + \sqrt{z^2 - 1}\right)^n + \left(z - \sqrt{z^2 - 1}\right)^n\right)$, $z \in \mathbb{C}$
3. $T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$, $T_0(z) = 1$, $T_1(z) = z$, $z \in \mathbb{C}$.

2. The univariate case

We begin with the classical univariate case of $K = [-1,1]$. As is well-known (see e.g. Rivlin [15, §2.7]), the Chebyshev polynomials have minimal growth *inside* $[-1,1]$ and maximal growth *outside* $[-1,1]$. Specifically

Proposition 2.1. *Suppose that $p \in \mathbb{R}[x]$ is a* **real** *(univariate) polynomial such that $\|p\|_{[-1,1]} \le 1$. Then, for $x_0 \in \mathbb{R}\backslash[-1,1]$,*

$$|p(x_0)| \le |T_n(x_0)|,\ n = \deg(p)$$

which is attained precisely for $p(x) = \pm T_n(x)$.

It is perhaps less well-known that, as shown by Erdos [10], the Chebyshev polynomials are also extremal in this sense at points $z \in \mathbb{C}$, $|z| \ge 1$.

Proposition 2.2. *Suppose that $p \in \mathbb{R}[x]$ is a* **real** *(univariate) polynomial such that $\|p\|_{[-1,1]} \le 1$. Then, for $z_0 \in \mathbb{C}$ such that $|z_0| \ge 1$,*

$$|p(z_0)| \le |T_n(z_0)|,\ n = \deg(p)$$

which is attained precisely for $p(x) = \pm T_n(x)$.

Proof. For completeness we give a self-contained proof, expanding on the indications given in [10]. Suppose then that $p \in \mathbb{R}[x]$ is such that $\|p\|_{[-1,1]} \le 1$ and let $n := \deg(p)$. The Chebyshev polynomial $T_n(z)$ has extreme points $x_k := \cos(k\pi/n)$, $k = 0, 1, \cdots, n$, in the interval $[-1,1]$. We will actually prove the more general statement that if $p \in \mathbb{R}[x]$ is such that $|p(x_k)| \le 1$, $k = 0, 1 \cdots, n$, then

$$|p(z_0)| \le |T_n(z_0)|,\ |z_0| \ge 1.$$

To see this we write $p(z)$ as its own interpolant at the x_k, i.e.,

$$p(z) = \sum_{k=0}^{n} p(x_k)\ell_k(z)$$

where

$$\ell_k(z) := \frac{\omega_n(z)}{\omega_n'(x_k)(z - x_k)},\ k = 0, 1, \cdots, n,$$

with $\omega_n(z) := C\prod_{k=0}^{n}(z - x_k)$ (for any constant $C \in \mathbb{C}\backslash\{0\}$) are the fundamental Lagrange polynomials. It is easy to check that we may take

$$\omega_n(z) = T_{n+1}(z) - T_{n-1}(z)$$

and that then

$$\omega_n'(x_k) = (-1)^k \begin{cases} 2n & k = 1, 2, \cdots, (n-1) \\ 4n & k = 0, n \end{cases}$$

so that

$$\ell_k(z) = \frac{\omega_n(z)}{2n(z - x_k)} \begin{cases} (-1)^k & k = 1, 2, \cdots, (n-1) \\ \frac{1}{2}(-1)^k & k = 0, n \end{cases}$$

and we may write

$$p(z) = \frac{\omega_n(z)}{2n} {\sum_{k=0}^{n}}' (-1)^k \frac{p(x_k)}{z - x_k}$$

and, since $T_n(x_k) = (-1)^k$,

$$T_n(z) = \frac{\omega_n(z)}{2n} \sum_{k=0}^{n}{}' \frac{1}{z - x_k}$$

where the prime on the sum denotes that the first and last terms are to be halved.

Now, as $|T_n(\pm 1)| = 1$, our claim is trivial for $z_0 = \pm 1$. Otherwise we must show that for $|z_0| \geq 1$

$$\left| \sum_{k=0}^{n}{}' (-1)^k \frac{p(x_k)}{z_0 - x_k} \right| \leq \left| \sum_{k=0}^{n}{}' \frac{1}{z_0 - x_k} \right|$$

for which it suffices to show that

$$\left| \sum_{k=0}^{n}{}' y_k \frac{z_0 - 1}{z_0 - x_k} \right| \leq \left| \sum_{k=0}^{n}{}' \frac{z_0 - 1}{z_0 - x_k} \right| \tag{3}$$

for any values y_k with $|y_k| \leq 1$, $k = 0, 1, \cdots, n$. Now, if we write $z_0 = x + iy$, $x, y \in \mathbb{R}$, then a brief calculation yields

$$\frac{z_0 - 1}{z_0 - x_k} = \frac{\{(x^2 + y^2) - (1 + x_k)x + x_k\} + i(1 - x_k)y}{|z_0 - x_k|^2}.$$

We notice that

$$\begin{aligned} &(x^2 + y^2) - (1 + x_k)x + x_k \\ &\geq \min\{(x^2 + y^2) - (1 + (-1))x + (-1), (x^2 + y^2) - (1 + (+1))x + (+1)\} \\ &= \min\{x^2 + y^2 - 1, (x - 1)^2 + y^2\} \geq 0 \end{aligned}$$

so that

$$\Re\left(\frac{z_0 - 1}{z_0 - x_k}\right) \geq 0$$

for all $k = 0, 1, \cdots, n$ and the sign of $\Im\left(\frac{z_0 - 1}{z_0 - x_k}\right)$ depends only on the sign of $\Im(z_0)$. Hence, if we write $a_k + ib_k := \frac{z_0 - 1}{z_0 - x_k}$, $a_k, b_k \in \mathbb{R}$ we have that the a_k are of constant sign and that the b_k are of constant sign.

With this notation, the condition (3) becomes

$$\left| \sum_{k=0}^{n}{}' y_k (a_k + ib_k) \right| \leq \left| \sum_{k=0}^{n}{}' (a_k + ib_k) \right|.$$

But,

$$\begin{aligned} \left| \left(\sum_{k=0}^{n}{}' a_k \right) + i \left(\sum_{k=0}^{n}{}' b_k \right) \right|^2 &= \left(\sum_{k=0}^{n}{}' a_k \right)^2 + \left(\sum_{k=0}^{n}{}' b_k \right)^2 \\ &= \left(\sum_{k=0}^{n}{}' |a_k| \right)^2 + \left(\sum_{k=0}^{n}{}' |b_k| \right)^2 \end{aligned}$$

and hence, since $|y_k| \le 1$,

$$\begin{aligned}\left|\sum_{k=0}^{n}{}' y_k(a_k+ib_k)\right|^2 &= \left|\sum_{k=0}^{n}{}'(y_ka_k+iy_kb_k)\right|^2\\ &= \left|\left(\sum_{k=0}^{n}{}' y_ka_k\right)+i\left(\sum_{k=0}^{n}{}' y_kb_k\right)\right|^2\\ &\le \left|\left(\sum_{k=0}^{n}{}' |y_ka_k|\right)+i\left(\sum_{k=0}^{n}{}' |y_kb_k|\right)\right|^2\\ &\le \left|\left(\sum_{k=0}^{n}{}' |a_k|\right)+i\left(\sum_{k=0}^{n}{}' |b_k|\right)\right|^2\\ &= \left|\sum_{k=0}^{n}{}'(a_k+ib_k)\right|^2.\end{aligned}$$

□

Remark. For $|z_0| < 1$ the Chebyshev polynomial $T_n(z)$ is not always extremal. Indeed for degree one, $T_1(z) = z$ and hence $|T_1(z_0)| < 1$ and is beaten by even $p(x) = 1$. In general, if $|z_0| < 1$, the extremal polynomial depends on the point z_0 and a formula for it does not seem to be known. □

The case of a polynomial with complex coefficients $p \in \mathbb{C}[z]$ is more complicated. In general $T_n(z)$ is not extremal, even at points $|z_0| \ge 1$. For example, take $p(z) := (1-iz)/\sqrt{2}$. Then for $x \in [-1,1]$, $|p(x)| = \sqrt{1+x^2}/\sqrt{2} \le 1$, but

$$|p(ir)| = \frac{1+r}{\sqrt{2}} > r = |T_1(ir)|, \text{ for } 0 < r < \sqrt{2}+1.$$

In general the extremal polynomials are not known. But nevertheless, at real points we have:

Corollary 2.3. *Suppose that $p \in \mathbb{C}[z]$ is such that $\|p\|_{[-1,1]} \le 1$. Then, setting $n := \deg(p)$,*

$$|p(x_0)| \le |T_n(x_0)|, \ x_0 \in \mathbb{R}\backslash[-1,1].$$

Proof. Write $p(z) = a(z) + ib(z)$ with $a, b \in \mathbb{R}[x]$. Then for $x \in [-1,1]$,

$$0 \le a^2(x) + b^2(x) = |p(x)|^2 \le 1.$$

It follows that $q(x) := 2(a^2(x) + b^2(x)) - 1$ is a polynomial of degree $2n$ such that $\|q\|_{[-1,1]} \le 1$. Then by Proposition 2.1 we have

$$|q(x_0)| \le |T_{2n}(x_0)|, \ x_0 \in \mathbb{R}\backslash[-1,1].$$

But

$$T_{2n}(x) = 2T_n^2(x) - 1$$

as this holds for $x = \cos(\theta)$ and we have

$$2|p(x_0)|^2 - 1 \le |2T_n^2(x_0) - 1| = 2T_n^2(x_0) - 1$$

from which the result follows easily. □

Remark. Proposition 2.2 shows that for *real* polynomials $p \in \mathbb{R}[x]$ of degree at most n and $z_0 \in \mathbb{C}\backslash[-1,1]$, $|z_0| \geq 1$,

$$\max_{\|p\|_{[-1,1]}\leq 1} |p(z_0)| = \max_{\|p\|_{X_n}\leq 1} |p(z_0)|$$

where $X_n \subset [-1,1]$ is the set of extreme points of the Chebyshev polynomial $T_n(x)$. For *complex* polynomials $p \in \mathbb{C}[z]$ this is not in general true (at least for $|z_0| < 1$). However, $\max_{\|p\|_{X_n}\leq 1} |p(z_0)|$ can always be easily determined for any $z_0 \in \mathbb{C}\backslash[-1,1]$. Indeed, write $p(z) = \sum_{k=0}^n p(x_k)\ell_k(z)$ in Lagrange form so that

$$|p(z_0)| \leq \sum_{k=0}^n |p(x_k)|\,|\ell_k(z_0)| \leq \sum_{k=0}^n |\ell_k(z_0)|$$

for $\|p\|_{X_n} \leq 1$. The extremal polynomial is given by

$$p(z) = \sum_{k=0}^n \frac{\overline{\ell_k(z_0)}}{|\ell_k(z_0)|}\ell_k(z).$$

Remark. We also mention a result of Duffin and Schaeffer ([9], see also [7], Theorem 5.2.1), in which Chebyshev polynomials may be used to give bounds on the derivatives of a real polynomial. If p is a polynomial of degree at most n and $\|p\|_{X_n} \leq 1$, then for each $m = 1, \ldots, n$, the m-th derivative of p satisfies

$$|p^{(m)}(z)| \leq |T_n^{(m)}(1 + \Im(z))|, \quad \text{whenever } \Re(z) \in [-1,1].$$

□

Despite the difficulty of identifying the precise extremal polynomials it is nevertheless possible to calculate the extremal function (1) providing the bound on polynomials (2). In fact, we show that although the Chebyshev polynomials are not always extremal, they do determine the function $V_{[-1,1]}(z)$.

Proposition 2.4. *(cf. Bernstein's Lemma,* [14, Thm. 5.5.7]*) At any point* $z \in \mathbb{C}\backslash[-1,1]$

$$V_{[-1,1]}(z) = \sup_{n\geq 1} \frac{1}{n} \log |T_n(z)| = \log |h(z)|$$

where we define

$$h(z) := z + \sqrt{z^2 - 1}$$

with the branch of the square root chosen so that $|h(z)| \geq 1$.
Consequently, for any (complex) polynomial $p \in \mathbb{C}[z]$ *and* $z \in \mathbb{C}\backslash[-1,1]$,

$$|p(z)| \leq \|p\|_{[-1,1]}|h(z)|^n \leq \|p\|_{[-1,1]}(2|T_n(z)| + 1).$$

Remark. To understand the function $h(z)$ a bit better, consider the so-called Joukowski transformation

$$J(z) := \frac{1}{2}\left(z + \frac{1}{z}\right).$$

For $|z| = r$,

$$J(re^{i\theta}) = \frac{1}{2}(r + 1/r)\cos(\theta) + i\frac{1}{2}(r - 1/r)\sin(\theta)$$

and we see that J maps the unit circle $|z| = 1$ to the interval $[-1,1]$ and the circle $|z| = r$, $r \neq 1$ to the ellipse $(x/a)^2 + (y/b)^2 = 1$ with $a = (1/2)(r + 1/r)$, $b = (1/2)|r - 1/r|$. In particular J maps the *exterior* of the unit disk to the exterior of the interval $[-1,1]$ and is one-to-one and onto there. The function $h(z)$ is precisely the inverse of $J(z)$, mapping the exterior of the interval to the exterior of the unit disk, i.e., $|h(z)| \geq 1$. Since $J(z)$ is analytic, $h(z)$ is also analytic. Also, the level sets $\{z \in \mathbb{C}\backslash[-1,1] : |h(z)| = \rho\}$ are exactly the ellipses described above with $r = \rho$. These having foci ± 1, may also be described as $\{z \in \mathbb{C} : |z-1| + |z+1| = 2a\}$, i.e.,

$$a = \frac{1}{2}(|z-1| + |z+1|) \geq 1$$

is a point on the same ellipse (level set of $|h(z)|$) as is z. In other words, we may evaluate

$$|h(z)| = h\left(\frac{|z-1| + |z+1|}{2}\right).$$

We also remark that the second definition of the Chebyshev polynomials given in the introduction may be interpreted as

$$T_n(z) = J(h^n(z)),\ z \in \mathbb{C}.$$

Perhaps it is also worth noting that from this it easily follows that

$$h^n(z) = T_n(z) + U_{n-1}(z)\sqrt{z^2-1}$$

where $U_{n-1}(z) = (1/n)T_n'(z)$ is the Chebyshev polynomial of the second kind. □

Proof of the Proposition. We have two things to prove:

(1) that if $p \in \mathbb{C}[z]$ is such that $\|p\|_{[-1,1]} \leq 1$ then $\dfrac{1}{\deg(p)} \log|p(z)| \leq \log|h(z)|$,

(2) that $\sup\limits_{n \geq 1} \dfrac{1}{n} \log|T_n(z)| = \log|h(z)|$,

both for $z \in \mathbb{C}\backslash[-1,1]$.

Let U be the domain $U := \mathbb{C}\backslash[-1,1]$.

To see (1) note that $\log|h(z)|$ is harmonic on U and zero on the boundary of U, i.e., $\partial U = [-1,1]$, while $\dfrac{1}{\deg(p)} \log|p(z)|$ is harmonic on U except at zeros of p inside U, making it technically *subharmonic* on U and ≤ 0 on ∂U. For $\alpha > 1$ consider then the function

$$f_\alpha(z) := \frac{1}{\deg(p)} \log|p(z)| - \alpha \log|h(z)|.$$

It then is also subharmonic on U, ≤ 0 on $[-1,1]$ and moreover

$$\lim_{|z| \to \infty} f_\alpha(z) = -\infty.$$

Hence, by the maximum principle for subharmonic functions (see e.g. Ransford [14]), we have that $f_\alpha(z) \leq 0$, $\forall z \in \overline{U}$, i.e.,

$$\frac{1}{\deg(p)} \log|p(z)| \leq \alpha \log|h(z)|,\ z \in \overline{U}.$$

Property (1) then follows from the fact that $\alpha > 1$ was arbitrary.

To see (2), we use the second formula for Chebyshev polynomials given in the Introduction

$$T_n(z) = \frac{1}{2}\left(\left(z + \sqrt{z^2-1}\right)^n + \left(z - \sqrt{z^2-1}\right)^n\right) = J(h^n(z)),\ z \in \mathbb{C}.$$

Clearly then

$$\lim_{n\to\infty}\frac{1}{n}\log|T_n(z)| = |h(z)|.$$

Since by (1) we also know that $(1/n)\log|T_n(z)| \le \log|h(z)|$, the result (2) follows.

The final inequality follows by noting that $h^n(z) = 2T_n(z) - h^{-n}(z)$ and that $|h(z)| \ge 1$. □

Remark. A C^2 subharmonic function $u(z)$ is characterized by having a non-negative Laplacian,

$$\Delta u(z) \ge 0.$$

If we change the usual Cartesian coordinates x, y to the complex coordinates $z, \overline{z}$ with partial derivatives defined by

$$\frac{\partial u}{\partial z} := \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right),\ \ \frac{\partial u}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)$$

then we may write the Laplacian as

$$\Delta u = 4\frac{\partial^2 u}{\partial z\partial\overline{z}}.$$

These second mixed partial derivatives play an important role in the multivariate case. □

We now consider the other standard univariate case, that of $K \subset \mathbb{C}$, the unit disk.

Proposition 2.5. *Suppose that* $K = \{z \in \mathbb{C} : |z| \le 1\}$ *is the unit disk and that* $p \in \mathbb{C}[z]$ *is such that* $\|p\|_K \le 1$. *Then for any* $z_0 \in \mathbb{C}\backslash K$, *i.e.,* $|z_0| > 1$,

$$|p(z_0)| \le |z_0^n| = |z_0|^n,\ \ n := \deg(p).$$

Moreover,

$$V_K(z) = \log|z|,\ \ z \in \mathbb{C}\backslash K.$$

Proof. The argument is really the same as for Proposition 2.4 (but easier), replacing $T_n(z)$ by z^n. Indeed, in this case, for any $\alpha > 1$, we set

$$f_\alpha(z) := \frac{1}{\deg(p)}\log|p(z)| - \alpha\log|z|$$

and note that $f_\alpha(z)$ is subharmonic outside the unit disk, is negative on the boundary of the disk, and

$$\lim_{|z|\to\infty} f_\alpha(z) = -\infty.$$

Hence, for $|z| \ge 1$, $f_\alpha(z) \le 0$, i.e.,

$$\frac{1}{\deg(p)}\log|p(z)| \le \alpha\log|z|,\ \ |z| \ge 1.$$

The result follows from the fact that $\alpha > 1$ was arbitrary. □

3. The case of $K \subset \mathbb{R}^d$ the unit ball of a norm

Suppose that

$$K = \{\mathbf{x} \in \mathbb{R}^d \,:\, \|\mathbf{x}\| \le 1\}$$

is the unit ball for some norm $\|\mathbf{x}\|$ on $\mathbb{R}^d$ (or, equivalently, that $K \subset \mathbb{R}^d$ is a convex body, symmetric with respect to the origin). The polar of K is defined as

$$K^\circ := \{\mathbf{y} \in \mathbb{R}^d \,:\, \mathbf{x}^t\mathbf{y} \le 1, \ \forall \mathbf{x} \in K\}. \tag{4}$$

(Here $\mathbf{x}^t$ denotes the transpose of $\mathbf{x}$ considered as a column vector, so that $\mathbf{x}^t\mathbf{y}$ is the (real) inner product of $\mathbf{x}$ and $\mathbf{y}$.) We note that the so-called *dual* form of the norm is

$$\|\mathbf{x}\| = \max\{\mathbf{y}^t\mathbf{x} \,:\, \mathbf{y} \in K^\circ\}.$$

Proposition 3.1. *Suppose that $K \subset \mathbb{R}^d$ is the unit ball of the norm $\|\mathbf{x}\|$ and that $p \in \mathbb{C}[\mathbf{x}]$, $\mathbf{x} \in \mathbb{R}^d$, is such that $\|p\|_K \le 1$. Then for any $\mathbf{x}_0 \in \mathbb{R}^d$ with $\|\mathbf{x}_0\| > 1$, we have*

$$|p(\mathbf{x}_0)| \le T_n(\|\mathbf{x}_0\|), \ n := \deg(p).$$

Further, let $\mathbf{y}_0 \in K^\circ$ be such that $\|\mathbf{x}_0\| = \mathbf{y}_0^t\mathbf{x}_0$. Then the polynomial

$$T_n(\mathbf{y}_0^t\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$$

is extremal.

Moreover,

$$V_K(\mathbf{x}_0) = \log h(\|\mathbf{x}_0\|).$$

Proof. First suppose that $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$, i.e., is a *real* polynomial. Consider the two points

$$\mathbf{a} := -\frac{\mathbf{x}_0}{\|\mathbf{x}_0\|} \text{ and } \mathbf{b} := \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|}.$$

Clearly $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$ and so $\mathbf{a}, \mathbf{b} \in K$. It is also easy to confirm that

$$\mathbf{x}_0 = \frac{1-t_0}{2}\mathbf{a} + \frac{1+t_0}{2}\mathbf{b}, \ t_0 := \|\mathbf{x}_0\| > 1.$$

Consider the line given by

$$\mathbf{x}(t) := \frac{1-t}{2}\mathbf{a} + \frac{1+t}{2}\mathbf{b}, \ t \in \mathbb{R};$$

we have (by the convexity of K) $\mathbf{x}(t) \in K$ for $-1 \le t \le 1$ and $\mathbf{x}(t_0) = \mathbf{x}_0$. Now let

$$q(t) := p(\mathbf{x}(t)) \in \mathbb{R}[t].$$

Then $\|q\|_{[-1,1]} \le \|p\|_K \le 1$ and so

$$|p(\mathbf{x}_0)| = |q(t_0)| \le T_n(t_0) = T_n(\|\mathbf{x}_0\|).$$

If $p \in \mathbb{C}[\mathbf{z}]$, i.e., is a *complex* polynomial, then exactly the same argument as used in the proof of Corollary 2.3 shows that, also in this case,

$$|p(\mathbf{x}_0)| \le T_n(\|\mathbf{x}_0\|).$$

If $p(\mathbf{x}) := T_n(\mathbf{y}_0^t\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ then, as $\mathbf{y}_0 \in K^\circ$, $|\mathbf{y}_0^t\mathbf{x}| \le 1$ for $\mathbf{x} \in K$ and so $\|p\|_K \le 1$. Further, $p(\mathbf{x}_0) = T_n(\mathbf{y}_0^t\mathbf{x}_0) = T_n(\|\mathbf{x}_0\|)$ showing that $T_n(\mathbf{y}_0^t\mathbf{x})$ is indeed extremal.

Finally we show that $V_K(\mathbf{x}_0) = \log h(\|\mathbf{x}_0\|)$. To see this, as the Chebyshev polynomials are extremal, we must show that

$$\sup_n (T_n(t_0))^{1/n} = h(t_0), \ t_0 \ge 1.$$

Note that we have already remarked that $\lim_{n\to\infty}(T_n(t_0))^{1/n} = h(t_0)$ but here we are asking for a bit more. However, the extremal property of Chebyshev polynomials for $K = [-1, 1]$ also yields $(T_n(t_0))^m \le T_{mn}(t_0)$ for all $m \in \mathbb{N}$, so that

$$T_n(t_0) \le (T_{mn}(t_0))^{1/m}.$$

Consequently,

$$(T_n(t_0))^{1/n} \le \lim_{m\to\infty}(T_{mn}(t_0))^{1/(mn)} = \lim_{k\to\infty}(T_k(t_0))^{1/k} = h(t_0).$$

□

Remark. By Proposition 2.2 we do know the extremal polynomial at *special* points $\mathbf{z}_0 \in \mathbb{C}^d \backslash K$. Indeed, if it is the case that

$$z_0 = \frac{1-t_0}{2}\mathbf{a} + \frac{1+t_0}{2}\mathbf{b}$$

for some $\mathbf{a}, \mathbf{b} \in K$ and $t_0 \in \mathbb{C}$ with $|t_0| \ge 1$, then for $\|p\|_K \le 1$,

$$|p(z_0)| \le |T_n(t_0)|.$$

However, in general, the value of the extremal function $V_K(\mathbf{z})$ at complex points $\mathbf{z} \in \mathbb{C}^d \backslash K$ is considerably more complicated. The interested reader may find some examples in Klimek [11]. We also mention that for K a real ball the study of the extremal function at complex points is one of the main topics of Baran's very interesting 1988 thesis [3].

We give the result (without proof) for $K = [-1, 1]^d$, a cube, i.e., the unit ball for the ℓ_∞ norm.

Proposition 3.2. *(Siciak, 1962,* [16]*) Suppose that* $K = [-1, 1]^d$. *Then*

$$V_K(\mathbf{z}) = \max_{1\le j\le d} \log |h(z_j)|.$$

At the end of this work we will briefly return to the case of $K \subset \mathbb{R}^d$, the Euclidean unit ball. □

4. The case of $K \subset \mathbb{C}^d$ the unit ball of a complex norm

Here we suppose that $\|\mathbf{z}\|$ is a norm on $\mathbb{C}^d$. In the complex case, the notion of polar set requires but the replacement of the transpose by the complex conjugate.

Proposition 4.1. *Suppose that* $K \subset \mathbb{C}^d$ *is the unit ball of the norm* $\|\mathbf{z}\|$ *and that* $p \in \mathbb{C}[\mathbf{z}]$, $\mathbf{z} \in \mathbb{C}^d$, *is such that* $\|p\|_K \le 1$. *Then for any* $\mathbf{z}_0 \in \mathbb{C}^d$ *with* $\|\mathbf{z}_0\| > 1$, *we have*

$$|p(\mathbf{z}_0)| \le \|\mathbf{z}_0\|^n, \ n := \deg(p).$$

Further, let $\mathbf{w}_0 \in K^\circ$ *be such that* $\|\mathbf{z}_0\| = \mathbf{w}_0^*\mathbf{z}_0$. *Then the polynomial*

$$(\mathbf{w}_0^*\mathbf{z})^n \in \mathbb{C}[\mathbf{z}]$$

is extremal.

Moreover, $V_K(\mathbf{z}_0) = \log \|\mathbf{z}_0\|$.

Proof. The proof is almost identical to that for the case of a real ball. We again let

$$\mathbf{a} := -\frac{\mathbf{z}_0}{\|\mathbf{z}_0\|} \text{ and } \mathbf{b} := \frac{\mathbf{z}_0}{\|\mathbf{z}_0\|}.$$

so that $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$ and so $\mathbf{a}, \mathbf{b} \in K$. As before, we have that

$$\mathbf{z}_0 = \frac{1-t_0}{2}\mathbf{a} + \frac{1+t_0}{2}\mathbf{b},\ t_0 := \|\mathbf{z}_0\| > 1$$

and now consider the *complex* line given by

$$\mathbf{z}(t) := \frac{1-t}{2}\mathbf{a} + \frac{1+t}{2}\mathbf{b},\ t \in \mathbb{C}.$$

We confirm that for $|t| \le 1$,

$$\begin{aligned}\|\mathbf{z}(t)\| &= \|\frac{1-t}{2}\mathbf{a} + \frac{1+t}{2}\mathbf{b}\| \\ &= \|\left(-\left(\frac{1-t}{2}\right) + \frac{1+t}{2}\right)\mathbf{b}\| \\ &= \|t\mathbf{b}\| = |t|\,\|\mathbf{b}\| \le 1.\end{aligned}$$

Thus $q(t) := p(\mathbf{z}(t)) \in \mathbb{C}[t]$ is a polynomial bounded by one on the unit disk and hence by Proposition 2.5 $|p(\mathbf{z}_0)| = |q(t_0)| \le |t_0|^n = \|\mathbf{z}_0\|^n$ and from this the rest of the Proposition follows easily. □

5. The real simplex at real points

Here we discuss the case of $K = S_d \subset \mathbb{R}^d$ a real simplex, i.e., the convex hull of $d+1$ vertices $\mathbf{v}_j \in \mathbb{R}^d$, $0 \le j \le d$, in general position. Note that S_d is *not* the unit ball of a norm. It is of course convex.

First some notation. For a point $\mathbf{x} \in \mathbb{R}^d$ let $\lambda_j(\mathbf{x})$, $0 \le j \le d$, denote the *barycentric coordinates of* $\mathbf{x}$ *with respect to* S_d. Recall that these are the linear functions of $\mathbf{x}$ characterized by

$$\mathbf{x} = \lambda_0(\mathbf{x})\mathbf{v}_0 + \cdots + \lambda_d(\mathbf{x})\mathbf{v}_d, \qquad \lambda_0(\mathbf{x}) + \cdots + \lambda_d(\mathbf{x}) = 1,$$

with $\lambda_j(\mathbf{x}) \ge 0$ for all $\mathbf{x} \in S_d$.

Set

$$I(\mathbf{x}) := \{j\,:\,\lambda_j(\mathbf{x}) > 0\} \text{ and } I'(\mathbf{x}) := \{j\,:\,\lambda_j(\mathbf{x}) \le 0\}.$$

We note that at any point $\mathbf{x} \in \mathbb{R}^d \backslash S_d$ both $I(\mathbf{x})$ and $I'(\mathbf{x})$ are non-empty. Further, we may write

$$\sum_{j=0}^d |\lambda_j(\mathbf{x})| = \sum_{j\in I(\mathbf{x})} \lambda_j(\mathbf{x}) - \sum_{j\in I'(x)} \lambda_j(\mathbf{x}) = 2\left(\sum_{j\in I(\mathbf{x})} \lambda_j(\mathbf{x})\right) - 1 \ge 1$$

as $\sum_{j=1}^d \lambda_j(\mathbf{x}) = 1$. In fact there is equality in the above if and only if $\mathbf{x} \in S_d$ so that, in particular, for $x \in \mathbb{R}^d \backslash S_d$,

$$2\left(\sum_{j\in I(\mathbf{x})} \lambda_j(\mathbf{x})\right) - 1 > 1.$$

Proposition 5.1. *(cf. Kroo and Schmidt* [12] *and also* [8]*) Suppose that* $p \in \mathbb{C}[\mathbf{x}]$, $\mathbf{x} \in \mathbb{R}^d$, *is such that* $\|p\|_{S_d} \le 1$ *and that* $\mathbf{x}_0 \in \mathbb{R}^d \backslash S_d$. *Then*

$$|p(\mathbf{x}_0)| \le T_n\Big(\sum_{j=0}^d |\lambda_j(\mathbf{x}_0)|\Big),\ n := \deg(p).$$

Moreover, this upper bound is attained by

$$p(x) := T_n\big(2 \sum_{j\in I(\mathbf{x}_0)} \lambda_j(\mathbf{x}) - 1\big).$$

Further, the extremal function at real points is given by

$$V_{S_d}(\mathbf{x}) = \log\big(h\big(\sum_{j=0}^{d} |\lambda_j(\mathbf{x})|\big)\big). \tag{5}$$

Proof. Just as in the univariate case, it suffices to prove this for $p \in \mathbb{R}[\mathbf{x}]$, i.e., a real polynomial, the argument being a consequence of the properties of the Chebyshev polynomials and not the underlying domain. Suppose then that $p \in \mathbb{R}[\mathbf{x}]$ and that $\|p\|_{S_d} \le 1$. The idea of the proof is, just as in the case of K a unit ball, to find two points $\mathbf{a}, \mathbf{b} \in S_d$ so that $\mathbf{a}, \mathbf{b}, \mathbf{x}_0$ are collinear and then restrict p to this line, obtaining a univariate problem to analyze.

Specifically, set

$$t_0 := 2 \sum_{j\in I(\mathbf{x}_0)} \lambda_j(\mathbf{x}_0) - 1 = \sum_{j=0}^{d} |\lambda_j(\mathbf{x}_0)| > 1$$

and

$$s_0 := \frac{1}{\sum_{j\in I(\mathbf{x}_0)} \lambda_j(\mathbf{x}_0)} = \frac{2}{1+t_0} \in (0,1).$$

We define the point $\mathbf{a}$ by assigning it barycentric coordinates

$$\mu_j = \begin{cases} 0 & \text{if } j \in I(\mathbf{x}_0) \\ \frac{s_0}{s_0-1}\lambda_j(\mathbf{x}_0) & \text{if } j \in I'(\mathbf{x}_0) \end{cases}.$$

Clearly each μ_j, so defined, is non-negative. We also confirm that

$$\begin{aligned}
\sum_{j=0}^{d} \mu_j &= \sum_{j\in I'(\mathbf{x}_0)} \mu_j \\
&= \frac{s_0}{s_0-1}\left(\sum_{j\in I'(\mathbf{x}_0)} \lambda_j(\mathbf{x}_0)\right) \\
&= \frac{s_0}{s_0-1}\left(1 - \sum_{j\in I(\mathbf{x}_0)} \lambda_j(\mathbf{x}_0)\right) \\
&= \frac{s_0}{s_0-1}\left(1 - \left(\frac{t_0+1}{2}\right)\right) \\
&= \frac{2/(1+t_0)}{2/(1+t_0)-1}\left(\frac{1-t_0}{2}\right) \\
&= \frac{2}{1-t_0}\,\frac{1-t_0}{2} = 1
\end{aligned}$$

so that the μ_j are indeed the barycentric coordinates of a point $\mathbf{a} \in S_d$.

We now define $\mathbf{b} := s_0\mathbf{x}_0 + (1-s_0)\mathbf{a}$ and calculate

$$\begin{aligned}\lambda_j(\mathbf{b}) &= s_0\lambda_j(\mathbf{x}_0) + (1-s_0)\mu_j\\ &= \begin{cases} s_0\lambda_j(\mathbf{x}_0) & \text{if } j \in I(\mathbf{x}_0)\\ s_0\lambda_j(\mathbf{x}_0) + (1-s_0)\frac{s_0}{s_0-1}\lambda_j(\mathbf{x}_0) & \text{if } j \in I'(\mathbf{x}_0)\end{cases}\\ &= \begin{cases} s_0\lambda_j(\mathbf{x}_0) & \text{if } j \in I(\mathbf{x}_0)\\ 0 & \text{if } j \in I'(\mathbf{x}_0)\end{cases}\\ &\geq 0.\end{aligned}$$

Hence $\mathbf{b} \in S_d$ as well. We note that both $\mathbf{a}, \mathbf{b}$ are on the boundary of S_d as they each have at least one zero barycentric coordinate.

By construction, $\mathbf{a}, \mathbf{b}$ and $\mathbf{x}_0$ are collinear and it is easy to verify that, in fact,

$$\mathbf{x}_0 = \frac{1-t_0}{2}\mathbf{a} + \frac{1+t_0}{2}\mathbf{b}.$$

We now let

$$q(t) := p(\frac{1-t}{2}\mathbf{a} + \frac{1+t}{2}\mathbf{b}),\ t \in \mathbb{R}$$

be the *univariate* restriction of p to the line defined by $\mathbf{a}, \mathbf{b}$ and $\mathbf{x}_0$. Note that $q(t)$ is a univariate polynomial of degree at most $n := \deg(p)$ and such that $q(-1) = p(\mathbf{a})$ and $q(+1) = p(\mathbf{b})$ while $q(t_0) = p(\mathbf{x}_0)$. Further, since the segment $[\mathbf{a}, \mathbf{b}] \subset S_d$, we have $\|q\|_{[-1,1]} \leq 1$. Hence, since $t_0 > 1$, by the univariate case of the interval,

$$|p(\mathbf{x}_0)| = |q(t_0)| \leq |T_n(t_0)| = T_n\left(\sum_{j=0}^{d} |\lambda_j(\mathbf{x}_0)|\right). \tag{6}$$

To show that this upper bound is attained by

$$p(\mathbf{x}) := T_n\big(2\sum_{j\in I(\mathbf{x}_0)} \lambda_j(\mathbf{x}) - 1\big)$$

we need only note that for $\mathbf{x} \in S_d$,

$$2\sum_{j\in I(\mathbf{x}_0)} \lambda_j(\mathbf{x}) - 1 \in [-1, 1]$$

and hence $\|p\|_{S_d} \leq 1$, while

$$p(\mathbf{x}_0) = T_n\big(2\sum_{j\in I(\mathbf{x}_0)} \lambda_j(\mathbf{x}_0) - 1\big) = T_n\big(\sum_{j=0}^{d} |\lambda_j(\mathbf{x}_0)|\big).$$

The formula for the extremal function (5) now follows by the arguments we have made about the Chebyshev polynomials in the univariate case. □

6. The real simplex at complex points

For any compact set $K \subset \mathbb{R}^d$ a "naive" candidate for the extremal function at complex points would always be the complexified version of the real formula. Unfortunately, this does not in general give the correct formula, notably in the case of the real ball (see the end of the paper). But, remarkably, it does work in the case of a simplex. Indeed we have, setting

$$\Sigma(z) := \sum_{j=0}^{d} |\lambda_j(\mathbf{z})|,$$

$$V_{S_d}(\mathbf{z}) = \log\big(h\big(\sum_{j=0}^{d} |\lambda_j(\mathbf{z})|\big)\big) = \log\big(h\big(\Sigma(z)\big)\big),\ \mathbf{z} \in \mathbb{C}^d \backslash S_d \tag{7}$$

as has been shown by Baran [1] (see also Klimek [11, Example 5.4.7]).

The proof given by Baran is somewhat indirect as it is based on the corresponding formula for the real ball and the so-called coordinate square map. In this section we give a direct verification of the formula (7). Indeed, we will show, just as in the univariate case, that for any $\alpha > 1$, and every polynomial $p \in \mathbb{C}[\mathbf{z}]$ such that $\|p\|_S \le 1$

$$\frac{1}{\deg(p)} \log|p(\mathbf{z}_0)| \le \alpha \log h(\Sigma(\mathbf{z}_0)),\ \mathbf{z}_0 \in \mathbb{C}^d \backslash S \tag{8}$$

(with $\log(0) := -\infty$). The argument depends on the following observations:

1. if $p(\mathbf{z}_0) = 0$ there is nothing to do.
2. if $\mathbf{z}_0$ is such that some $\lambda_j(\mathbf{z}_0) = 0$ then $\mathbf{z}_0$ is in the hyperplane of S defined by $\lambda_j(\mathbf{z}) = 0$. The intersection of S with this hyperplane is a lower dimensional simplex, and hence by an induction argument we can assume that (8) holds there.
3. at any point where $\lambda_j(\mathbf{z}) \neq 0$ for $j = 0, 1, \cdots, d$, $\log h(\Sigma)$ is a real C^2 function and there exists a direction vector $\mathbf{u} \in \mathbb{C}^d$ such that $\log h(\Sigma)$ restricted to the line $L_{\mathbf{z}} := \{\mathbf{y} \in \mathbb{C}^d : \mathbf{y} = \mathbf{z} + t\mathbf{u},\ t \in \mathbb{C}\}$ satisfies
$$\frac{\partial^2}{\partial t \partial \bar{t}} \log h(\Sigma(\mathbf{z} + t\mathbf{u}))\Big|_{t=0} = 0.$$
(We remark that the technical term for this property is that $\log h(\Sigma)$ is *maximal*.) Moreover, the direction vector $\mathbf{u} = \mathbf{u}(\mathbf{z})$ depends *continuously* on the point $\mathbf{z}$.

Actually, the verification of 3. will require somewhat lengthy, but elementary, calculations, that although not trivial, have a certain elegance. We give them below, but for the moment suppose that 3. holds. We first show how then (8) follows. Indeed, consider

$$f_\alpha(\mathbf{z}) := \frac{1}{\deg(p)} \log|p(\mathbf{z})| - \alpha \log h((\Sigma(\mathbf{z})).$$

On S itself, $f_\alpha(\mathbf{z}) \le 0$ as $\log h(\Sigma) = 0$ there and $\|p\|_S \le 1$. Further,

$$\lim_{|\mathbf{z}| \to \infty} f_\alpha(\mathbf{z}) = -\infty.$$

Hence by (semi-)continuity, $f_\alpha(\mathbf{z})$ will attain its maximum at some point $\mathbf{z}_0 \in \mathbb{C}^d$. By observation 2., if $\lambda_j(\mathbf{z}_0) = 0$ for some j then this maximum is negative and we will be done. Otherwise, we may apply 3.. We may use this to determine a continuous complex curve $\gamma(t)$, defined by $\gamma'(t) = \mathbf{u}(\gamma(t))$, $\gamma(0) = \mathbf{z}_0$ in a neighbourhood of $t = 0 \in \mathbb{C}$. By construction $t \mapsto \log h(\Sigma(\mathbf{u}(\gamma(t))))$ is harmonic and, moreover, as by observation 1. we may assume that $p(\mathbf{z}_0) \neq 0$, $f_\alpha(\gamma(t))$ is also harmonic in a neighbourhood of $t = 0 \in \mathbb{C}$ and has maximum value at $t = 0$, a contradiction. □

We now proceed to verify 3.. In general,

$$\frac{\partial^2}{\partial t\partial\bar{t}}f(\mathbf{z}+t\mathbf{u}) = \sum_{j,k=1}^{d}\frac{\partial^2 f}{\partial z_j\partial\overline{z}_k}u_j\overline{u}_k = \mathbf{u}^*H^{(d)}\mathbf{u}$$

where $H^{(d)}\in\mathbb{C}^{d\times d}$ is the *complex Hessian* of f defined by

$$H_{jk}^{(d)} := \frac{\partial^2 f}{\partial z_j\partial\overline{z}_k}, \quad 1\le j,k\le d$$

Hence 3. follows from the existence of a vector $\mathbf{u}=\mathbf{u}(\mathbf{z})\in\mathbb{C}^d$ in the kernel of H with continuous dependence on $\mathbf{z}$. This is what we show below.

There being $d+1$ barycentric coordinates, some of the calculations will be done in $\mathbb{C}^{d+1}$ and others in $\mathbb{C}^d$. To keep things straight we will use capital letters to denote vectors in $\mathbb{C}^{d+1}$ and lower case letters for vectors in $\mathbb{C}^d$.

To begin let us write

$$F(\Sigma) := \log(h(\Sigma))$$

for convenience. Differentiating in barycentric coordinates,

$$\frac{\partial}{\partial\lambda_j}F(\Sigma) = \frac{1}{2}F'(\Sigma)\bar{\lambda}_j/|\lambda_j|.$$

The second partials are given by

$$\frac{\partial^2 F(\Sigma)}{\partial\lambda_j\partial\bar{\lambda}_k} = \frac{1}{4}F''(\Sigma)\frac{\bar{\lambda}_j\lambda_k}{|\lambda_j||\lambda_k|} \quad \text{if } k\neq j, \tag{9}$$

$$\frac{\partial^2 F(\Sigma)}{\partial\lambda_j\partial\bar{\lambda}_j} = \frac{1}{4}F''(\Sigma)+\frac{1}{4}\frac{F'(\Sigma)}{|\lambda_j|} \tag{10}$$

for all $j=0,\ldots,d$.

A calculation gives the following formulas:

$$F'(\Sigma)=\frac{1}{\sqrt{\Sigma^2-1}},\quad F''(\Sigma)=-\frac{\Sigma}{(\Sigma^2-1)^{3/2}},\quad\text{and } \frac{F''(\Sigma)}{F'(\Sigma)}=-\left(\frac{\Sigma}{\Sigma^2-1}\right). \tag{11}$$

Let $H_{jk}^{(d+1)}=\partial^2F(\Sigma)/\partial\lambda_j\partial\bar{\lambda}_k$ and write $H^{(d+1)}=[H_{jk}]_{j,k=0}^d\in\mathbb{C}^{(d+1)\times(d+1)}$ to denote the complex Hessian in barycentric coordinates, considered as independent coordinates. Indeed, (9) and (10) may be written as

$$\begin{aligned}4H^{(d+1)} &= F''(\Sigma)\begin{bmatrix}\frac{\bar{\lambda}_0}{|\lambda_0|}\\ \vdots\\ \frac{\bar{\lambda}_d}{|\lambda_d|}\end{bmatrix}\begin{bmatrix}\frac{\lambda_0}{|\lambda_0|} & \cdots & \frac{\lambda_d}{|\lambda_d|}\end{bmatrix} + F'(\Sigma)\begin{bmatrix}\frac{1}{|\lambda_0|} & 0 & \cdots & 0\\ 0 & \frac{1}{|\lambda_1|} & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{1}{|\lambda_d|}\end{bmatrix}\\ &=: F''(\Sigma)\mathbf{U}\mathbf{U}^*+F'(\Sigma)D.\end{aligned}$$

Then, in particular, it follows that

$$
\begin{aligned}
\mathbf{Z}^* H^{(d+1)}\mathbf{Z} &= \frac{1}{4}\left(F''(\Sigma)\mathbf{Z}^*\mathbf{U}\mathbf{U}^*\mathbf{Z} + F'(\Sigma)\mathbf{Z}^* D\mathbf{Z}\right)\\
&= \frac{1}{4}\left(F''(\Sigma)|\mathbf{Z}^*\mathbf{U}|^2 + F'(\Sigma)\mathbf{Z}^* D\mathbf{Z}\right)\\
&= \frac{1}{4}F'(\Sigma)\left(\frac{F''(\Sigma)}{F'(\Sigma)}|\mathbf{Z}^*\mathbf{U}|^2 + \mathbf{Z}^* D\mathbf{Z}\right)\\
&= \frac{1}{4}F'(\Sigma)\left(-\left(\frac{\Sigma}{\Sigma^2-1}\right)|\mathbf{U}^*\mathbf{Z}|^2 + \mathbf{Z}^* D\mathbf{Z}\right) \qquad (12)
\end{aligned}
$$

for $\mathbf{Z} \in \mathbb{C}^{d+1}$ (considered as a column vector).

We need to reduce the Hessian $H^{(d+1)}$ in barycentric coordinates in $\mathbb{C}^{d+1}$ to affine coordinates in $\mathbb{C}^d$. Define

$$f(z_1, \ldots, z_d) = f(\mathbf{z}) := F(\lambda_0(\mathbf{z}), \lambda_1(\mathbf{z}), \ldots, \lambda_d(\mathbf{z}))$$

with $\lambda_0(\mathbf{z}) = 1 - z_1 - \cdots - z_d$ and $\lambda_j(\mathbf{z}) = z_j$ for $j = 1, \ldots, d$. The chain rule yields

$$
\begin{aligned}
\frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} &= \frac{\partial^2 F}{\partial \lambda_j \partial \bar{\lambda}_k} - \frac{\partial^2 F}{\partial \lambda_j \partial \bar{\lambda}_0} + \frac{\partial^2 F}{\partial \lambda_0 \partial \bar{\lambda}_0} - \frac{\partial^2 F}{\partial \lambda_0 \partial \bar{\lambda}_k}\\
&= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 F}{\partial \lambda_0 \partial \bar{\lambda}_0} & \frac{\partial^2 F}{\partial \lambda_0 \partial \bar{\lambda}_k} \\ \frac{\partial^2 F}{\partial \lambda_j \partial \bar{\lambda}_0} & \frac{\partial^2 F}{\partial \lambda_j \partial \bar{\lambda}_k} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.
\end{aligned}
$$

Let $H_f^{(d)} := [\partial^2 f/\partial z_j \partial \bar{z}_k]_{j,k=1}^d$ denote the Hessian of f; then the above calculation yields

$$H_f^{(d)} = B^* H^{(d+1)} B, \quad \text{where } B = \begin{bmatrix} -1 & -1 & \cdots & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{C}^{(d+1)\times d}.$$

Proposition 6.1. *Let $\mathbf{z} \in \mathbb{C}^d \setminus S_d$. The complex Hessian $H_f^{(d)} = B^* H^{(d+1)} B \in \mathbb{C}^{d\times d}$ is non-negative definite and singular at $\mathbf{z}$. In particular, the vector $\mathbf{v} \in \mathbb{C}^d$ with components $v_j = |\lambda_j| - \bar{\lambda}_j \Sigma$, where $\lambda_j = \lambda_j(\mathbf{z})$ for each $j = 1, \ldots, d$, is in the kernel of $H_f^{(d)}$.*

Proof. For $\mathbf{w} = (w_1, \cdots, w_d) \in \mathbb{C}^d$ let

$$w_0 := -\sum_{j=1}^d w_j, \text{ and } \mathbf{W} := [w_0, \mathbf{w}] \in \mathbb{C}^{d+1}$$

so that

$$\mathbf{w}^* H_f^{(d)} \mathbf{w} = \mathbf{W}^* H^{(d+1)} \mathbf{W}.$$

Note that for the specific vector $\mathbf{v}$, defined above,

$$\begin{aligned} v_0 &= -\sum_{j=1}^{d}\{|\lambda_j| - \bar{\lambda}_j\Sigma\} \\ &= -\{(\Sigma - |\lambda_0|) - (1-\bar{\lambda}_0)\Sigma\} \\ &= |\lambda_0| - \bar{\lambda}_0\Sigma. \end{aligned}$$

Hence the vector $\mathbf{V} := [v_0, \mathbf{v}] \in \mathbb{C}^{d+1}$ is given by

$$V_j = |\lambda_j| - \bar{\lambda}_j\Sigma,\ \ 0 \le j \le d.$$

Now define the weighted scalar product

$$\langle \mathbf{A}, \mathbf{B}\rangle_\lambda := \sum_{j=0}^{d} \tfrac{1}{|\lambda_j|} A_j \bar{B}_j,\ \ \mathbf{A}, \mathbf{B} \in \mathbb{C}^{d+1}.$$

Note that, with this notation,

$$\begin{aligned} \langle \mathbf{W}, \mathbf{V}\rangle_\lambda &= \sum_{j=0}^{d} \frac{W_j(|\lambda_j| - \lambda_j\Sigma)}{|\lambda_j|} \\ &= \sum_{j=0}^{d} W_j - \left(\sum_{j=0}^{d} \frac{\lambda_j W_j}{|\lambda_j|}\right)\Sigma \\ &= 0 - \left(\sum_{j=0}^{d} \frac{\lambda_j W_j}{|\lambda_j|}\right)\Sigma \\ &= -(\mathbf{U}^*\mathbf{W})\Sigma \end{aligned}$$

while

$$\mathbf{W}^* D\mathbf{W} = \sum_{j=0}^{d} \frac{|W_j|^2}{|\lambda_j|} = \langle \mathbf{W}, \mathbf{W}\rangle_\lambda.$$

Hence, by (12), we may write

$$\begin{aligned} \mathbf{W}^* H^{(d+1)}\mathbf{W} &= \frac{1}{4}F'(\Sigma)\left(-\left(\frac{\Sigma}{\Sigma^2-1}\right)|\mathbf{U}^*\mathbf{W}|^2 + \mathbf{W}^* D\mathbf{W}\right) \\ &= \frac{1}{4}F'(\Sigma)\left(-\left(\frac{1}{\Sigma(\Sigma^2-1)}\right)|\langle \mathbf{W}, \mathbf{V}\rangle_\lambda|^2 + \langle \mathbf{W}, \mathbf{W}\rangle_\lambda\right). \end{aligned} \tag{13}$$

Now, by the Cauchy-Schwartz inequality,

$$|\langle \mathbf{W}, \mathbf{V}\rangle_\lambda|^2 \le \langle \mathbf{V}, \mathbf{V}\rangle_\lambda \langle \mathbf{W}, \mathbf{W}\rangle_\lambda \tag{14}$$

where

$$\begin{aligned}
\langle \mathbf{V},\mathbf{V}\rangle_\lambda &= \sum_{j=0}^{n} \frac{|V_j|^2}{|\lambda_j|} \\
&= \sum_{j=0}^{d} \frac{(|\lambda_j|-\bar\lambda_j\Sigma)(|\lambda_j|-\lambda_j\Sigma)}{|\lambda_j|} \\
&= \sum_{j=0}^{d} \frac{|\lambda_j|^2-|\lambda_j|(\bar\lambda_j+\lambda_j)\Sigma+|\lambda_j|^2\Sigma}{|\lambda_j|} \\
&= (1+\Sigma^2)\sum_{j=0}^{d}|\lambda_j| - \left(\Sigma_{j=0}^{d}\bar\lambda_j+\Sigma_{j=0}^{d}\lambda_j\right)\Sigma \\
&= (1+\Sigma^2)\Sigma - 2\Sigma = \Sigma(\Sigma^2-1).
\end{aligned}$$

In other words,

$$\left(\frac{1}{\Sigma(\Sigma^2-1)}\right)|\langle \mathbf{W},\mathbf{V}\rangle_\lambda|^2 \le \langle \mathbf{W},\mathbf{W}\rangle_\lambda$$

so by (13),

$$\mathbf{w}^* H_f^{(d)}\mathbf{w} = \mathbf{W}^* H^{(d+1)}\mathbf{W} \ge 0.$$

If $\mathbf{W}=\mathbf{V}$, i.e., $\mathbf{w}=\mathbf{v}$, then we have equality in (14) and $\mathbf{W}^*H^{(d+1)}\mathbf{W}=0$, i.e., $\mathbf{v}$ is in the kernel of $H_f^{(d)}$. □

7. The case of K a convex body

Consider $K\subset\mathbb{R}^d$ a convex body, i.e., a compact, covex set with non-empty interior.

As mentioned previously, the case of $K\subset\mathbb{R}^d$ a convex body, symmetric with respect to the origin, corresponds to that of K the unit ball for some norm and is handled in Section 3. It turns out that one may also find the extremal polynomials at *real* points for not necessarily centrally symmetric convex bodies.

First recall that a real hyperplane $H_{\mathbf{a}}\subset\mathbb{R}^d$ is a supporting hyperplane for K at $\mathbf{a}\in\partial K$ if $\mathbf{a}\in H_{\mathbf{a}}$ and K lies entirely in one of the two half-spaces determined by H_a, i.e., if $H_{\mathbf{a}}$ is given by

$$H_{\mathbf{a}} = \{\mathbf{x}\in\mathbb{R}^d \,:\, \ell(\mathbf{x})=\ell(\mathbf{a})\}$$

where $\ell:\mathbb{R}^d\to\mathbb{R}$ is a linear functional on $\mathbb{R}^d$, then $\ell(\mathbf{x}-\mathbf{a})$ is of constant sign for $\mathbf{x}\in K$. The important geometric property that we require is given by Kroo and Schmidt [12, Cor. 1].

Theorem 7.1. *(Kroo and Schmidt) Let $K\subset\mathbb{R}^d$ be a convex body. Then for each $\mathbf{x}_0\in\mathbb{R}^d\backslash K$, there exists two points $\mathbf{a},\mathbf{b}\in\partial K$ with respective supporting hyperplanes $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$ such that (i) $\mathbf{a},\mathbf{b}$ and $\mathbf{x}_0$ are collinear and (ii) $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$ are parallel, i.e., K is contained in the bi-infinite "strip" between the two parallel supporting hyperplanes.*

Note that we may rephrase this as: for each $\mathbf{x}_0\in\mathbb{R}^d\backslash K$ there are points $\mathbf{a},\mathbf{b}\in\partial K$ and a linear functional $\ell:\mathbb{R}^d\to\mathbb{R}$ such that $\mathbf{a},\mathbf{b}$ and $\mathbf{x}_0$ are collinear and that $\ell(K)=[\ell(\mathbf{a}),\ell(\mathbf{b})]$.

From this we may easily provide a formula for extremal polynomials.

Proposition 7.2. *([12, Thm. 1A]) Let $K\subset\mathbb{R}^d$ be a convex body. Suppose that $p\in\mathbb{C}[x]$ is such that $\|p\|_K\le 1$, and that $\mathbf{x}_0\in\mathbb{R}^d\backslash K$. Then for the $\mathbf{a},\mathbf{b}$ and ℓ provided by Theorem 7.1,*

$$|p(\mathbf{x}_0)|\le |T_n(2\lambda(\mathbf{x}_0)-1)|,\quad n:=\deg(p),$$

where

$$\lambda(\mathbf{x}):=\frac{\ell(\mathbf{x})-\ell(\mathbf{a})}{\ell(\mathbf{b})-\ell(\mathbf{a})}$$

is the barycentric coordinate of $\ell(\mathbf{x})$ with respect to the interval $[\ell(\mathbf{a}),\ell(\mathbf{b})]$. Moreover, this upper bound is attained for

$$p(\mathbf{x}):=T_n(2\lambda(\mathbf{x})-1).$$

Proof. Suppose that $p\in\mathbb{R}[x]$ is such that $\|p\|_K\le 1$, and that $\mathbf{x}_0\in\mathbb{R}^d\backslash K$ First note that, as $\mathbf{a},\mathbf{b}$ and $\mathbf{x}_0$ are collinear, we have

$$\mathbf{x}_0=\lambda(\mathbf{x}_0)\mathbf{b}+(1-\lambda(\mathbf{x}_0))\mathbf{a}.$$

Now let

$$q(t):=p(\left(\frac{1+t}{2}\right)\mathbf{b}+\left(\frac{1-t}{2}\right)\mathbf{a}),\quad t\in\mathbb{R}$$

be the restriction of p to the line through $\mathbf{a},\mathbf{b}$ and $\mathbf{x}_0$. Note that $q(t_0)=p(\mathbf{x}_0)$ for $t_0:=2\lambda(\mathbf{x}_0)-1$. Further, by convexity, $\left(\frac{1+t}{2}\right)\mathbf{b}+\left(\frac{1-t}{2}\right)\mathbf{a})\in K$ for $-1\le t\le 1$. Hence $\|q\|_{[-1,1]}\le\|p\|_k\le 1$ and so by the extremal property of Chebyshev polynomials,

$$|p(\mathbf{x}_0)|=|q(t_0)|\le|T_n(t_0)|=|T_n(2\lambda(\mathbf{x}_0)-1)|.$$

If we let $p(\mathbf{x}):=T_n(2\lambda(\mathbf{x})-1)$, then for $\mathbf{x}\in K$, $\ell(\mathbf{x})\in[\ell(\mathbf{a}),\ell(\mathbf{b})]$ and consequently, $2\lambda(\mathbf{x})-1\in[-1,1]$. It follows that $\|p\|_K=1$, and obviously, $p(\mathbf{x}_0)=T_n(2\lambda(\mathbf{x}_0)-1)$. □

Although this formula has the appearance of being quite simple, in reality it requires the sometimes difficult determination of the special points $\mathbf{a}$ and $\mathbf{b}$. It turns out that the extremal polynomial may be expressed in variational form solely in terms of the linear maps $\ell(\mathbf{x})$.

Proposition 7.3. *Let $K\subset\mathbb{R}^d$ be a convex body. Suppose that $p\in\mathbb{C}[x]$ is such that $\|p\|_K\le 1$, and that $\mathbf{x}_0\in\mathbb{R}^d\backslash K$. Then*

$$|p(\mathbf{x}_0)|\le\max_{\ell:\mathbb{R}^d\to\mathbb{R},\ \text{linear}}|T_n(2\lambda(\ell(\mathbf{x}_0))-1)|,\quad n:=\deg(p), \tag{15}$$

where $\lambda(\ell(\mathbf{x}))$ is the barycentric coordinate of $\ell(\mathbf{x})$ with respect to the interval $\ell(K)$. Moreover, this upper bound is attained for

$$p(\mathbf{x}):=T_n(2\lambda(\ell_{\mathbf{x}_0}(\mathbf{x}))-1)$$

where $\ell_{\mathbf{x}_0}$ is the linear mapping for which the maximum in (15) *is attained.*

Proof. For any linear map $\ell:\mathbb{R}^d\to\mathbb{R}$ consider the polynomial

$$q(\mathbf{x}):=T_n(2\lambda(\ell_{\mathbf{x}_0}(\mathbf{x}))-1).$$

Then $q(\mathbf{x})$ is of degree at most n and is such that $\|q\|_K\le 1$. Hence,

$$\max_{\|p\|_K\le 1,\ \deg(p)\le n}|p(\mathbf{x}_0)|\ge\max_{\ell:\mathbb{R}^d\to\mathbb{R}\ \text{linear}}|T_n(2\lambda(\ell(\mathbf{x}_0))-1)|.$$

On the other hand, in view of the special choice of ℓ given in Proposition 7.2,

$$\max_{\|p\|_K\le 1,\ \deg(p)\le n}|p(\mathbf{x}_0)|\le \max_{\ell:\mathbb{R}^d\to\mathbb{R}\text{ linear}}|T_n(2\lambda(\ell(\mathbf{x}_0))-1)|.$$

□

Remark. Note that $\lambda(\ell(\mathbf{x}))$ remains the same if ℓ is replaced by a scalar multiple, so we may assume $\|\ell\|_K\le 1$. If we identify linear maps $\ell(\mathbf{x})=\mathbf{y}^t\mathbf{x}$, with their "normal vectors" $\mathbf{y}\in\mathbb{R}^d$, this means that we may express (15) using the polar of K, K°, as

$$\begin{aligned}|p(\mathbf{x}_0)| &\le \max_{\ell\in K^\circ}|T_n(2\lambda(\ell(\mathbf{x}_0))-1)|,\quad n:=\deg(p),\\ &= T_n(\max_{\ell\in K^\circ}|2\lambda(\ell(\mathbf{x}_0))-1)|)\end{aligned}\tag{16}$$

where the second statement follows from the monotinicity of $|T_n(x)|$ outside the interval $[-1,1]$.

In the special case that K is centrally symmetric with respect to the origin then $\ell(K)$ will be a symmetric interval of the form $[-a,a]$ and the expression (16) will be scale invariant. Thus it suffices to take the maximum in (16) over ∂K° and even, since the maximum is a linear problem, over the extreme points of K°. A calculation also yields

$$\ell(K)=[-1,1]\text{ and }2\lambda(\ell(\mathbf{x}_0))-1=\ell(\mathbf{x}_0),\quad\text{for all }\ell\in\partial K^\circ.$$

Therefore, in the case that K is centrally symmetric,

$$|p(\mathbf{x}_0)|\le T_n(\max_{\ell\in\operatorname{ext}(K^\circ)}|\ell(\mathbf{x}_0)|).$$

□

In the case of K a polytope, even if K is not necessarily centrally symmetric, the variational problem of Proposition 7.3 can often be simplified to finding the maximum over a *finite* set of linear maps ℓ.

Example. Consider K the triangle with vertices $\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2\in\mathbb{R}^2$. The linear maps ℓ that need to be considered are, by Proposition 7.2, those that provide parallel supporting hyperplanes at two points $\mathbf{a},\mathbf{b}\in K$, collinear with $\mathbf{x}_0\in\mathbb{R}^2\backslash K$. If one of them, say $\mathbf{a}$, is in the interior of one of the sides of the triangle K, then ℓ must necessarily be that which gives the equation of that side. If instead, both $\mathbf{a}$ and $\mathbf{b}$ are vertices, then the hyperplanes $H_{\mathbf{a}}$ and $H_{\mathbf{b}}$ need not be along a side, but they can be rotated until one of the planes aligns with a side. Now, the equations of the three sides are given by $\ell_j(\mathbf{x}):=\lambda_j(\mathbf{x})=1$, $j=0,1,2$ and so, in particular, $\ell_j(K)=[0,1]$, $j=0,1,2$, so that the barycentric coordinate of $\ell_j(\mathbf{x}_0)$ with respect to $\ell_j(K)=[0,1]$ is just $\ell_j(\mathbf{x}_0)$ itself. Hence,

$$\max_{\ell:\mathbb{R}^d\to\mathbb{R},\text{ linear}}|2\lambda(\ell(\mathbf{x}_0))-1|=\max_{j=0,1,2}|2\lambda_j(\mathbf{x}_0)-1|.$$

It is interesting to compare this expression with the corresponding one given in Proposition 5.1, $|\lambda_0(\mathbf{x}_0)|+|\lambda_1(\mathbf{x}_0)|+|\lambda_2(\mathbf{x}_0)|$. Although not immediately obvious they are indeed the same for exterior points $\mathbf{x}_0\in\mathbb{R}^2\backslash K$. To see this, first note that

$$2\lambda_j(\mathbf{x}_0)-1=2\lambda_j(\mathbf{x}_0)-\sum_{k=0}^{2}\lambda_k(\mathbf{x}_0)$$

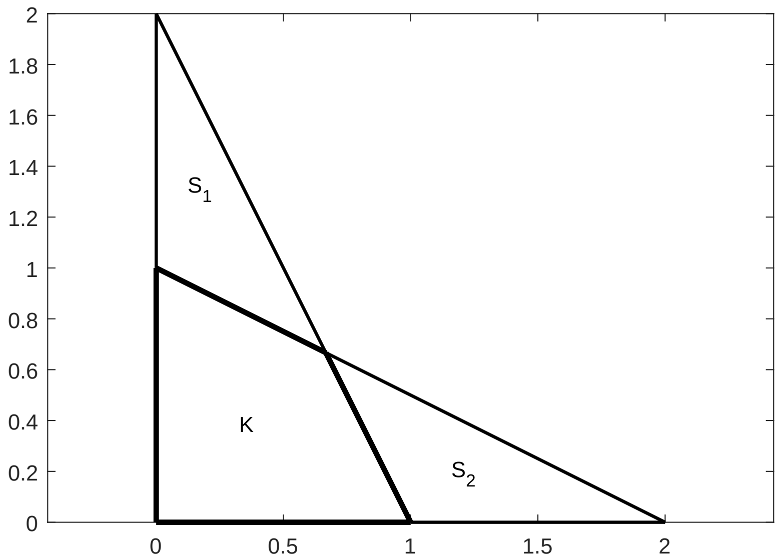

FIGURE 1. The Quadrilateral K

so that

$$\begin{aligned}|2\lambda_0(\mathbf{x}_0)-1| &= \pm(\lambda_0(\mathbf{x}_0)-\lambda_1(\mathbf{x}_0)-\lambda_2(\mathbf{x}_0)),\\ |2\lambda_1(\mathbf{x}_0)-1| &= \pm(-\lambda_0(\mathbf{x}_0)+\lambda_1(\mathbf{x}_0)-\lambda_2(\mathbf{x}_0)),\\ |2\lambda_2(\mathbf{x}_0)-1| &= \pm(-\lambda_0(\mathbf{x}_0)-\lambda_1(\mathbf{x}_0)+\lambda_2(\mathbf{x}_0)).\end{aligned}$$

For an exterior point $\mathbf{x}_0 \in \mathbb{R}^2\backslash K$, the barycentric coordinates cannot all be positive, and they cannot also all be negative as their sum is 1. Hence

$$\max_{j=0,1,2}|2\lambda_j(\mathbf{x}_0)-1| = \max_{\epsilon\in\{-1,+1\}^3}\sum_{j=0}^{d}\epsilon j\lambda_j(\mathbf{x}_0) = \sum_{j=0}^{2}|\lambda_j(\mathbf{x}_0)|.$$

□

Similarly, for a convex polygon in $\mathbb{R}^2$ one need only maximize over the linear functionals that correspond to the edges of the polygon. A similar statement may be made for convex polytopes in $\mathbb{R}^d$, but it will be necessary to consider certain "complementary" pairs of lower dimensional faces. We omit the (combinatorial) details.

Finally, we mention that it is sometimes also possible to obtain simplified formulas for extremal polynomials by decomposing K into simplices, instead of just the "strips" given by parallel supporting hyperplanes.

Example. Consider $K \subset \mathbb{R}^2$ the quadrilateral with vertices $(0,0)$, $(0,1)$, $(2/3,2/3)$ and $(1,0)$, shown in Figure 1.

Clearly K is the intersection of the two triangles S_1 and S_2 shown in the figure. Note also that each edge of K lies in an edge of either S_1 or else S_2. Hence, if we let $\lambda_j(\mathbf{x})$ be the barycentric coordinates with respect to S_1 and $\mu_j(\mathbf{x})$ those with respect to S_2, by the extremal formula for the triangle,

$$|p(\mathbf{x}_0)| \le \max\left\{T_n(\sum_{j=0}^2 |\lambda_j(\mathbf{x}_0)|), T_n(\sum_{j=0}^2 |\mu_j(\mathbf{x}_0)|\right\}$$

for $p \in \mathbb{C}[\mathbf{x}]$ with $\|p\|_K \le 1$ and $n := \deg(p)$. This approach can also be used to give formulas for the extremal function (even at complex points). The details will be given in the forthcoming paper [13]. $\square$

8. The case of K a real Euclidean ball

Suppose now that K is the real Euclidean unit ball, i.e., $K = B := \{\mathbf{x} \in \mathbb{R}^d : \|x\|_2 \le 1\}$. It turns out that given the extremal function for the real simplex one may easily derive that for the real ball. We remark that in the literature (see e.g. Klimek [11, Thm. 5.4.6]) what is usually done is to first compute the extremal function for the ball, making good use of Pluripotential Theory, and then using that to give the formula for the simplex. Ours is the reverse process, which we maintain can be understood by more elementary means.

Proposition 8.1. *For $K = B$ the real Euclidean unit ball,*

$$V_B(\mathbf{z}) = \frac{1}{2}\log\big(h\big(\|z\|_2^2 + \big|\sum_{j=1}^d z_j^2 - 1\big|\big)\big),\ \mathbf{z} \in \mathbb{C}^d.$$

Proof. The proof is based on the following observations:

(1) for any $\mathbf{z}_0 \in \mathbb{C}^d \backslash K$, there is a sequence of polynomials P_k, with

$$\lim_{k\to\infty} \deg(P_k) = \infty \text{ and } \lim_{k\to\infty} \frac{1}{\deg(P_k)} \log\big(|P_k(\mathbf{z}_0)|\big) = V_B(\mathbf{z}_0).$$

(2) it may be assumed that the $P_k(\mathbf{z})$ are symmetric in each variable separately, i.e., that $P_k(z_1, \cdots, z_d) = p_k(z_1^2, \cdots, z_d^2)$ for some polynomial $p_k(\mathbf{z})$.

(3) if $P_k(z_1, \cdots, z_d) = p_k(z_1^2, \cdots, z_d^2)$ then $\|P_k\|_B = \|p_k\|_S$ where now S denotes the *standard* unit simplex

$$S = \{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{j=1}^d x_j \le 1, x_j \ge 0\ \forall j\}.$$

(4) as $\deg(P_k) = 2\deg(p_k)$

$$V_B(z_1 \cdots, z_d) = \frac{1}{2} V_S(z_1^2, \cdots, z_d^2).$$

Proof of (1). For fixed $\mathbf{z}_0 \in \mathbb{C}^d \backslash B$, let

$$P_k(\mathbf{z}_0) := \text{argmax}\, \{\log|P(\mathbf{z}_0)| : \|P\|_B \le 1,\ \deg(P) \le k\}.$$

Then

$$V_B(\mathbf{z}_0) = \sup_{\|P\|_B\le 1} \frac{1}{\deg(P)} \log\big(|P(\mathbf{z}_0)|\big) = \sup_{k\ge 1} \frac{1}{k} \log\big(|P_k(\mathbf{z}_0)|\big).$$

We claim that there is in fact a subsequence $\{k_n\}$ of the integers such that $k_n \to \infty$ and

$$\lim_{n\to\infty} \frac{1}{k_n} \log\bigl(|P_{k_n}(\mathbf{z}_0)|\bigr) = V_B(\mathbf{z}_0).$$

Indeed, if there exists a k_0 such that

$$V_B(\mathbf{z}_0) = \frac{1}{k_0} \log\bigl(|P_{k_0}(\mathbf{z}_0)|\bigr)$$

then, for any $m \geq 1$,

$$\begin{aligned} V_B(\mathbf{z}_0) &= \frac{1}{k_0} \log\bigl(|P_{k_0}(\mathbf{z}_0)|\bigr) \\ &= \frac{1}{mk_0} \log\bigl(|(P_{k_0}(\mathbf{z}_0))^m|\bigr) \\ &\leq \frac{1}{mk_0} \log\bigl(|P_{mk_0}(\mathbf{z}_0)|\bigr) \\ &\leq V_B(\mathbf{z}_0). \end{aligned}$$

In other words,

$$\frac{1}{mk_0} \log\bigl(|P_{mk_0}(\mathbf{z}_0)|\bigr) = V_B(\mathbf{z}_0)$$

for any $m \geq 1$, and consequently

$$\frac{1}{m^j k_0} \log\bigl(|P_{m^j k_0}(\mathbf{z}_0)|\bigr) = V_B(\mathbf{z}_0)$$

for all $j \geq 1$, $m > 1$. $\square$

Proof of (2). Considering the jth variable z_j, let

$$P_k^e(z_1, \cdots, z_d) := \frac{P_k(z_1, \cdots, z_j, \cdots, z_d) + P_k(z_1, \cdots, -z_j, \cdots, z_d)}{2},$$

$$P_k^o(z_1, \cdots, z_d) := \frac{P_k(z_1, \cdots, z_j, \cdots, z_d) - P_k(z_1, \cdots, -z_j, \cdots, z_d)}{2}$$

be the even and odd symmetrizations of $P_k(\mathbf{z})$, respectively. Note that $\deg(P_k^e) \leq \deg(P_k)$ and $\deg(P_k^o) \leq \deg(P_k)$ with either $\deg(P_k^e) = \deg(P_k)$ or $\deg(P_k^o) = \deg(P_k)$ (or both). Note also that, as $\|P_k\|_B \leq 1$,

$$\|P_k^e\|_B \leq 1 \text{ and} \|P_k^o\|_B \leq 1.$$

We claim that either

$$\lim_{k\to\infty} \frac{1}{\deg(P_k^e)} \log\bigl(|P_k^e(\mathbf{z}_0)|\bigr) = V_B(\mathbf{z}_0) \text{ with } \lim_{k\to\infty} \deg(P_k^e) = \infty$$

or

$$\lim_{k\to\infty} \frac{1}{\deg(P_k^o)} \log\bigl(|P_k^o(\mathbf{z}_0)|\bigr) = V_B(\mathbf{z}_0) \text{ with } \lim_{k\to\infty} \deg(P_k^o) = \infty$$

(or both). Indeed,

$$\begin{aligned} \frac{1}{\deg(P_k)} \log\bigl(|P_k(\mathbf{z}_0)|\bigr) &= \frac{1}{\deg(P_k)} \log\bigl(|P_k^e(\mathbf{z}_0) + P_k^o(\mathbf{z}_0)|\bigr) \\ &\leq \frac{1}{\deg(P_k)} \log\bigl(2\max\{|P_k^e(\mathbf{z}_0)|, |P_k^o(\mathbf{z}_0)\}|\bigr) \end{aligned}$$

Consequently, as $\deg(P_k)\to\infty$, we have $\frac{1}{\deg(P_k)}\log 2\to 0$ and

$$\lim_{k\to\infty}\frac{1}{\deg(P_k)}\log\bigl(\max\{|P_k^e(\mathbf{z}_0)|,|P_k^o(\mathbf{z}_0)|\}\bigr)=V_B(\mathbf{z}_0).$$

But,

$$\begin{aligned}&\frac{1}{\deg(P_k)}\log\bigl(\max\{|P_k^e(\mathbf{z}_0)|,|P_k^o(\mathbf{z}_0)|\}\bigr)\\ &\quad\le\max\left\{\frac{1}{\deg(P_k)}\log\bigl(|P_k^e(\mathbf{z}_0)|\bigr),\frac{1}{\deg(P_k)}\log\bigl(|P_k^o(\mathbf{z}_0)|\bigr)\right\}\\ &\quad\le\max\left\{\frac{1}{\deg(P_k^e)}\log\bigl(|P_k^e(\mathbf{z}_0)|\bigr),\frac{1}{\deg(P_k^o)}\log\bigl(|P_k^o(\mathbf{z}_0)|\bigr)\right\}\\ &\quad\le V_B(\mathbf{z}_0).\end{aligned}$$

Hence, passing to a subsequence if necessary, we have that indeed either

$$\lim_{k\to\infty}\frac{1}{\deg(P_k^e)}\log\bigl(|P_k^e(\mathbf{z}_0)|\bigr)=V_B(\mathbf{z}_0)$$

or

$$\lim_{k\to\infty}\frac{1}{\deg(P_k^o)}\log\bigl(|P_k^o(\mathbf{z}_0)|\bigr)=V_B(\mathbf{z}_0)$$

(or both), as claimed. If necessary, we may use the same argument as for (1) and conclude that the limits of the degrees are also infinity.

Now, if it is the case that the *odd* polynomials $P_k^o(\mathbf{z})$ give the limit, note that we may write $P_k^o(\mathbf{z})=z_jQ_k(\mathbf{z})$ for some polynomial $Q_k(\mathbf{z})$ which is *even* in z_j. Note that then $(\mathbf{z}_0)_j\neq 0$ for otherwise $P_k^o(\mathbf{z}_0)=0$ and is not a competitor for the extremal function. Hence

$$\begin{aligned}V_B(\mathbf{z}_0)&=\lim_{k\to\infty}\frac{1}{\deg(P_k^o)}\log\bigl(|P_k^o(\mathbf{z}_0)|\bigr)\\ &=\lim_{k\to\infty}\frac{1}{\deg(P_k^o)}\log\bigl(|(\mathbf{z}_0)_j|\,|Q_k(\mathbf{z}_0)|\bigr)\\ &=\lim_{k\to\infty}\frac{1}{\deg(P_k^o)}\{\log\bigl(|Q_k(\mathbf{z}_0)|\bigr)+\log|(\mathbf{z}_0)_j|\}\\ &=\lim_{k\to\infty}\frac{1}{\deg(P_k^o)}\log\bigl(|Q_k(\mathbf{z}_0)|\bigr)\\ &=\lim_{k\to\infty}\frac{1}{\deg(Q_k)}\log\bigl(|Q_k(\mathbf{z}_0)|\bigr).\end{aligned}$$

There is one more technical point to check. The candidates for the extremal function are those with max norm on B at most 1. What can be said about $Q_k(\mathbf{z})$? Note that, as $P_k^o(\mathbf{z})=z_jQ_k(\mathbf{z})$, we may express

$$Q_k(\mathbf{z})=\frac{P_k^o(\mathbf{z})}{z_j}=\int_0^1\frac{\partial P_k^o}{\partial z_j}(z_1,\cdots,tz_j,\cdots,z_d)dt$$

and so by Markov's inequality on the ball (cf. [4]),

$$\|Q_k\|_B\le(\deg(P_k^o))^2\|P_k^o\|_B\le(\deg(P_k^o))^2.$$

Replacing Q_k by $Q_k/(\deg(P_k^o))^2$ has no effect on the limit and hence we may assume that $\|Q_k\|_B\le 1$.

In other words, even if the limit is given by the *odd* polynomials, we may replace them by a sequence of *even* polynomials.

Repeating the procedure for each variable we may conclude that there is a sequence of polynomials $P_k(\mathbf{z})$ which are even in each variable separately and such that

$$\lim_{k\to\infty}\frac{1}{\deg(P_k)}\log\big(|P_k(\mathbf{z}_0)|\big)=V_B(\mathbf{z}_0).$$

Another way of expressing this is that

$$V_B(\mathbf{z}_0)=\sup\left\{\frac{1}{\deg(P)}\log\big(|P(\mathbf{z}_0)|\big)\ :\ \|P\|_B\le 1,\ P(z_1,\cdots,z_d)=p(z_1^2,\cdots,z_d^2)\right\}.$$

□

Proof of (3) and (4). This is simple. Just note that if $p(\mathbf{u})$ is such that $P(z_1,\cdots,z_d)=p(z_1^2,\cdots,z_d^2)$ then for the change of variables $u_j=z_j^2$, $1\le j\le d$, $\mathbf{u}\in S$ if and only if $\mathbf{z}\in B$. Hence we may write

$$\begin{aligned}V_B(\mathbf{z}_0)&=\sup\left\{\frac{1}{\deg(P)}\log\big(|P(\mathbf{z}_0)|\big)\ :\ \|P\|_B\le 1,\ P(z_1,\cdots,z_d)=p(z_1^2,\cdots,z_d^2)\right\}\\&=\sup\left\{\frac{1}{2\deg(p)}\log\big(|p(\mathbf{z}_0^2)|\big)\ :\ \|p\|_S\le 1\right\}\\&=\frac{1}{2}V_S(\mathbf{z}_0^2).\end{aligned}$$

□

The Proposition now follows by noting that since $\lambda_0=1-\sum_{j=1}^d\lambda_j$, we have

$$\begin{aligned}V_S(\mathbf{z}_0^2)&=\log\Big(h\big(\sum_{j=0}^d|\lambda_j|\big)\Big)\\&=\log\Big(h\big(\sum_{j=1}^d|(\mathbf{z}_0)_j|^2+\big|1-\sum_{j=1}^d(\mathbf{z}_0)_j^2\big|\big)\Big)\\&=\log\Big(h\big(\|\mathbf{z}_0\|_2^2+\big|1-\sum_{j=1}^d(\mathbf{z}_0)_j^2\big|\big)\Big).\end{aligned}$$

□

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