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Multivariate Bernstein operators and redundant systems

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Abstract

The Bernstein operator B_n for a simplex in $\mathbb{R}^d$ is naturally defined via the Bernstein basis obtained from the barycentric coordinates given by its vertices. Here we consider a generalisation of this basis and the Bernstein operator, which is obtained from generalised barycentric coordinates that are given for more general configurations of points in $\mathbb{R}^d$. We call the associated polynomials a Bernstein frame, as they span the polynomials of degree $\leq n$, but may not be a basis. By using this redundant system we are able to give geometrically motivated proofs of some basic properties of the corresponding generalised Bernstein operator, such as the fact it is degree reducing and converges for all polynomials. We also consider the conditions for polynomials in this Bernstein form to join smoothly.

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1. Introduction

The Bernstein operator and its variants have been actively studied for over a century [23,17]. Initially, it was used to give a constructive proof of the Weierstrass density theorem, which culminated in Korovkin's theorem on approximation by positive linear operators [18]. Numerous

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examples have since been given [1], with most being univariate, often with many parameters (akin to the very general families of orthogonal polynomials in the Askey scheme). Over the last forty years, the shape preserving properties of the multivariate Bernstein operator have led to important applications, most notably Bézier curves and surfaces [27,16,15] used in geometric design.

By far the most studied multivariate generalisation is the Bernstein–Durrmeyer operator on a simplex [22,2,9,3,13]. In [24] it was shown that it is not possible to extend the Bernstein operator, and all its properties, to regions which are not simplices (or tensor products of them). Our generalisation is based on a redundant "Bernstein basis", and relaxes the condition of being positive on all of the region.

The Bernstein operator B_n for a simplex in $\mathbb{R}^d$ is defined via the Bernstein basis for $\Pi_n(\mathbb{R}^d)$ (the d-variate polynomials of degree $\leq n$). This basis is obtained by taking powers of the barycentric coordinates given by the vertices of the simplex. In the next section, we outline the basic properties of the affine generalised barycentric coordinates introduced in [29], which are given for more general configurations of points in $\mathbb{R}^d$, e.g., the vertices of a convex polygon. These lead naturally to an analogue of the Bernstein basis, a set of polynomials of degree n which span $\Pi_n(\mathbb{R}^d)$. These are *not* a basis if they are given by more than $d+1$ points, and so we refer to this possibly redundant system as a Bernstein frame (cf. [4]).

In Section 3, we define the generalised Bernstein operator given by a Bernstein frame. We give geometrically motivated proofs of some basic properties of it. These include showing that it is degree reducing and converges for all polynomials, that it reproduces the linear polynomials, and more generally has the same spectral structure as the classical Bernstein operator. Similar arguments in terms of a basis would be far more cumbersome. Finally, we explore some applications of our generalised Bernstein operator. These include a de Casteljau algorithm, shape preservation properties (Section 4), and smoothness conditions in terms of the control points of the associated Bézier surfaces (Section 5).

2. The Bernstein frame

Let V consist of $d+1$ affinely independent points in $\mathbb{R}^d$, i.e., be the vertices of a d-simplex. The **barycentric coordinates** (cf. [10,21]) of a point $x \in \mathbb{R}^d$ with respect to V are the unique coefficients $(\xi_v(x))_{v\in V} \in \mathbb{R}^V$ for which x can be written as an affine combination of the points in V, i.e.,

$$x = \sum_{v\in V} \xi_v(x) v, \qquad \sum_{v\in V} \xi_v(x) = 1. \tag{2.1}$$

We follow [10] and index the barycentric coordinates by the points $v \in V$ that they correspond to, and use standard multi-index notation. It follows, from (2.1), that the ξ_v are linear polynomials which are a basis for $\Pi_1(\mathbb{R}^d)$. More generally, for any $n \geq 1$, the polynomials

$$B_\alpha := \binom{|\alpha|}{\alpha} \xi^\alpha, \quad |\alpha| = n (\alpha \in \mathbb{Z}_+^V)$$

are a basis for $\Pi_n(\mathbb{R}^d)$. Here $|\alpha| = \sum_v \alpha_v$, $\binom{n}{\alpha} = \frac{n!}{\alpha!}$, and $\xi^\alpha = \prod_v \xi_v^{\alpha_v}$.

From now on, let V be a sequence (or multiset) of $m = |V|$ points with affine hull $\mathbb{R}^d$, so that each point $x \in \mathbb{R}^d$ can be written as an affine combination

$$x = \sum_{v\in V} a_v v, \qquad \sum_{v\in V} a_v = 1, \tag{2.2}$$

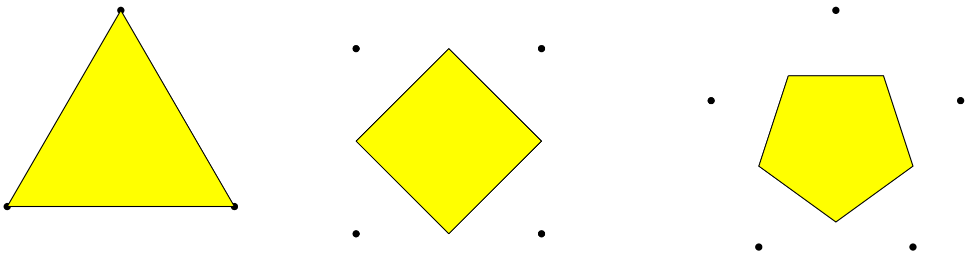

Fig. 1. The region of nonnegativity N_V for V given by the vertices of a triangle, square and pentagon.

where the coefficients $a = (a_v)_{v\in V}$ are unique if and only if V consists of $d+1$ points. Following [29], we call the unique minimal ℓ_2-norm coefficients $a \in \mathbb{R}^V$ satisfying (2.2) the (**generalised barycentric**) **coordinates** given by V, and denote them by $(\xi_v(x))_{v\in V}$. By construction, they satisfy (2.1). Each ξ_v is a linear polynomial, and they span $\Pi_1(\mathbb{R}^d)$, since (2.1) gives

$$1 = \sum_{v\in V} \xi_v(x), \qquad x_j = \sum_{v\in V} \xi_v(x) v_j, \quad j = 1, \ldots, d, \tag{2.3}$$

which is equivalent to the following reproduction formula for affine functions

$$f = \sum_{v\in V} f(v)\xi_v, \quad \forall f \in \Pi_1(\mathbb{R}^d). \tag{2.4}$$

From the formula for ξ_v given in [29] it is easy to see:

- The coordinates of the barycentre $c := \frac{1}{m}\sum_{v\in V} v$ of V are $\xi_v(c) = \frac{1}{m}$, $\forall v$.
- ξ_v is constant (equal to $\frac{1}{m}$) if and only if v is the barycentre c.
- Repeated points have the same coordinates, i.e., $\xi_v = \xi_w$ if and only if $v = w$.
- The ξ_v are continuous functions of the points $v \in V$ (with affine hull $\mathbb{R}^d$).

These imply that the set of points where the coordinates are nonnegative

$$N_V := \{x \in \mathbb{R}^d : \xi_v(x) \geq 0, \forall v \in V\} \tag{2.5}$$

is a convex polytope, with the barycentre of V as an interior point. We call N_V the **region of nonnegativity** for the coordinates given by V (see Fig. 1).

We write $V \setminus w$ for the sequence (or multiset) obtained by removing the point w from V (once), and $\mathrm{Aff}(V)$ for the affine hull of the points in V. We recall:

Proposition 2.6 (*[29]*). *The generalised barycentric coordinates satisfy*

(a) $\frac{1}{m} < \xi_v(v) \leq 1$.
(b) $\xi_v(w) = \xi_w(v)$.
(c) $\xi_v(v) = 1$ *if and only if* $v \notin \mathrm{Aff}(V \setminus v)$, *in which case* $\xi_v = 0$ *on* $\mathrm{Aff}(V \setminus v)$.
(d) $\sum_v \xi_v(v) = d + 1$.

Expanding the monomial basis for $\Pi_n(\mathbb{R}^d)$ in terms of ξ using (2.3), shows that the polynomials

$$B_\alpha := \binom{|\alpha|}{\alpha} \xi^\alpha, \quad |\alpha| = n \tag{2.7}$$

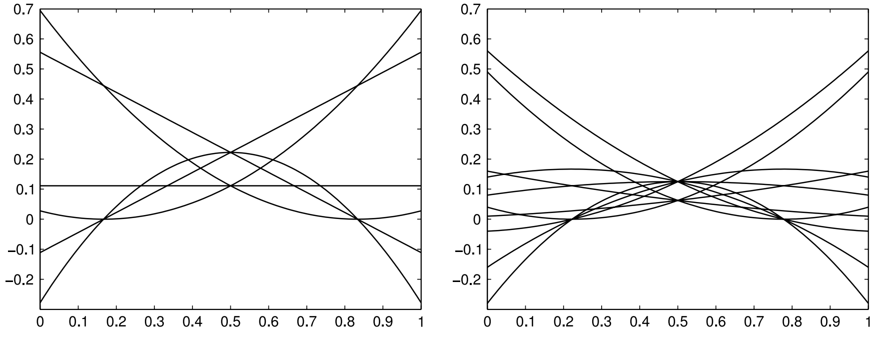

Fig. 2. The Bernstein frame for $\Pi_2(\mathbb{R})$ for $V = \{0, \frac{1}{2}, 1\}$ and $V = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$.

span $\Pi_n(\mathbb{R}^d)$. By a dimension count, these $\binom{n+m-1}{m-1}$ polynomials are basis if and only if V consists of $d+1$ affinely independent points. Thus, we refer to $\{B_\alpha : |\alpha| = n\}$ as the **Bernstein frame** given by the points V. This is a partition of unity, since applying the multinomial theorem to (2.1) gives

$$\sum_{|\alpha|=n} B_\alpha = \sum_{|\alpha|=n} \binom{n}{\alpha} \xi^\alpha = \Big(\sum_{v\in V} \xi_v\Big)^n = 1. \tag{2.8}$$

A Bernstein frame is nonnegative on N_V, the region of nonnegativity given by (2.5). There have been studies of the approximation properties of linear operators given by partitions of unity which may take negative values on the region of interest, see, e.g., [11].

Example 2.9 (*See* Fig. 2)**.** For $V = \{0, \frac{1}{2}, 1\} \subset \mathbb{R}$ the generalised barycentric coordinates are

$$\xi_0(x) = \frac{5}{6} - x, \qquad \xi_{\frac{1}{2}}(x) = \frac{1}{3}, \qquad \xi_1(x) = x - \frac{1}{6}.$$

Here, some polynomials in the Bernstein frame have degree $< n$. This is the case if and only if the barycentre of V is a point of V. The coordinates for $V = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ are

$$\xi_0(x) = \frac{7}{10} - \frac{9}{10}x, \qquad \xi_{\frac{1}{3}}(x) = \frac{2}{5} - \frac{3}{10}x, \qquad \xi_{\frac{2}{3}}(x) = \frac{3}{10}x + \frac{1}{10},$$
$$\xi_1(x) = \frac{9}{10}x - \frac{1}{5}.$$

We say a polynomial $p \in \Pi_n(\mathbb{R}^d)$ is in (Bernstein–Bézier) **B-form** if

$$p = \sum_{|\alpha|=n} c_\alpha B_\alpha. \tag{2.10}$$

The **mesh function** $c : \alpha \mapsto c_\alpha$ is unique if and only if V consists of $d+1$ points. The mesh function with minimal ℓ_2-norm gives a canonical B-form, i.e., what [28] calls the **canonical coordinates** of p with respect to $\{B_\alpha\}_{|\alpha|=n}$.

Many familiar formulas for the Bernstein basis extend to a Bernstein frame. Here are a couple of examples (also see Section 4). Let e_v be the multi-index given by

$$e_v(w) := \begin{cases} 1, & v = w; \\ 0, \end{cases}$$

and define $B_\alpha := 0$ if $\alpha \not\geq 0$.

Proposition 2.11. *The Bernstein frame $\{B_\alpha\}_{|\alpha|=n}$ can be calculated recursively via*

$$B_\alpha = \sum_{v\in V} \xi_v B_{\alpha-e_v}, \qquad B_0 = 1, \tag{2.12}$$

and expressed in terms of the Bernstein frame for polynomials of degree $n+1$ via

$$B_\alpha = \sum_{v\in V} \frac{\alpha_v+1}{|\alpha|+1} B_{\alpha+e_v}. \tag{2.13}$$

Proof. We calculate

$$\sum_{v\in V} \xi_v B_{\alpha-e_v} = \sum_{v\in V} \binom{|\alpha|-1}{\alpha-e_v} \xi^\alpha = \sum_{v\in V} \frac{\alpha_v}{|\alpha|} \binom{|\alpha|}{\alpha} \xi^\alpha = B_\alpha,$$

and, using $\sum_v \xi_v = 1$, that

$$\begin{aligned} B_\alpha &= B_\alpha \sum_{v\in V} \xi_v = \sum_{v\in V} \frac{|\alpha|!}{\alpha!} \xi^\alpha \xi_v = \sum_{v\in V} \frac{\alpha_v+1}{|\alpha|+1} \frac{|\alpha+e_v|!}{(\alpha+e_v)!} \xi^{\alpha+e_v} \\ &= \sum_{v\in V} \frac{\alpha_v+1}{|\alpha|+1} B_{\alpha+e_v}. \qquad \square \end{aligned}$$

Proposition 2.14 (*Differentiation*)**.** *For $u, v, w \in V$, we have*

$$D_{v-w}\xi_u = \xi_u(v) - \xi_u(w) = \xi_v(u) - \xi_w(u).$$

Thus the Bernstein frame satisfies

$$D_{v-w} B_\alpha = |\alpha| \sum_{u\in V} \big(\xi_u(v) - \xi_u(w)\big) B_{\alpha-e_u}.$$

Proof. Since ξ_u is affine

$$\begin{aligned} (D_{v-w}\xi_u)(x) &= \lim_{t\to 0} \frac{\xi_u(x+t(v-w)) - \xi_u(x)}{t} \\ &= \lim_{t\to 0} \frac{\xi_u(x) + t\xi_u(v) - t\xi_u(w) - \xi_u(x)}{t} = \xi_u(v) - \xi_u(w). \end{aligned}$$

By the product and chain rules, we have

$$\begin{aligned} D_{v-w} B_\alpha &= \frac{|\alpha|!}{\alpha!} D_{v-w} \prod_{u\in V} \xi_u^{\alpha_u} = \frac{|\alpha|!}{\alpha!} \sum_{u\in V} \alpha_u \xi_u^{\alpha_u-1} \big(\xi_u(v) - \xi_u(w)\big) \xi^{\alpha-\alpha_u e_u} \\ &= |\alpha| \sum_{u\in V} \big(\xi_u(v) - \xi_u(w)\big) \frac{(|\alpha|-1)!}{(\alpha-e_u)!} \xi^{\alpha-e_u} \\ &= |\alpha| \sum_{u\in V} \big(\xi_u(v) - \xi_u(w)\big) B_{\alpha-e_u}. \qquad \square \end{aligned}$$

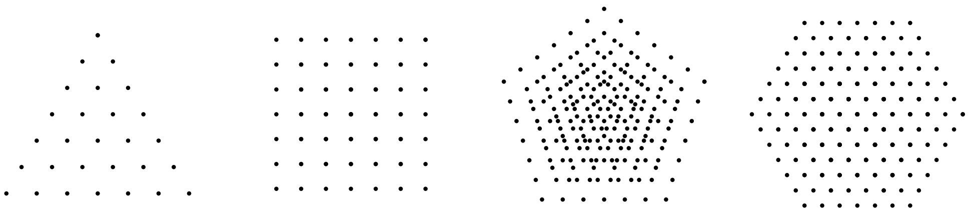

Fig. 3. The points $\{v_\alpha\}_{|\alpha|=n}$ used in the definition of $B_{n,V}f$, where $n = 7$ and V is the vertices of a triangle, square, pentagon and hexagon (respectively).

3. The generalised Bernstein operator

For a Bernstein frame (2.7) given by points V in $\mathbb{R}^d$, we define a (**generalised**) **Bernstein operator** $B_n = B_{n,V}$ of degree $n \geq 1$ by the usual formula

$$B_n(f) := \sum_{|\alpha|=n} B_\alpha f(v_\alpha), \qquad v_\alpha := \sum_{v\in V} \frac{\alpha_v}{|\alpha|} v, \tag{3.1}$$

(see Fig. 3), which is equivalent to

$$B_n(f) = \sum_{v_1\in V} \cdots \sum_{v_n\in V} f\Big(\frac{v_1 + \cdots + v_n}{n}\Big)\xi_{v_1}\cdots\xi_{v_n}.$$

This maps functions f which are nonnegative at the points $(v_\alpha)_{|\alpha|=n}$ (which are contained in $T = \operatorname{conv}(V)$, the convex hull of the points in V) to polynomials of degree $\leq n$ which are nonnegative on the convex polytope (region of nonnegativity) N_V given by (2.5), so that

$$f \geq 0 \quad \text{on } T \implies B_n f \geq 0 \quad \text{on } N_V, \tag{3.2}$$

and reproduces the linear polynomials (cf. Theorem 3.19).

We now show that the generalised Bernstein operator is **degree reducing**, i.e.,

$$B_n(f) \in \Pi_k(\mathbb{R}^d), \qquad \forall f \in \Pi_k \ (k = 0, 1, \ldots).$$

Define the univariate and multivariate (falling) shifted factorials by

$$[x]^n := x(x-1)\cdots(x-n+1), \qquad [\alpha]^\beta := \prod_{v\in V}[\alpha_v]^{\beta_v},$$

and the multivariate Stirling numbers of the second kind by

$$S(\tau,\beta) := \prod_{v\in V} S(\tau_v, \beta_v),$$

where $S(\tau_v, \beta_v)$ are the Stirling numbers of the second kind. We note that

$$S(\tau,\beta) = 0, \qquad \beta \not\leq \tau, \tag{3.3}$$

and define

$$\binom{|\alpha|}{\alpha} := 0, \qquad \alpha \not\geq 0. \tag{3.4}$$

These are related by

$$\alpha^\tau = \sum_{\beta\le\tau} S(\tau,\beta)[\alpha]^\beta. \tag{3.5}$$

Lemma 3.6. *For any τ and n, we have*

$$\sum_{|\alpha|=n} \alpha^\tau \binom{|\alpha|}{\alpha} \xi^\alpha = \sum_{\beta\le\tau} S(\tau,\beta)[n]^{|\beta|}\xi^\beta. \tag{3.7}$$

Proof. Since $[|\alpha|]^{|\beta|}\binom{|\alpha-\beta|}{\alpha-\beta} = \binom{|\alpha|}{\alpha}[\alpha]^\beta$ (without restriction on α and β), (3.5) gives

$$\sum_{\beta\le\tau} S(\tau,\beta)[|\alpha|]^{|\beta|}\binom{|\alpha-\beta|}{\alpha-\beta} = \binom{|\alpha|}{\alpha}\sum_{\beta\le\tau} S(\tau,\beta)[\alpha]^\beta = \binom{|\alpha|}{\alpha}\alpha^\tau.$$

Thus, we calculate

$$\begin{aligned}\sum_{|\alpha|=n} \alpha^\tau \binom{|\alpha|}{\alpha}\xi^\alpha &= \sum_{|\alpha|=n}\sum_{\beta\le\tau} S(\tau,\beta)[|\alpha|]^{|\beta|}\binom{|\alpha-\beta|}{\alpha-\beta}\xi^\alpha \\ &= \sum_{\beta\le\tau} S(\tau,\beta)[n]^{|\beta|}\xi^\beta \sum_{\substack{|\alpha|=n\\ \alpha\ge\beta}} \binom{|\alpha-\beta|}{\alpha-\beta}\xi^{\alpha-\beta} = \sum_{\beta\le\tau} S(\tau,\beta)[n]^{|\beta|}\xi^\beta,\end{aligned}$$

with the last equality given by the multinomial identity. □

Theorem 3.8 (*Degree Reducing*). *The generalised Bernstein operator B_n is degree reducing. More precisely,*

$$B_n(\xi^\beta) = \frac{[n]^{|\beta|}}{n^{|\beta|}}\xi^\beta + \sum_{0<|\gamma|<|\beta|} \frac{[n]^{|\gamma|}}{n^{|\beta|}} a(\gamma,\beta)\xi^\gamma, \tag{3.9}$$

where $w_1,\ldots,w_m$ is the sequence of points in V, and

$$\begin{aligned}a(\gamma,\beta) := \sum_{|\tau_1|=\beta_{w_1}} \cdots \sum_{|\tau_m|=\beta_{w_m}} \binom{\beta_{w_1}}{\tau_1}\xi^{\tau_1}(w_1)\cdots\binom{\beta_{w_m}}{\tau_m}\xi^{\tau_m}(w_m) \\ \times S(\tau_1+\cdots+\tau_m,\gamma).\end{aligned} \tag{3.10}$$

Proof. Since each ξ_w is an affine function, and $\xi_w(v) = \xi_v(w)$, we have

$$\xi_w(v_\alpha) = \xi_w\Big(\sum_{v\in V}\frac{\alpha_v}{|\alpha|}v\Big) = \sum_{v\in V}\frac{\alpha_v}{|\alpha|}\xi_w(v) = \sum_{v\in V}\frac{\alpha_v}{|\alpha|}\xi_v(w),$$

and the multinomial identity gives

$$\begin{aligned}(\xi^\beta)(v_\alpha) &= \prod_{w\in V}\Big(\sum_{v\in V}\frac{\alpha_v}{|\alpha|}\xi_v(w)\Big)^{\beta_w} = \prod_{w\in V}\sum_{|\tau|=\beta_w}\binom{\beta_w}{\tau}\frac{\alpha^\tau}{|\alpha|^{\beta_w}}\xi^\tau(w) \\ &= \sum_{|\tau_1|=\beta_{w_1}}\cdots\sum_{|\tau_m|=\beta_{w_m}}\binom{\beta_{w_1}}{\tau_1}\cdots\binom{\beta_{w_m}}{\tau_m}\xi^{\tau_1}(w_1)\cdots\xi^{\tau_m}(w_m)\frac{\alpha^{\tau_1+\cdots+\tau_m}}{|\alpha|^{|\beta|}}.\end{aligned}$$

Thus, by rearranging (3.1) and Lemma 3.6, we have

$$\begin{aligned} B_n(\xi^\beta) &= \sum_{|\tau_1|=\beta_{w_1}} \cdots \sum_{|\tau_m|=\beta_{w_m}} \binom{\beta_{w_1}}{\tau_1} \xi^{\tau_1}(w_1) \cdots \binom{\beta_{w_m}}{\tau_m} \xi^{\tau_m}(w_m) \\ &\quad \times \sum_{|\alpha|=n} \frac{\alpha^{\tau_1+\cdots+\tau_m}}{n^{|\beta|}} \binom{n}{\alpha} \xi^\alpha \\ &= \sum_{|\tau_1|=\beta_{w_1}} \cdots \sum_{|\tau_m|=\beta_{w_m}} \binom{\beta_{w_1}}{\tau_1} \xi^{\tau_1}(w_1) \cdots \binom{\beta_{w_m}}{\tau_m} \xi^{\tau_m}(w_m) \\ &\quad \times \sum_{\gamma \le \tau_1+\cdots+\tau_m} \frac{[n]^{|\gamma|}}{n^{|\beta|}} S(\tau_1+\cdots+\tau_m, \gamma) \xi^\gamma. \end{aligned}$$

Here $B_n(\xi^\beta)$ is written as a polynomial in ξ of degree $\le |\beta|$, so that B_n is degree reducing. The terms of degree $|\beta|$ can be simplified using the multinomial identity, $\xi_v(w_j) = \xi_{w_j}(v)$, and (2.4), as follows

$$\begin{aligned} &\sum_{|\tau_1|=\beta_{w_1}} \cdots \sum_{|\tau_m|=\beta_{w_m}} \binom{\beta_{w_1}}{\tau_1} \xi^{\tau_1}(w_1) \cdots \binom{\beta_{w_m}}{\tau_m} \xi^{\tau_m}(w_m) \frac{[n]^{|\beta|}}{n^{|\beta|}} \xi^{\tau_1+\cdots+\tau_m} \\ &= \frac{[n]^{|\beta|}}{n^{|\beta|}} \prod_{j=1}^m \Big(\sum_{|\tau_j|=\beta_{w_j}} \binom{\beta_{w_j}}{\tau_j} \xi^{\tau_j}(w_j) \xi^{\tau_j} \Big) = \frac{[n]^{|\beta|}}{n^{|\beta|}} \prod_{j=1}^m \Big(\sum_{v\in V} \xi_v(w_j) \xi_v \Big)^{\beta_{w_j}} \\ &= \frac{[n]^{|\beta|}}{n^{|\beta|}} \prod_{j=1}^m \Big(\sum_{v\in V} \xi_{w_j}(v) \xi_v \Big)^{\beta_{w_j}} = \frac{[n]^{|\beta|}}{n^{|\beta|}} \prod_{j=1}^m \xi_{w_j}^{\beta_{w_j}} = \frac{[n]^{|\beta|}}{n^{|\beta|}} \xi^\beta. \end{aligned}$$

By collecting the terms of degree $< |\beta|$, we obtain (3.9). Here, (3.3) allows us to remove the restriction $\gamma \le \tau_1 + \cdots + \tau_m$, and there are no terms of degree 0 since $S(1, 0) = 0$. □

Since $[n]^{|\gamma|} = 0$, $|\gamma| > n$, the formula (3.9) implies that B_n is degree reducing.

Remark 3.11. If V consists of $d + 1$ affinely independent points, then

$$a(\gamma, \beta) = \begin{cases} S(\beta, \gamma), & \gamma \le \beta; \\ 0, & \gamma \not\le \beta, \end{cases}$$

and (3.9) simplifies to

$$B_n(\xi^\beta) = \frac{[n]^{|\beta|}}{n^{|\beta|}} \xi^\beta + \sum_{\gamma<\beta} \frac{[n]^{|\gamma|}}{n^{|\beta|}} S(\beta, \gamma) \xi^\gamma.$$

This was proved in [7, Lemma 2.1] for the case when $\beta_{v_0} = 0$ for some $v_0 \in V$.

Example 3.12 (*Linear Reproduction*)**.** For $|\beta| = 1$, we have

$$B_n(\xi_v) = \xi_v, \quad \forall v \in V, \tag{3.13}$$

i.e., B_n reproduces the linear polynomials $\Pi_1(\mathbb{R}^d) = \operatorname{span}\{\xi_v\}_{v\in V}$. This is equivalent to

$$x = \sum_{|\alpha|=n} B_\alpha(x) v_\alpha, \qquad \sum_{|\alpha|=n} B_\alpha(x) = 1, \quad x \in \mathbb{R}^d. \tag{3.14}$$

Example 3.15 (*Quadratics*)**.** For $|\beta| = 2$, we recall $S(1,0) = 0,\ S(2,1) = 1$, so that

$$\begin{aligned} &a(e_u, 2e_w) = \xi_u^2(v) = \xi_v^2(u), \\ &a(e_u, e_v + e_w) = \xi_u(v)\xi_u(w) = (\xi_v\xi_w)(u), \quad v \neq w, \end{aligned} \tag{3.16}$$

and we obtain

$$B_n(\xi^\beta) = \Big(1 - \frac{1}{n}\Big)\xi^\beta + \frac{1}{n}\sum_{u\in V} \xi^\beta(u)\xi_u \to \xi^\beta, \quad \text{as } n \to \infty,\ |\beta| = 2. \tag{3.17}$$

Since B_n is not a positive operator in general, the application of the Korovkin theory is more involved (see Theorem 3.31). We easily obtain the following encouraging result.

Corollary 3.18 (*Convergence*)**.** *For all polynomials f, $B_n(f) \to f$, as $n \to \infty$.*

Proof. It suffices to consider $f = \xi^\beta$. For $n \geq |\beta|$, (3.9) gives

$$B_n(\xi^\beta) - \xi^\beta = \Big(\frac{[n]^{|\beta|}}{n^{|\beta|}} - 1\Big)\xi^\beta - \sum_{|\gamma|<|\beta|} \frac{[n]^{|\gamma|}}{n^{|\beta|}} a(\gamma,\beta)\xi^\gamma,$$

where $\frac{[n]^{|\beta|}}{n^{|\beta|}} - 1,\ \frac{[n]^{|\gamma|}}{n^{|\beta|}} = O(\frac{1}{n})$, as $n \to \infty$. $\square$

The remaining eigenstructure of B_n is as follows.

Theorem 3.19 (*Diagonalisation*)**.** *The generalised Bernstein operator B_n given by points $V \subset \mathbb{R}^d$ is diagonalisable, with eigenvalues*

$$\lambda_k^{(n)} := \frac{[n]^k}{n^k}, \quad k = 1,\dots,n,\ 1 = \lambda_1^{(n)} > \lambda_2^{(n)} > \cdots > \lambda_n^{(n)} > 0.$$

Let $P_{k,V}^{(n)}$ denote the $\lambda_k^{(n)}$-eigenspace. Then

$$P_{1,V}^{(n)} = \Pi_1(\mathbb{R}^d), \quad \forall n. \tag{3.20}$$

For $k > 1$, $P_{k,V}^{(n)}$ consists of polynomials of exact degree k, and is spanned by

$$p_{\xi^\beta}^{(n)} = \xi^\beta + \sum_{0<|\alpha|<|\beta|} c(\alpha,\beta,n)\xi^\alpha,\ |\beta| = k, \tag{3.21}$$

where the coefficients can be calculated using (3.10) and the recurrence formula

$$\begin{aligned} c(\alpha,\beta,n) &:= \frac{a(\alpha,\beta)}{1-|\beta|}, \quad |\alpha| = |\beta| - 1, \\ c(\alpha,\beta,n) &:= \frac{[n]^{|\alpha|}}{\lambda_{|\beta|}^{(n)} - \lambda_{|\alpha|}^{(n)}}\Big(\frac{a(\alpha,\beta)}{n^{|\beta|}} + \sum_{|\alpha|<|\gamma|<|\beta|} c(\gamma,\beta,n)\frac{a(\alpha,\gamma)}{n^{|\gamma|}}\Big), \quad |\alpha| < |\beta| - 1. \end{aligned} \tag{3.22}$$

Proof. By Example 3.12, the $\lambda_1^{(n)} = 1$ eigenspace $P_{1,V}^{(n)}$ contains $\Pi_1(\mathbb{R}^d)$. Recall, from (3.9), that $B_n(\xi^\beta)$ has the form

$$B_n(\xi^\beta) = \lambda_{|\beta|}^{(n)}\xi^\beta + \sum_{0<|\gamma|<|\beta|} \frac{[n]^{|\gamma|}}{n^{|\beta|}} a(\gamma,\beta)\xi^\gamma. \tag{3.23}$$

Motivated by this, we seek $\lambda_k^{(n)}$-eigenfunctions of the form

$$f = \xi^\beta + \sum_{0<|\alpha|<|\beta|} c(\alpha,\beta,n)\xi^\alpha, \quad |\beta| = k > 1.$$

We observe that for such an eigenfunction the coefficients $c(\alpha,\beta,n)$ are *not unique*—even when V consists of $d+1$ points. Expanding $B_n(f) = \lambda_k^{(n)} f$ using (3.23) gives

$$\begin{aligned} B_n(f) &= B_n(\xi^\beta) + \sum_{0<|\gamma|<|\beta|} c(\gamma,\beta,n) B_n(\xi^\gamma) \\ &= \lambda_{|\beta|}^{(n)}\xi^\beta + \sum_{0<|\alpha|<|\beta|} \frac{[n]^{|\alpha|}}{n^{|\beta|}} a(\alpha,\beta)\xi^\alpha \\ &\quad + \sum_{0<|\gamma|<|\beta|} c(\gamma,\beta,n)\Big(\lambda_{|\gamma|}^{(n)}\xi^\gamma + \sum_{0<|\alpha|<|\gamma|} \frac{[n]^{|\alpha|}}{n^{|\gamma|}} a(\alpha,\gamma)\xi^\alpha\Big) \\ &= \lambda_{|\beta|}^{(n)}\xi^\beta + \sum_{0<|\alpha|<|\beta|} \lambda_{|\beta|}^{(n)} c(\alpha,\beta,n)\xi^\alpha. \end{aligned} \tag{3.24}$$

Equating the ξ^α, $0 < |\alpha| < |\beta|$ coefficients gives

$$\begin{aligned} \lambda_{|\beta|}^{(n)} c(\alpha,\beta,n) &= \frac{[n]^{|\alpha|}}{n^{|\beta|}} a(\alpha,\beta) + c(\alpha,\beta,n)\lambda_{|\alpha|}^{(n)} \\ &\quad + \sum_{|\alpha|<|\gamma|<|\beta|} c(\gamma,\beta,n)\frac{[n]^{|\alpha|}}{n^{|\gamma|}} a(\alpha,\gamma). \end{aligned} \tag{3.25}$$

Since $\lambda_{|\alpha|}^{(n)} > \lambda_{|\beta|}^{(n)}$, this is equivalent to

$$c(\alpha,\beta,n) = \frac{1}{\lambda_{|\beta|}^{(n)} - \lambda_{|\alpha|}^{(n)}}\Big(\frac{[n]^{|\alpha|}}{n^{|\beta|}} a(\alpha,\beta) + \sum_{|\alpha|<|\gamma|<|\beta|} c(\gamma,\beta,n)\frac{[n]^{|\alpha|}}{n^{|\gamma|}} a(\alpha,\gamma)\Big).$$

From this we can define suitable $c(\alpha,\beta,n)$ recursively, as in (3.22), starting from

$$c(\alpha,\beta,n) := \frac{1}{\lambda_{|\beta|}^{(n)} - \lambda_{|\alpha|}^{(n)}}\frac{[n]^{|\alpha|}}{n^{|\beta|}} a(\alpha,\beta) = \frac{a(\alpha,\beta)}{1-|\beta|}, \quad |\alpha| = |\beta| - 1.$$

A simple dimension count shows that the eigenfunction $\{p_{\xi^\beta}\}_{|\beta|=k}$, so defined, span a space $P_{k,V}^{(n)}$ of dimension $\binom{k+d-1}{d-1}$. Again, by dimension counting, we conclude that B_n is diagonalisable, with $P_{1,V}^{(n)} = \Pi_1(\mathbb{R}^d)$. □

Example 3.26 (*Quadratic Eigenfunctions*)**.** Using (3.16), we have

$$p_{\xi^\beta}^{(n)} = \xi^\beta - \sum_{u\in V} \xi^\beta(u)\xi_u, \quad |\beta| = 2. \tag{3.27}$$

In general, $p_{\xi^\alpha}^{(n)}$ does depend on n (cf. [6]).

Despite the fact the coefficients in (3.21) are not unique, we can take their limit as $n \to \infty$. This indicates that the redundant expansion (3.21) is natural.

Corollary 3.28 (*Limits of the Eigenfunctions*)**.** *For* $0 < |\alpha| < |\beta|$,

$$\lim_{n\to\infty} c(\alpha,\beta,n) = c^*(\alpha,\beta),$$

where

$$\begin{aligned} c^*(\alpha,\beta) &:= \frac{a(\alpha,\beta)}{1-|\beta|}, \quad |\alpha| = |\beta|-1, \\ c^*(\alpha,\beta) &:= \frac{2}{(|\beta|-|\alpha|)(-|\alpha|-|\beta|+1)} \sum_{|\gamma|=|\alpha|+1} c^*(\gamma,\beta)a(\alpha,\gamma), \quad |\alpha| < |\beta|-1. \end{aligned} \tag{3.29}$$

Thus, the eigenfunctions of (3.21) *satisfy*

$$p^{(n)}_{\xi^\beta} \to p^*_{\xi^\beta} := \xi^\beta + \sum_{0<|\alpha|<|\beta|} c^*(\alpha,\beta)\xi^\alpha, \quad \text{as } n\to\infty. \tag{3.30}$$

Proof. Fix β. We use strong induction on $j = |\beta|-|\alpha| = 1,\ldots,|\beta|$ to prove that the limit exists. For $|\alpha| = |\beta|-1$ ($j=1$) the limit is clear. Suppose the limit of $c(\gamma,\beta,n)$ exists for all γ with $|\alpha| < |\gamma| < |\beta|$. Then taking the limit of (3.32) gives

$$\lim_{n\to\infty} c(\alpha,\beta,n) = \frac{2}{(|\beta|-|\alpha|)(|\beta|-|\alpha|-2|\beta|+1)} \sum_{|\gamma|=|\alpha|+1} c^*(\gamma,\beta)a(\alpha,\gamma).$$

This follows from the calculations

$$\begin{aligned} \lambda^{(n)}_{|\beta|} - \lambda^{(n)}_{|\alpha|} &= \frac{[n]^{|\alpha|}[n-|\alpha|]^{|\beta|-|\alpha|}}{n^{|\beta|}} - \frac{[n]^{|\alpha|}}{n^{|\alpha|}} = \frac{[n]^{|\alpha|}}{n^{|\beta|}}([n-|\alpha|]^{|\beta|-|\alpha|} - n^{|\beta|-|\alpha|}), \\ [n-|\alpha|]^{|\beta|-|\alpha|} - n^{|\beta|-|\alpha|} \\ &= \frac{1}{2}(|\beta|-|\alpha|)(-|\alpha|-|\beta|+1)n^{|\beta|-|\alpha|-1} + \text{lower order powers of } n. \end{aligned}$$

Since $p^{(n)}_{\xi^\beta}, p^*_{\xi^\beta} \subset \Pi_{|\beta|}(\mathbb{R}^d)$, we have the convergence asserted in (3.30). □

We now prove the strongest Korovkin theorem that the restricted positivity property (3.2) allows. Since this requires a modification of the usual argument, which is not stated in the literature, we give a self contained proof. This result supercedes Corollary 3.18.

Theorem 3.31 (*Korovkin*)**.** *Let* $T = T_V$ *be the convex hull of* V, *and* N_V *be the region of nonnegativity. For* $f \in C(T)$, $B_n f \to f$ *uniformly on* N_V.

Proof. Let $\varepsilon > 0$, and M be the maximum of f over T. Since f is uniformly continuous on the compact set T, there is a $\delta > 0$ such that $|f(s)-f(t)| < \varepsilon$, $\forall \|s-t\| < \delta$. Thus, we obtain the estimate

$$|f(s)-f(t)| \le \varepsilon + 2M\frac{\|s-t\|^2}{\delta^2}, \quad \forall s,t\in T.$$

For fixed t, we have

$$-\varepsilon - \frac{2M}{\delta^2}\|\cdot - t\|^2 \le f - f(t) \le \varepsilon + \frac{2M}{\delta^2}\|\cdot - t\|^2 \quad \text{on } T,$$

and so applying B_n (which reproduces constants) and using (3.2) gives

$$-\varepsilon - \frac{2M}{\delta^2} B_n\big(\| \cdot - t\|^2\big) \le B_n f - f(t) \le \varepsilon + \frac{2M}{\delta^2} B_n\big(\| \cdot - t\|^2\big) \quad \text{on } N_V.$$

This last step is the main difference in argument. For $t \in N_V$, evaluating at t gives

$$|B_n f(t) - f(t)| \le \varepsilon + \frac{2M}{\delta^2} B_n\big(\| \cdot - t\|^2\big)(t), \qquad \forall t \in N_V. \tag{3.32}$$

We now estimate $B_n(\| \cdot - t\|^2)(t)$. From (2.1), we obtain

$$\| \cdot - t\|^2 = \Big\langle \sum_v \{\xi_v v - \xi_v t\}, \sum_w \{\xi_w w - \xi_w t\} \Big\rangle = \sum_v \sum_w \xi_v \xi_w \langle v - t, w - t\rangle.$$

Thus, (3.17) gives

$$\begin{aligned} B_n\big(\| \cdot - t\|^2\big) &= \sum_v \sum_w \Big\{ \Big(1 - \frac{1}{n}\Big) \xi_v \xi_w + \frac{1}{n} \sum_{u \in V} \xi_v(u) \xi_w(u) \xi_u \Big\} \langle v - t, w - t\rangle \\ &= \Big(1 - \frac{1}{n}\Big) \| \cdot - t\|^2 + \frac{1}{n} \sum_v \sum_w \sum_{u \in V} \xi_v(u) \xi_w(u) \xi_u \langle v - t, w - t\rangle \\ &= \Big(1 - \frac{1}{n}\Big) \| \cdot - t\|^2 + \frac{1}{n} \sum_{u \in V} \xi_u \langle u - t, u - t\rangle, \end{aligned}$$

so that

$$B_n\big(\| \cdot - t\|^2\big)(t) = \frac{1}{n} \sum_{u \in V} \|u - t\|^2 \xi_u(t) = \frac{1}{n} \Big(\sum_{u \in V} \|u\|^2 \xi_u(t) - \|t\|^2 \Big) \le \frac{1}{n} D^2,$$

where $D := \operatorname{diam}(T)$ is the diameter of T. Thus, we obtain the uniform estimate

$$|B_n f(t) - f(t)| \le \varepsilon + \frac{1}{n} \frac{2MD^2}{\delta^2}, \qquad t \in N_V,$$

and conclude that $B_n f \to f$ uniformly on N_V. □

4. Applications to CAGD

We now consider how our generalised Bernstein operator $B_n = B_{n,V}$ might be used in CAGD (computer aided geometric design) to describe polynomials defined on convex polyhedra which are not simplices. We first observe that the Korovkin theory does not allow a Bernstein type operator which is positive on the entire region $T = \operatorname{conv}(V)$ if the points of V are not the vertices of a simplex or a cube.

Theorem 4.1 ([24]). *Let T be a convex polygon with five or more vertices. There is no positive linear operator $L : C(T) \to C(T)$ which reproduces $\Pi_1(\mathbb{R}^2)$ and maps $\Pi_2(\mathbb{R}^2)$ to itself other than the identity.*

We recall that our B_n reproduces $\Pi_1(\mathbb{R}^2)$ and maps $\Pi_2(\mathbb{R}^2)$, but has the restricted positivity property (3.2), i.e.,

$$f \ge 0 \quad \text{on } T \implies B_n f \ge 0 \text{ on } N_V,$$

where N_V is the region of nonnegativity (2.5). In the multivariate case, the description of other shape preserving properties is involved (cf. [26]). We mention some which do generalise easily.

These suggest that the target region for representing polynomials on should perhaps be the convex polyhedron N_V, rather than $T = \text{conv}(V)$.

Proposition 4.2 (*Shape Preservation*). *Let $T = T_V$ be the convex hull of V, and N_V be the region of nonnegativity. If f is convex on T, then*

$$B_n f \geq B_{n+1} f \geq \cdots \geq f \quad \text{on } N_V. \tag{4.3}$$

Proof. Fix $x \in N_V$. By (2.1), and a calculation, we have the convex combinations

$$x = \sum_{|\alpha|=n} B_\alpha(x) v_\alpha, \qquad v_\beta = \sum_{v \in V} \frac{\beta_v}{|\beta|} v_{\beta - e_v}.$$

Hence for f convex, Jensen's inequality gives

$$B_n f(x) = \sum_{|\alpha|=n} B_\alpha(x) f(v_\alpha) \geq f(x), \qquad \sum_{v \in V} \frac{\beta_v}{|\beta|} f(v_{\beta - e_v}) \geq f(v_\beta). \tag{4.4}$$

By the degree raising formula (2.13), we have

$$B_n f = \sum_{|\alpha|=n} B_\alpha f(v_\alpha) = \sum_{v \in V} \sum_{|\alpha|=n} \frac{\alpha_v + 1}{|\alpha| + 1} B_{\alpha + e_v} f(v_\alpha),$$

so that

$$B_n f - B_{n+1} f = \sum_{|\beta|=n+1} c_\beta B_\beta, \qquad c_\beta := \sum_{v \in V} \frac{\beta_v}{|\beta|} f(v_{\beta - e_v}) - f(v_\beta).$$

By (4.4), $c_\beta \geq 0$, so that $B_n f \geq B_{n+1} f$ on N_V. □

A polynomial in B-form can be calculated via the de Casteljau algorithm. We present this in terms of the **blossom** (**polar form**) of a polynomial $p \in \Pi_n(\mathbb{R}^d)$, which we recall (cf. [25,12,8]) is the unique symmetric n-affine function $\overset{\omega}{p}$ with

$$\overset{\omega}{p}(t, \ldots, t) = p(t), \quad \forall t \in \mathbb{R}^d. \tag{4.5}$$

Proposition 4.6 (*de Casteljau Algorithm*). *Suppose that $p \in \Pi_n(\mathbb{R}^d)$ has the B-form*

$$p = \sum_{|\alpha|=n} c_\alpha B_\alpha. \tag{4.7}$$

Then the blossom of p at $t_1, \ldots, t_n$ can be calculated from $(c_\alpha)_{|\alpha|=n}$ via

for $j = 1$ *to* n *do*

$$c_\alpha := \sum_{v \in V} \xi_v(t_j) c_{\alpha + e_v}, \qquad |\alpha| = n - j$$

end for

with $c_0 = \overset{\omega}{p}(t_1, \ldots, t_n)$. In particular, taking $t_1 = \cdots = t_n = t$ gives $c_0 = p(t)$.

Proof. Suppose $p \in \Pi_n(\mathbb{R}^d)$ is in the B-form (4.7), i.e., equivalently

$$p = \sum_{v_1} \cdots \sum_{v_n} c_{e_{v_1} + \cdots + e_{v_n}} \xi_{v_1} \cdots \xi_{v_n}. \tag{4.8}$$

Then its blossom is given by

$$P(t_1,\ldots,t_n) := \sum_{v_1}\cdots\sum_{v_n} c_{e_{v_1}+\cdots+e_{v_n}}\xi_{v_1}(t_1)\cdots\xi_{v_n}(t_n), \tag{4.9}$$

since P clearly defines an n-affine function, which, by (4.8), satisfies (4.5). The n steps of the algorithm are the averagings given by the n sums in (4.9). □

In the terminology of [8], this is a *symmetric simplicial algorithm* (here the points V need not be the vertices of a simplex).

The coefficients c_α in the B-form (4.7) of p are not unique, unless V is the vertices of a simplex. Using (2.1) to expand, we have

$$\begin{aligned} p(t) &= \overset{\omega}{p}(t,\ldots,t) = \overset{\omega}{p}\left(\sum_{v_1}\xi_{v_1}(t)v_1,\ldots,\sum_{v_n}\xi_{v_n}(t)v_n\right) \\ &= \sum_{v_1}\cdots\sum_{v_n}\xi_{v_1}(t)\cdots\xi_{v_n}(t)\,\overset{\omega}{p}(v_1,\ldots,v_n), \end{aligned}$$

and so the coefficients c_α in (4.7) can be chosen to be the blossoms

$$b_\alpha = \overset{\omega}{p}(\alpha\cdot V), \qquad \alpha\cdot V := (\ldots,\underbrace{v,\ldots,v}_{\alpha_v \text{ times}},\ldots), \tag{4.10}$$

which we call the **blossoming coefficients**.

Proposition. *For any p of the form (4.7), the blossoming coefficients $b=(b_\alpha)_{|\alpha|=n}$ are given by the matrix multiplication*

$$b = Qc, \qquad Q = [q_{\alpha\beta}]_{|\alpha|=n,|\beta|=n}, \quad q_{\alpha\beta} := \overset{\omega}{B_\beta}(\alpha\cdot V).$$

For $n>1$, our calculations show that the blossoming coefficients are not the ℓ_2-norm minimising choice (which is given by multiplication by an orthogonal projection matrix).

Remark 4.11. The previous discussion extends to $p=(p_1,\ldots,p_s):\mathbb{R}^d\to\mathbb{R}^s$, where $p_j\in \Pi_n(\mathbb{R}^d)$ and $c_\alpha\in\mathbb{R}^s$, by considering coordinates. Since vector valued p are used in practice, e.g., Bézier curves in $\mathbb{R}^3$, we henceforth state our results in this setting.

We define the **control points** of the curve ($d=1$), surface ($d=2$), etc., given by $t\mapsto p(t)\in\mathbb{R}^s$, where p has B-form (4.7), to be

$$\left\{\begin{pmatrix} v_\alpha \\ c_\alpha\end{pmatrix} : |\alpha|=n\right\} \subset \mathbb{R}^{d+s}.$$

Since the B-form is not unique when V has more than $d+1$ points, a given curve (surface, etc.) may be given by different sets of control points. This redundancy has advantages from the point of view of design, as a given surface could be arrived at by a number of different choices of control points. Equivalently, each control point has less influence on the surface, and so moving a control point causes a small change of shape making the fine tuning of a surface easier. Once a suitable surface has been obtained it can be presented in terms of a set of canonical control points—if so desired.

By combining (3.14) and (4.7), we have

$$\begin{pmatrix} x \\ p(x) \end{pmatrix} = \sum_{|\alpha|=n} \mathbf{c}_\alpha B_\alpha(x), \quad \mathbf{c}_\alpha := \begin{pmatrix} v_\alpha \\ c_\alpha \end{pmatrix}. \tag{4.12}$$

Thus a point $\big(x, p(x)\big)$ on the curve (surface, etc.) can be calculated by the de Casteljau algorithm applied to the vectors $(\mathbf{c}_\alpha)_{|\alpha|=n} \subset \mathbb{R}^{d+s}$. Many basic properties of Bézier curves and surfaces follow from (4.12). We now list these using the terminology of [16].

Convex hull property: Since $\sum_{|\alpha|=n} B_\alpha(x) = 1$ and $B_\alpha(x) \geq 0$, $x \in N_V$, the point $\big(x, p(x)\big)$ is an affine combination (also called a barycentric combination in CAGD) of the control points $(\mathbf{c}_\alpha)_{|\alpha|=n}$, which is a convex combination if $x \in N_V$.

Affine invariance: For $A : \mathbb{R}^s \to \mathbb{R}^\ell$ an affine map, we have

$$A\big(p(x)\big) = \sum_{|\alpha|=n} (Ac_\alpha) B_\alpha(x).$$

Invariance under affine parameter transformations: For $A : \mathbb{R}^d \to \mathbb{R}^d$ an invertible affine map, $B_\alpha(Ax) = A(B_\alpha(x))$, so that

$$p(Ax) = \sum_{|\alpha|=n} c_\alpha A\big(B_\alpha(x)\big).$$

Symmetry: Suppose that A is an affine map which maps (the multiset) V to V. Then $\xi_v(x) = \xi_{Av}(Ax)$ (see [29]), so that

$$B_\alpha(x) = B_{\alpha\circ A^{-1}}(Ax),$$

where $\alpha \circ A^{-1} : V \to \mathbb{Z}_+$ denotes the multi-index $v \mapsto \alpha_{A^{-1}v}$. Using this we obtain

$$p(x) = \sum_{|\alpha|=n} c_\alpha B_\alpha(x) = \sum_{|\alpha|=n} c_\alpha B_{\alpha\circ A^{-1}}(Ax) = \sum_{|\beta|=n} c_{\beta\circ A} B_\beta(Ax).$$

Invariance under barycentric combinations: The affine combination of two curves (surfaces, etc.) is given by corresponding affine combination of the control points, i.e.,

$$\lambda \sum_{|\alpha|=n} \mathbf{b}_\alpha B_\alpha(x) + (1-\lambda) \sum_{|\alpha|=n} \mathbf{c}_\alpha B_\alpha(x) = \sum_{|\alpha|=n} \big\{\lambda \mathbf{b}_\alpha + (1-\lambda)\mathbf{c}_\alpha\big\} B_\alpha(x), \quad \lambda \in \mathbb{R}.$$

Endpoint interpolation: If $v \in V$ satisfies $v \notin \mathrm{Aff}(V \setminus v)$, then Proposition 2.6 gives

$$(B_n f)(v) = f(v).$$

The remaining property given in [16] is **pseudolocal control**, i.e., the fact that B_α is peaked at the point v_α, and so moving the control point $\mathbf{c}_\alpha$ has the most influence on the curve (surface, etc.) near the point v_α. Numerical calculations indicate some degree of localisation of the B_α near v_α, but we do not make any quantitative statements here.

To summarise, the main features of our Bernstein polynomial approximations on a nonsimplicial convex polytope are:

- A surface may be determined by several different choices of control points.
- On N_V the surface is a convex combination of the control points.
- Convex functions are monotonely approximated from above by B_n on N_V.

Typical applications, that require a single polynomial defined on a convex polytope, include the description and construction of finite elements on regular polygons and the design of hexagonal lenses (cf. [14]).

5. Smoothness and multivariate splines

We now consider the possible application of our results to multivariate spline theory, where surfaces are constructed by joining polynomial pieces together smoothly. There is a highly developed theory (and associated software) that involves polynomials defined on simplices, based on the description of the smoothness conditions in terms of the control points (cf. [10,21]). We assume that the reader is familiar with this, and we investigate how it extends to partitions involving nonsimplical cells. There was work in this direction (see [19,5]) on *mixed grid partitions* where in the bivariate case the cells were *triangles* and *parallelograms*, and in the trivariate case they were *tetrahedrons*, *prisms* and *parallelopipeds*. The B-form developed for nonsimplicial cells was for polynomials of *coordinate degree n* (as were the spline spaces), rather than total degree n, though the positioning of the control points is the same as we propose. Bivariate C^1-quadratics on a polygonal partition were considered in [31].

We will give our results in terms of the blossoming coefficients. The following theorem is adapted from [10,25,20]. The paper [30] outlines why these approaches are all equivalent. If (c_α) are the blossoming coefficients, then we refer to the $(\mathbf{c}_\alpha)$ of (4.12) as the **blossoming control points** of $p = \sum c_\alpha B_\alpha$. Let

$$r \cdot x := \underbrace{x, \ldots, x}_{r \text{ times}}.$$

Theorem 5.1. *Let W be a set of points in $\mathbb{R}^d$, with $L = \mathrm{Aff}(W)$. Then the polynomials $f, g \in \Pi_n(\mathbb{R}^d)$ have all derivatives of order $\leq r$ on L equal if and only if*

$$\overset{\omega}{f}(w_1, \ldots, w_{n-r}, r \cdot x) = \overset{\omega}{g}(w_1, \ldots, w_{n-r}, r \cdot x), \quad w_1, \ldots, w_{n-r} \in W,\ x \in \mathbb{R}^d. \quad (5.2)$$

Lemma 5.3. *Let c be the blossoming coefficients of (4.10) for $p \in \Pi_n(\mathbb{R}^d)$. Then*

$$\overset{\omega}{p}(\alpha \cdot V, r \cdot x) = \sum_{|\beta|=r} c_{\alpha+\beta} B_\beta(x), \quad |\alpha| = n - r,\ r = 0, \ldots, n.$$

Proof. Since the blossom $\overset{\omega}{p}$ is n-affine, we have

$$\begin{aligned}\overset{\omega}{p}(\alpha \cdot V, r \cdot x) &= \overset{\omega}{p}\left(\alpha \cdot V, \sum_{v_1 \in V} \xi_{v_1}(x) v_1, \ldots, \sum_{v_r \in V} \xi_{v_r}(x) v_r\right) \\ &= \sum_{v_1 \in V} \cdots \sum_{v_r \in V} \xi_{v_1}(x) \cdots \xi_{v_r}(x)\, \overset{\omega}{p}(\alpha \cdot V, v_1, \ldots, v_r) \\ &= \sum_{|\beta|=r} c_{\alpha+\beta} B_\beta(x). \quad \square\end{aligned}$$

Combining Theorem 5.1 and Lemma 5.3 gives smoothness conditions for the C^r-joining of polynomials in terms of their blossoming B-form coefficients.

We now illustrate this for C^1-quadratics on a region consisting of a triangle and a quadrilateral with a common edge.

Example 5.4 (C^1-*Quadratics on Triangle and Quadrilateral*)**.** Without loss of generality, suppose the vertices of the triangle and quadrilateral are

$$V = \{v_1, v_2, v_3\} = \{0, e_1, a\}, \quad a_2 \neq 0, \qquad \tilde{V} = \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4\} = \{0, e_1, b, -e_2\},$$

so the common edge has endpoints $v_1 = \tilde{v}_1 = 0$ and $v_2 = \tilde{v}_2 = e_1 = (1, 0)$. We index the generalised barycentric coordinates ξ and $\tilde{\xi}$ by both their order and the points themselves, e.g., ξ_3 and $\xi_a = \xi_{(a_1,a_2)}$, whichever is the most convenient. We have

$$\xi_1(x, y) = (1 - x) + \frac{a_1 - 1}{a_2} y, \qquad \xi_2(x, y) = x - \frac{a_1}{a_2} y, \qquad \xi_3(x, y) = \frac{1}{a_2} y,$$

$$\tilde{\xi}_{(0,0)}(x, y) = \frac{-(b_2^2 + b_1 b_2 + b_1 + 1)x + (b_1^2 + b_1 b_2 - b_1 + 1)y + b_1^2 + b_2^2 + 1}{2(b_1^2 + b_2^2 - b_1 b_2 + b_2 - b_1 + 1)}$$

$$\tilde{\xi}_{(1,0)}(x, y) = \frac{(2b_2^2 - b_1 b_2 + 2b_2 - b_1 + 2)x + (b_1^2 - 2b_1 b_2 - b_1)y + b_1^2 - b_1 b_2 - b_1}{2(b_2^2 + b_2 + 1 - b_1 b_2 - b_1 + b_1^2)},$$

$$\tilde{\xi}_{(b_1,b_2)}(x, y) = \frac{(2b_1 - b_2 - 1)x + (2b_2 - b_1 + 1)y + b_2 - b_1 + 1}{2(b_2^2 + b_2 + 1 - b_1 b_2 - b_1 + b_1^2)},$$

$$\tilde{\xi}_{(0,-1)}(x, y) = \frac{(-b_2^2 + 2b_1 b_2 - b_2)x + (-2b_1^2 + b_1 b_2 + 2b_1 - b_2 - 2)y + b_2^2 - b_1 b_2 + b_2}{2(b_2^2 + b_2 + 1 - b_1 b_2 - b_1 + b_1^2)}.$$

Suppose that $f, g \in \Pi_2(\mathbb{R}^2)$ are quadratics with blossoming coefficient B-forms

$$f = \sum_{\substack{|\alpha|=2 \\ \alpha \in \mathbb{Z}_+^V}} c_\alpha B_\alpha, \qquad g = \sum_{\substack{|\alpha|=2 \\ \alpha \in \mathbb{Z}_+^{\tilde{V}}}} \tilde{c}_\alpha \tilde{B}_\alpha.$$

In Theorem 5.1, we take $n = 2$, and

$$W = \{v_1, v_2\} = \{\tilde{v}_1, \tilde{v}_2\} = \{0, e_1\}.$$

For $r = 0$, the condition for f and g to have a continuous join on the line $L = \mathrm{Aff}(W)$ is given by the two element sequences from $W = \{v_1, v_2\}$, i.e.,

$$\overset{\omega}{f}(v_1, v_1) = \overset{\omega}{g}(v_1, v_1), \qquad \overset{\omega}{f}(v_1, v_2) = \overset{\omega}{g}(v_1, v_2), \qquad \overset{\omega}{f}(v_2, v_2) = \overset{\omega}{g}(v_2, v_2) \tag{5.5}$$

which by Lemma 5.3 is

$$c_{(2,0,0)} = \tilde{c}_{(2,0,0,0)}, \qquad c_{(1,1,0)} = \tilde{c}_{(1,1,0,0)}, \qquad c_{(0,2,0)} = \tilde{c}_{(0,2,0,0)}. \tag{5.6}$$

For $r = 1$, the condition for f and g to have a C^1-join on L is given by the one element sequences from $W = \{v_1, v_2\}$, i.e.,

$$\overset{\omega}{f}(v_1, x) = \overset{\omega}{g}(v_1, x), \qquad \overset{\omega}{f}(v_2, x) = \overset{\omega}{g}(v_2, x).$$

By Lemma 5.3, these equalities of linear polynomials in x can be written as

$$c_{(2,0,0)}\xi_1 + c_{(1,1,0)}\xi_2 + c_{(1,0,1)}\xi_3 = \tilde{c}_{(2,0,0,0)}\tilde{\xi}_1 + \tilde{c}_{(1,1,0,0)}\tilde{\xi}_2 + \tilde{c}_{(1,0,1,0)}\tilde{\xi}_3 + \tilde{c}_{(1,0,0,1)}\tilde{\xi}_4,$$

$$c_{(1,1,0)}\xi_1 + c_{(0,2,0)}\xi_2 + c_{(0,1,1)}\xi_3 = \tilde{c}_{(1,1,0,0)}\tilde{\xi}_1 + \tilde{c}_{(0,2,0,0)}\tilde{\xi}_2 + \tilde{c}_{(0,1,1,0)}\tilde{\xi}_3 + \tilde{c}_{(0,1,0,1)}\tilde{\xi}_4.$$

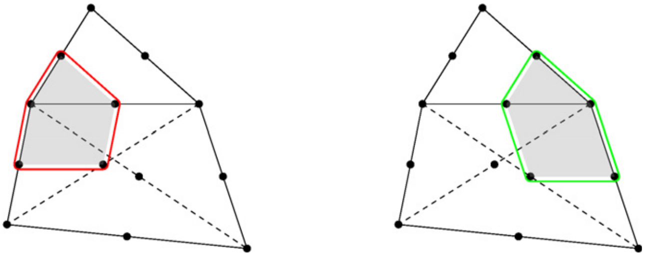

Fig. 4. The C^1-smoothness conditions of Example 5.4.

Since the blossoming control points for the quadrilateral are simply those for corresponding triangles, it follows that these smoothness conditions have the usual geometric interpretation: that all the control points involved (3 and 4 respectively, see Fig. 4) lie in a common plane.

Remark 5.7. Smoothness conditions across an affine subspace L can be developed so long as the points on L which are in common have affine span L. For example, if L is the line through v_1 and v_2, and V and $\tilde{V}$ have at least two distinct common points on L, say v_1 and v_2, then the C^0-smoothness condition across L is (5.5), which by Lemma 5.3 gives something similar to (5.6), depending the other points in V and $\tilde{V}$. If a point of $\{v_1, v_2\}$ is repeated in V or $\tilde{V}$, then it may be that $\alpha \cdot V = \beta \cdot V$, without $\alpha = \beta$, in which case $c_\alpha = c_\beta$, and either may be taken in the smoothness condition.

6. Future work

For applications where it is desirable to have generalised barycentric coordinates which are nonnegative on the convex hull of V, one could modify the definition of $(\xi_v(x))_{v\in V}$ for $x \in \operatorname{conv}(V)$ to be the unique minimal ℓ_2-norm coefficients $a \in \mathbb{R}^V$ satisfying

$$x = \sum_{v\in V} a_v v, \qquad \sum_{v\in V} a_v = 1, \quad a_v \geq 0. \tag{6.1}$$

This is well defined, since the set of $a \in \mathbb{R}^V$ satisfying (6.1) is a nonempty closed convex set, and so has a unique element of minimal norm. With this definition, ξ_v is a continuous piecewise linear polynomial defined on $\operatorname{conv}(V)$, which coincides with the original on N_V.

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