Affine generalised barycentric coordinates

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Abstract

For a given set of points in $\mathbb{R}^d$, there may be many ways to write a point x in their affine hull as an affine combination of them. We show there is a *unique* way which minimises the sum of the squares of the coefficients. It turns out that these coefficients, which are given by a simple formula, are affine functions of x, and so generalise the barycentric coordinates. These *affine* generalised barycentric coordinates have many nice properties, e.g., they depend continuously on the points, and transform naturally under symmetries and affine transformations of the points. Because of this, they are well suited to representing polynomials on polytopes. We give a brief discussion of the corresponding Bernstein–Bézier form and potential applications, such as finite elements and orthogonal polynomials.

Key Words: barycentric coordinates, Wachspress coordinates, mean value coordinates, multivariate Bernstein polynomials, least squares method

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1 Introduction

A sequence $p_1, \ldots, p_n$ of $n = d+1$ points in $\mathbb{R}^d$ is affinely independent if and only if each point $x \in \mathbb{R}^d$ can be written uniquely as an *affine combination* of them, i.e.,

$$x = \sum_j \lambda_j(x)\, p_j, \qquad \sum_j \lambda_j(x) = 1. \tag{1.1}$$

The functions λ_j, so defined, are called *barycentric coordinates.* They are affine functions (linear polynomials), which are nonnegative on the simplex given by the convex hull of the points. They satisfy natural symmetry and affine transformation properties. Because of these properties, they are used extensively to describe polynomials on simplices, e.g., simplicial finite elements, orthogonal polynomials on a triangle, and in CAGD (computer aided geometric design), see, e.g., [1]. Most generalisations of the barycentric coordinates to date have been driven by the CAGD applications (see the end of this section).

For a finite sequence of points, which are possibly not affinely independent, we propose a generalisation of the barycentric coordinates: the affine coordinates of minimal ℓ_2–norm. These new coordinates are again affine functions, but they do not have the nonnegativity property on all of the convex hull of the points – this is too much to hope for in general (see [12]). They are however nonnegative on a convex polytope which contains the barycentre of the points, and they transform naturally under affine maps.

The rest of the paper is set out as follows. We now give an overview of the properties of *generalised barycentric coordinates* desired for CAGD applications, and indicate the corresponding property (or lack of) for our *affine generalised barycentric coordinates.* In Section 2, we prove the main results about the existence, construction and properties of these coordinates. We then describe the region on which the coordinates are nonnegative (Section 3), and how the coordinates transform under affine transformations (Section 4). We conclude with some illustrative examples, and further discussion.

For points $P = (p_1, \ldots, p_n)$ in $\mathbb{R}^d$, a sequence of functions $\lambda_j : \Omega \to \mathbb{R}$, $j = 1, \ldots, n$ (defined on some $\Omega \subset \mathbb{R}^d$ containing P) are called **generalised barycentric coordinates** if (1.1) holds for all $x \in \Omega$ (cf [6], [8]). This condition is equivalent to the following reproduction formula for all affine functions $f : \mathbb{R}^d \to \mathbb{R}$ (linear polynomials)

$$f = \sum_j f(p_j)\lambda_j, \qquad \forall f \in \Pi_1. \tag{1.2}$$

Other additional properties desired of such coordinates λ_j include (cf [8])

(a) **Domain** $\Omega = \mathbb{R}^d$. The λ_j should be defined on as large a domain $\Omega \subset \mathbb{R}^d$ as possible.

(b) **Nonnegativity**. The λ_j should be nonnegative, e.g., on the convex hull of the points.

(c) **Smoothness.** The coordinates λ_j should be as smooth as possible.

(d) **Lagrange property.** The function $\sum_j f(p_j)\lambda_j$ should interpolate f at the points.

Our coordinates λ_j are affine functions, and so satisfy (a) and (c) in the strongest possible way (typically generalisations are rational functions or piecewise polynomials). This precludes the Lagrange property (d), except when the number of distinct points in P is $d+1$. For (b), we show that the λ_j are nonnegative on a convex polytope which contains the barycentre. This is pleasing, as this property is known to be incompatible with (a) and (c), see, e.g., [12] and the comments in [8]. Further, our construction depends *only on the points* (p_j), and not on a polytope of which they are vertices. Thus they need not be the vertices of a convex polytope (they may even be repeated), and no connectivity information is required to calculate them. This is in contrast to most methods, e.g., the *Wachspress coordinates* ([11]), *mean value coordinates* ([5]), and *convex set coordinates* ([10]).

The Lagrange property is central to current applications in CAGD, and so perhaps our coordinates are better suited to applications involving finite elements and orthogonal polynomials, where interpolation is not a key issue.

2 Affine coordinates with minimal ℓ_2–norm

The main result, presented here, holds for any sequence of points $P = (p_1, \ldots, p_n)$ (points may be repeated) in an affine space. For simplicity, we consider the Hilbert space $\mathcal{H} = \mathbb{R}^d$, and allow only affine combinations of those vectors we call points, and inner products to be taken only between vectors.

For a sequence $(v_1, \ldots, v_n)$ of vectors in $\mathcal{H}$, we define the *synthesis map* by

$$V = [v_1, \ldots, v_n] : \mathbb{R}^n \to \mathcal{H} : a \mapsto \sum_j a_j v_j.$$

This takes the coefficients (a_j) to the linear combination $\sum_j a_j v_j$ of the vectors in (v_j), as the matrix notation suggests. Thus V is onto if and only if the vectors (v_j) span the space $\mathcal{H}$. Its adjoint is $V^* : \mathcal{H} \to \mathbb{R}^n : f \mapsto (\langle f, v_j \rangle)_{j=1}^n$.

Definition 2.1 *For points $(p_j)_{j=1}^n$ in $\mathbb{R}^d$, the unique coefficients $(\lambda_j(x))_{j=1}^n$ of minimal ℓ_2–norm for which*

$$x = \sum_j \lambda_j(x)\, p_j, \qquad \sum_j \lambda_j(x) = 1$$

are called the **affine generalised barycentric coordinates** *of the point $x \in \mathbb{R}^d$.*

The following explicit formula makes many properties of these coordinates apparent.

Theorem 2.1 *(Formula) Let $P = (p_j)_{j=1}^n$ be points in $\mathbb{R}^d$, with affine hull $\mathbb{R}^d$, and barycentre $c = c_P := \frac{1}{n}\sum_j p_j$. Let $v_j := p_j - c$. Each $x \in \mathbb{R}^d$ can be written uniquely as*

$$x = \sum_{j=1}^n \lambda_j(x) p_j, \qquad \sum_{j=1}^n \lambda_j(x) = 1,$$

where coefficients $a = (\lambda_j(x)) \in \mathbb{R}^n$ have minimal ℓ_2–norm, and are given by

$$\lambda_j(x) = \langle x - c, (VV^*)^{-1} v_j \rangle + \frac{1}{n}, \qquad V = [v_1, \ldots, v_n], \tag{2.3}$$

Moreover, if $x = \sum_j \xi_j(x) p_j$ for some coefficients $\xi_j(x)$ with $\sum_j \xi_j(x) = 1$, then

$$\sum_j |\xi_j(x)|^2 = \sum_j |\lambda_j(x)|^2 + \sum_j |\xi_j(x) - \lambda_j(x)|^2. \tag{2.4}$$

Proof: Let $\vec{1} \in \mathbb{R}^n$ be the vector of 1's. We seek to minimise $a \in \mathbb{R}^n$, where

$$\sum_j a_j p_j = x, \qquad \sum_j a_j = 1.$$

By making the substitutions $a_j = b_j + \frac{1}{n}$ and $p_j = v_j + c$, this can be written

$$Vb = \sum_j b_j v_j = x - c, \qquad \sum_j b_j = 0.$$

The minimal ℓ_2–norm solution to the first equation is

$$b = V^+(x-c), \qquad V^+ \text{ the (Moore–Penrose) pseudoinverse}$$

which automatically satisfies $\sum_j b_j = 0$. We now simplify $b = V^+(x-c)$, and show that the corresponding a is the minimal norm solution of the original system.

Since the affine hull of $\{p_j\}$ is $\mathbb{R}^d$, the vectors $v_j = p_j - c$ span $\mathbb{R}^d$, i.e., $V : \mathbb{R}^n \to \mathbb{R}^d$ is onto, so that $V^+ = V^*(VV^*)^{-1}$, and

$$b_j = \langle V^*(VV^*)^{-1}(x-c), e_j \rangle = \langle x-c, (VV^*)^{-1}Ve_j \rangle = \langle x-c, (VV^*)^{-1}v_j \rangle.$$

This establishes the proposed formula for $a_j = \lambda_j(x)$.

We now show that $a = \lambda(x)$ is indeed the ℓ_2–norm minimiser. Suppose

$$x = \sum_j \xi_j(x) p_j, \quad \sum \xi_j(x) = 1, \qquad \text{i.e.,} \quad V\xi(x) + c = x, \quad \langle \xi(x), \vec{1} \rangle = 1$$

Since $\sum_j v_j = 0$ can be written as $V\vec{1} = 0$, and $\langle \vec{1}, \vec{1} \rangle = n$, we calculate

$$\langle \lambda(x), \xi(x) \rangle = \langle V^*(VV^*)^{-1}(x-c) + \frac{1}{n}\vec{1}, \xi(x) \rangle = \langle (VV^*)^{-1}(x-c), x-c \rangle + \frac{1}{n},$$

$$\begin{aligned}\langle \lambda(x), \lambda(x) \rangle &= \langle V^*(VV^*)^{-1}(x-c) + \frac{1}{n}\vec{1}, V^*(VV^*)^{-1}(x-c) + \frac{1}{n}\vec{1} \rangle \\ &= \langle (VV^*)^{-1}(x-c), x-c \rangle + \frac{1}{n}.\end{aligned}$$

Thus the vectors $\lambda(x)$ and $\xi(x) - \lambda(x)$ are orthogonal, and so Pythagoras gives (2.4). □

The expansion (2.3) is the affine space analogue of the dual frame expansion (which minimises the ℓ_2–norm of the coefficients of a linear combination). Indeed, the vectors $\tilde{v}_j := (VV^*)^{-1}v_j$ give the *dual frame* for the frame (v_j) for $\mathbb{R}^d$ (cf [3]).

From the formula in (2.3), we observe that

- The coordinates of the barycentre c are $\lambda_j(c) = \frac{1}{n}$, $\forall j$.
- λ_j is constant (equal to $\frac{1}{n}$) if and only if p_j is the barycentre c.
- $\lambda_j = \lambda_k$ if and only if $p_j = p_k$.

- The λ_j are continuous functions of the points $p_1, \ldots, p_n$ (with affine hull $\mathbb{R}^d$).

These imply that the (closed) set of points where the coordinates are nonnegative

$$N = N_P := \{x \in \mathbb{R}^d : \lambda_j(x) \geq 0, \ \forall j\} \tag{2.5}$$

is a convex polytope, with the barycentre as an interior point (see §3).

We write the sequence obtained by removing the point p_j from $P = (p_1, \ldots, p_n)$ as

$$P \setminus p_j := (p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n),$$

and the affine hull of the points in P as

$$\mathrm{Aff}(P) := \{\sum_j \xi_j p_j : \sum_j \xi_j = 1\}.$$

The generalised barycentric coordinates also have some less obvious properties.

Proposition 2.1 *The coordinates* (λ_j) *of Theorem 2.1 satisfy the following*

(a) $\frac{1}{n} \leq \lambda_j(p_j) \leq 1$.

(b) $\lambda_j(p_k) = \lambda_k(p_j)$.

(c) $\lambda_j(p_j) = 1$ *if and only if* $p_j \notin \mathrm{Aff}(P \setminus p_j)$*, in which case* $\lambda_j = 0$ *on* $\mathrm{Aff}(P \setminus p_j)$.

(d) $\sum_j \lambda_j(p_j) = d + 1$.

Proof: (a) Recall that $(VV^*)^{-1}$ is positive definite, so that

$$\lambda_j(p_j) = \langle v_j, (VV^*)^{-1} v_j \rangle + \frac{1}{n} \geq \frac{1}{n}. \tag{2.6}$$

Since p_j can be written as the affine combination $1p_j + \sum_{k \neq j} 0 p_k$, we have

$$|\lambda_j(p_j)|^2 \leq \sum_k |\lambda_k(p_j)|^2 \leq 1^2 + \sum_{k \neq j} 0^2 = 1 \quad \Longrightarrow \quad \lambda_j(p_j) \leq 1.$$

(b) Similarly, since the inner product is symmetric

$$\lambda_j(p_k) = \langle v_k, (VV^*)^{-1} v_j \rangle + \frac{1}{n} = \langle (VV^*)^{-1} v_k, v_j \rangle + \frac{1}{n} = \langle v_j, (VV^*)^{-1} v_k \rangle + \frac{1}{n} = \lambda_k(p_j).$$

(c) Suppose $p_j \in \mathrm{Aff}(P \setminus p_j)$, so that p_j can be written as an affine combination $p_j = \sum_{k \neq j} \xi_k p_k$. Then p_j can also be expressed as the affine combination

$$p_j = (1 - t) p_j + \sum_{k \neq j} t \xi_k p_k, \qquad t \in \mathbb{R}.$$

The sum of the squares of the coefficients above is

$$(1 - t)^2 + \sum_{k \neq j} t^2 |\xi_k|^2 = 1 - 2t + t^2 \Big(1 + \sum_{k \neq j} |\xi_k|^2\Big), \tag{2.7}$$

which is strictly less than 1 for $t > 0$ sufficiently small, and so $\lambda_j(p_j) < 1$.

Conversely, suppose $p_j \notin \mathrm{Aff}(P \setminus p_j)$. Then the only way p_j can be expressed as an affine combination of the points in P is $1p_j + \sum_{k\neq j} 0p_k$, so that $\lambda_j(p_j) = 1$, and $\lambda_k(p_j) = 0$, $k \neq j$. In this case $\lambda_j(p_k) = \lambda_k(p_j) = 0$, $k \neq j$, so that $\lambda_j = 0$ on $\mathrm{Aff}(P \setminus p_j)$.

(d) Take the sum of (2.6) over j, and use $\mathrm{trace}(I_{\mathbb{R}^d}) = d$, to get

$$\sum_j \lambda_j(p_j) = \sum_j \Big(\langle v_j, (VV^*)^{-1} v_j \rangle + \frac{1}{n} \Big) = \sum_j \langle (VV^*)^{-1} v_j, v_j \rangle + 1$$
$$= \mathrm{trace}(V^*(VV^*)^{-1}V) + 1 = \mathrm{trace}(VV^*(VV^*)^{-1}) + 1 = d + 1.$$

□

We observe that (c) and (d) imply that the linear polynomial

$$L_P(f) := \sum_j f(p_j)\lambda_j \tag{2.8}$$

interpolates f at any point p_j for which $p_j \notin \mathrm{Aff}(P \setminus p_j)$. By variation of the above arguments, this polynomial also interpolates at points p^* which occur in P with some multiplicity, if p^* is not in the affine hull of the set of points in P minus the point p^*.

Let $S = S_P$ be the sum of the squares of the affine generalised barycentric coordinates (the quantity they minimise), i.e.,

$$S(x) = S_P(x) := \sum_j |\lambda_j(x)|^2.$$

This has some properties analogous to those of the λ_j.

Proposition 2.2 *The quadratic polynomial $S = S_P$ satisfies the following*

(a) $S(x) \geq \frac{1}{n}$, *with equality if and only if x is the barycentre c.*

(b) $S(p_j) \leq 1$, *with equality if and only if $p_j \notin \mathrm{Aff}(P \setminus p_j)$.*

(c) $\sum_j S(p_j) = d + 1$.

Proof: (a) The nonnegative quadratic S has a unique minimum when

$$\nabla S(x) = 2 \sum_j \lambda_j(x)(VV^*)^{-1} v_j = 0 \quad \Longleftrightarrow \quad \sum_j \lambda_j(x) v_j = \sum_j \lambda_j(x)(p_j - c) = x - c = 0,$$

and the minimum value is $S(c) = \sum_j (\frac{1}{n})^2 = \frac{1}{n}$.

(b) Recall the proof of Proposition 2.1. If $p_j \in \mathrm{Aff}(P \setminus p_j)$, then (2.7) implies

$$S(p_j) \leq (1-t)^2 + \sum_{k \neq j} t^2 |\xi_k|^2 < 1, \quad \text{for } t > 0 \text{ sufficiently small},$$

and if $p_j \notin \mathrm{Aff}(P \setminus p_j)$, then

$$S(p_j) = |\lambda_j(p_j)|^2 + \sum_{k\neq j} |\lambda_k(p_j)|^2 = 1^2 + \sum_{k \neq j} 0^2 = 1.$$

(c) Since $\sum_j v_j = 0$, we obtain

$$\sum_j S(p_j) = \sum_j \sum_k \Big| \langle v_j, (VV^*)^{-1} v_k \rangle + \frac{1}{n} \Big|^2 = \sum_j \sum_k |\langle (VV^*)^{-1} v_j, v_k \rangle|^2 + 1.$$

The second double sum above can be expressed in terms of the Frobenius norm

$$\sum_j \sum_k |\langle (VV^*)^{-1} v_j, v_k \rangle|^2 = \|V^*(VV^*)^{-1} V\|_F^2 = \mathrm{trace}(I_{\mathbb{R}^d}) = d,$$

which completes the proof. □

Example 1. Let V be a set of $d+1$ affinely independent points in $\mathbb{R}^d$, with barycentric coordinates $\xi = (\xi_v)_{v \in V}$. Suppose P is a sequence of points in V, with each v appearing with multiplicity $m_v \geq 1$. Then the coordinates with $p_j = v$ are equal, and they add to ξ_v, so that

$$\lambda_j = \frac{1}{m_v} \xi_v \quad \text{when} \quad p_j = v. \tag{2.9}$$

Thus

$$S_P(x) = \sum_j \lambda_j^2 = \sum_{v \in V} m_v \Big(\frac{\xi_v}{m_v} \Big)^2 = \sum_{v \in V} \frac{\xi_v^2}{m_v}, \qquad c_P = \frac{1}{n} \sum_{v \in V} m_v v. \tag{2.10}$$

3 The geometry of the region of nonnegativity

Recall from (2.5) the set $N = N_P$ where our coordinates λ for P are nonnegative

$$N = N_P := \{ x \in \mathbb{R}^d : \lambda_j(x) \geq 0, \ \forall j \},$$

which we will refer to as the **region of nonnegativity**.

For p_j not equal to the barycentre c_P, the affine generalised barycentric coordinate λ_j is not constant. It is most easily visualised in terms of the hyperplane

$$Z_j = Z_{j,P} := \{ x \in \mathbb{R}^d : \lambda_j(x) = 0 \} \qquad \text{(the zero set of } \lambda_j),$$

or, equivalently, the half space

$$H_j = H_{j,P} := \{ x \in \mathbb{R}^d : \lambda_j(x) \geq 0 \} \qquad \text{(the set where } \lambda_j \geq 0).$$

Since $\lambda_j(c_P) = \frac{1}{n}$, the coordinate λ_j is completely determined by the pair consisting of the point c_P and either of Z_j or H_j. The region of nonnegatively can be written

$$N_P = \bigcap_{j : p_j \neq c_P} H_j,$$

and so is a (bounded) convex polytope containing the barycentre (as previously claimed).

For the purposes of illustration, we also consider the ellipsoid

$$E = E_P := \{x \in \mathbb{R}^d : \sum_j |\lambda_j(x)|^2 = 1\} = S^{-1}(1),$$

which by Proposition 2.2 has centre c_P, and its convex hull

$$F = F_P := \{x \in \mathbb{R}^d : \sum_j |\lambda_j(x)|^2 \le 1\} = S^{-1}([0,1]).$$

The solid ellipsoid F_P contains the region of nonnegativity N_P, since

$$x \in N_P \quad \Longrightarrow \quad 0 \le \lambda_j(x) \le 1 \quad \Longrightarrow \quad \sum_j |\lambda_j(x)|^2 \le \sum_j \lambda_j(x) = 1.$$

Further, by Proposition 2.2, F_P contains the points P, and a point p_j lies on the ellipsoid E_P if and only if $p_j \notin \mathrm{Aff}(P \backslash p_j)$. Thus the distance of a point p_j from E_P can be thought of as a measure of how affinely independent it is from the other the other points in P.

Example 2. Let $P = (p_1, p_2, p_3)$ be three affinely independent points in $\mathbb{R}^2$, i.e., the vertices of a triangle. Then (λ_j) are the barycentric coordinates for this triangle, and E_P is the unique ellipse passing through the points of P, which therefore has the equation

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1.$$

The region of nonnegativity N_P is the solid triangle given by the convex hull of the points.

If the three points are repeated with some multiplicities, then by (2.9) the region of nonnegativity stays the same, but by (2.10) the ellipse enlarges (it can be made arbitrarily large). As a consequence, we have that any point in $\mathbb{R}^2$ (or $\mathbb{R}^d$) can be written as an affine combination of points from a fixed bounded set (with interior) where the sum of the squares of the coefficients is arbitrarily small.

4 Transformation of the coordinates under an affine map

The coordinates $\lambda = (\lambda_j)$ transform naturally under the action of an affine map.

Proposition 4.1 *(Affine maps). Let $P = (p_1, \ldots, p_n)$ be points with affine hull $\mathbb{R}^d$, $A : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible affine map, and $Q := AP = (Ap_1, \ldots, Ap_n)$. Denote the affine generalised barycentric coordinates for P and Q by λ_P and λ_Q. Then*

$$\lambda_Q(Ax) = \lambda_P(x), \qquad \forall x \in \mathbb{R}^d.$$

In particular, the region of nonnegativity, etc, transform as follows

$$N_{AP} = AN_P, \qquad Z_{j,AP} = AZ_{j,P}, \qquad H_{j,AP} = AH_{j,P},$$

$$S_{AP} = AS_P, \qquad E_{AP} = AE_P, \qquad F_{AP} = AF_P.$$

Proof: Write $Ax = Lx + b$, where L is a linear map. Then the barycentre of Q is

$$c_Q = \frac{1}{n}\sum_j (Lp_j + b) = L\Big(\frac{1}{n}\sum_j p_j\Big) + b = Lc_P + b,$$

so that $Ap_j = L(v_j + c_P) + b = Lv_j + c_Q$. Thus we have

$$\begin{aligned}\lambda_Q(Ax) &= (LV)^*(LV(LV)^*)^{-1}(Ax - c_Q) + \frac{1}{n}\vec{1} \\ &= V^*(VV^*)^{-1}L^{-1}(Lx - Lc_P) + \frac{1}{n}\vec{1} = \lambda_P(x).\end{aligned}$$

Since A is onto $\mathbb{R}^d$, this then gives

$$N_Q = \{y : \lambda_Q(y) \geq 0\} = \{Ax : \lambda_Q(Ax) = \lambda_P(x) \geq 0\} = A\{x : \lambda_P(x) \geq 0\} = AN_P.$$

For the remaining sets the arguments are similar. □

5 Further examples

We now illustrate the geometric nature of our coordinates via some examples.

Example 3. *Points in* $\mathbb{R}^1$. Suppose $d = 1$, and that $p_1 \leq p_2 \leq \cdots \leq p_n$. Then

$$\lambda_j(x) = \frac{p_j - c}{\sum_k |p_k - c|^2}(x - c) + \frac{1}{n}, \qquad p_1 < c := \frac{1}{n}\sum_k p_k < p_n.$$

The region of nonnegativity $N_P = [a, b]$, is given by $\lambda_n(a) = 0$, $\lambda_1(b) = 0$, i.e.,

$$a = c - \frac{1}{n}\frac{\sum_k |p_k - c|^2}{p_n - c}, \qquad b = c + \frac{1}{n}\frac{\sum_k |p_k - c|^2}{c - p_1}.$$

For large numbers of points chosen randomly from the interval $[0, 1]$, the limiting values are

$$a \to \frac{1}{2} - \frac{\int_0^1 |t - \frac{1}{2}|^2\,dt}{\frac{1}{2}} = \frac{1}{3}, \qquad b \to \frac{1}{2} - \frac{\int_0^1 |t + \frac{1}{2}|^2\,dt}{\frac{1}{2}} = \frac{2}{3}, \qquad n \to \infty,$$

and

$$S(x) = \sum_j |\lambda_j(x)|^2 = \frac{|x - c|^2}{\sum_k |p_k - c|^2} + \frac{1}{n} \to 0, \qquad n \to \infty.$$

Thus the region of nonnegativity does not shrink as the number of the points increases.

Example 4. *Four points in* $\mathbb{R}^2$ (see Figs. 1 and 2). Since N_P, E_P, etc transform naturally under invertible affine maps, we suppose, without loss of generality, that

$$P = \Big(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}a\\b\end{pmatrix}\Big),$$

where there are no restrictions on (a,b). By direct computation

$$\begin{aligned}
\lambda_{(0,0)}(x,y) &= \frac{(ab-1-a-b^2)x+(ab-1-b-a^2)y+1+a^2+b^2}{2(1+ab+a^2-a+b^2-b)}, \\
\lambda_{(1,0)}(x,y) &= \frac{(2+ab-a+2b^2-2b)x+(a-a^2-2ab)y+ab+a^2-a}{2(1+ab+a^2-a+b^2-b)}, \\
\lambda_{(0,1)}(x,y) &= \frac{(b-b^2-2ab)x+(2+ab-b+2a^2-2a)y+ab+b^2-b}{2(1+ab+a^2-a+b^2-b)}, \\
\lambda_{(a,b)}(x,y) &= \frac{(2a+b-1)x+(a+2b-1)y+1-a-b}{2(1+ab+a^2-a+b^2-b)}, \qquad (5.11)
\end{aligned}$$

where the coordinates are indexed by the points. We observe that these coordinates depend continuously on (a,b).

If the convex hull of the points is a quadrilateral Q, i.e., $a,b>0$ and $a+b>1$, then the polytope N_P (which depends continuously on P) has four vertices, one of which lies on the edge from $(0,0)$ to $(0,1)$, namely

$$\lambda_{(0,1)}(x,y)=\lambda_{(a,b)}(x,y)=0 \quad\iff\quad (x,y)=\big(\tfrac{a+b-1}{2a+b-1},0\big).$$

Thus we conclude that N_P circumscribes boundary of Q, with one point on each edge. The barycentre of the vertices of N_P is not $c_P=\frac{1}{4}(a+1,b+1)$ in general, since the vertices are

$$\begin{pmatrix}\frac{a+b-1}{2a+b-1}\\ 0\end{pmatrix},\quad \begin{pmatrix}0\\ \frac{a+b-1}{a+2b-1}\end{pmatrix},\quad \begin{pmatrix}\frac{ab}{b+1}\\ \frac{b^2+1}{b+1}\end{pmatrix},\quad \begin{pmatrix}\frac{a^2+1}{a+1}\\ \frac{ab}{a+1}\end{pmatrix}.$$

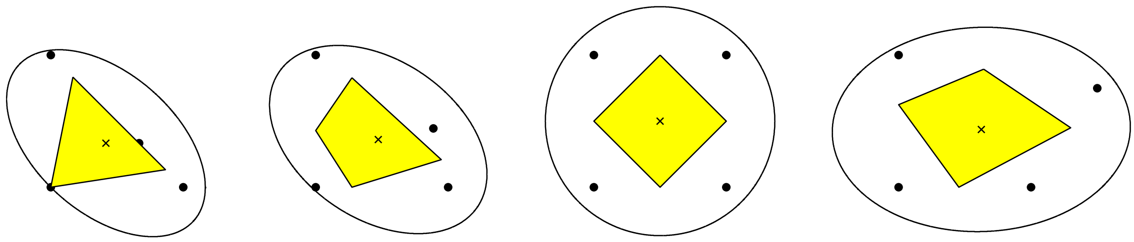

Figure 1: The points P, E_P,N_P,c_P for Ex. 4 with $(a,b)=(\frac{2}{3},\frac{1}{3}),(\frac{8}{9},\frac{4}{9}),(1,1),(\frac{3}{2},\frac{3}{4})$.

If the convex hull of the points is a triangle, i.e., one point is in the convex hull of the other three, then N_P is a triangle. If the interior point is (a,b), then the vertices of N_P are

$$\begin{pmatrix}\frac{ab}{b+1}\\ \frac{b^2+1}{b+1}\end{pmatrix},\quad \begin{pmatrix}\frac{a^2+1}{a+1}\\ \frac{ab}{a+1}\end{pmatrix},\quad \begin{pmatrix}\frac{a(a+b-1)}{a+b-2}\\ \frac{b(a+b-1)}{a+b-2}\end{pmatrix}.$$

The formulas in (5.11) can be expressed in terms of the barycentric coordinates for three of them as follows. Let $(\xi_v)_{v\in V}$ be the barycentric coordinates for three points in $\mathbb{R}^2$, and a be another. Then the coordinates λ for the four points $V\cup a$ are

$$\lambda_v=\xi_v-\xi_v(a)\lambda_a, \qquad \lambda_a=\frac{1}{S_V(a)+1}\sum_v \xi_v(a)\xi_v, \qquad S_V:=\sum_v|\xi_v|^2.$$

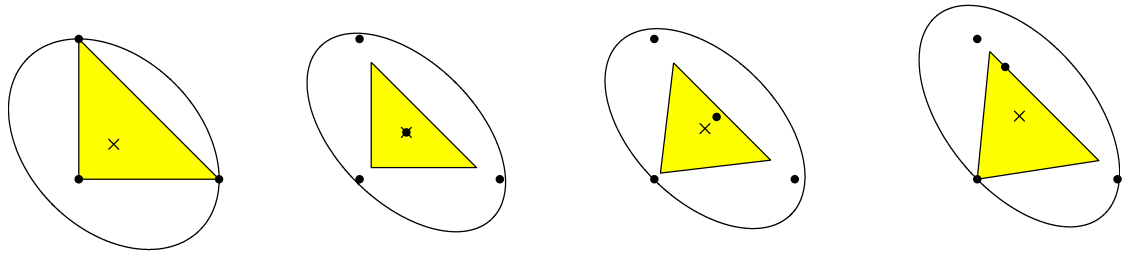

Figure 2: The points P, E_P,N_P,c_P for Ex. 4 with $(a,b)=(0,0),(\frac{1}{3},\frac{1}{3}),(\frac{4}{9},\frac{4}{9}),(\frac{1}{5},\frac{4}{5})$.

The values of the coordinates at the points are

$$\lambda_a(a)=\frac{S_V(a)}{S_V(a)+1},\qquad \lambda_v(v)=1-\frac{|\xi_v(a)|^2}{S_V(a)+1},\qquad \lambda_a(v)=\lambda_v(a)=\frac{\xi_v(a)}{S_V(a)+1}.$$

Example 5. *The vertices of a regular polygon* (see Fig. 3). Fix a centre c, and let (v_j) be n equally spaced unit vectors in $\mathbb{R}^2$, say

$$v_j=\begin{pmatrix}\cos\frac{2\pi}{n}j\\ \sin\frac{2\pi}{n}j\end{pmatrix},\qquad j=1,\ldots,n.$$

Then $p_j:=v_j+c$ are the vertices of a regular n–gon. Since $VV^*=\frac{n}{2}I$, we have

$$\lambda_j(x)=\langle x-c,\tfrac{2}{n}v_j\rangle+\tfrac{1}{n},\qquad \tilde{p}_j=\tfrac{2}{n}v_j+c.$$

This implies $\lambda_j(p_j)=\frac{3}{n}$ and that λ_j is zero on the line through the point $c-\frac{1}{2}v_j$, which is orthogonal to the vector v_j. By trigonometry, the lines Z_j and Z_{j+1} intersect at the point

$$w_j:=c-\frac{1}{2\cos\frac{\pi}{n}}\begin{pmatrix}\cos\frac{2\pi}{n}(j+\frac{1}{2})\\ \sin\frac{2\pi}{n}(j+\frac{1}{2})\end{pmatrix},$$

and N_P is the n–sided regular polygon (inscribing a circle of radius $1/(2\cos\frac{\pi}{n})$) given by

$$N_P=\operatorname{conv}\{w_j:j=1,\ldots,n\}.$$

The ellipse E_P is the disc of radius $r=\sqrt{\frac{n-1}{2}}$ centred at c, since

$$\sum_j|\lambda_j(x)|^2=\sum_j\Bigl|\langle x-c,\frac{2}{n}v_j\rangle+\frac{1}{n}\Bigr|^2=\sum_j|\langle x-c,\frac{2}{n}v_j\rangle|^2+\frac{1}{n}=\frac{2}{n}\|x-c\|^2+\frac{1}{n}.$$

By writing our expansion as

$$x=\frac{1}{n}\sum_{j=1}^n(n\lambda_j(x))\,p_j,\qquad \frac{1}{n}\sum_{j=1}^n(n\lambda_j(x))=1,$$

we can obtain the limiting case (of points on the unit circle)

$$x=\frac{1}{2\pi}\int_0^{2\pi}\lambda_\theta(x)\,p_\theta\,d\theta,\qquad \frac{1}{2\pi}\int_0^{2\pi}\lambda_\theta(x)\,d\theta=1,$$

where

$$p_\theta := v_\theta + c, \quad v_\theta := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \lambda_\theta(x) := 2\langle x - c, v_\theta \rangle + 1, \qquad 0 \le \theta \le 2\pi.$$

Here the coordinates λ_θ are nonnegative on the disc with centre c and radius $\frac{1}{2}$.

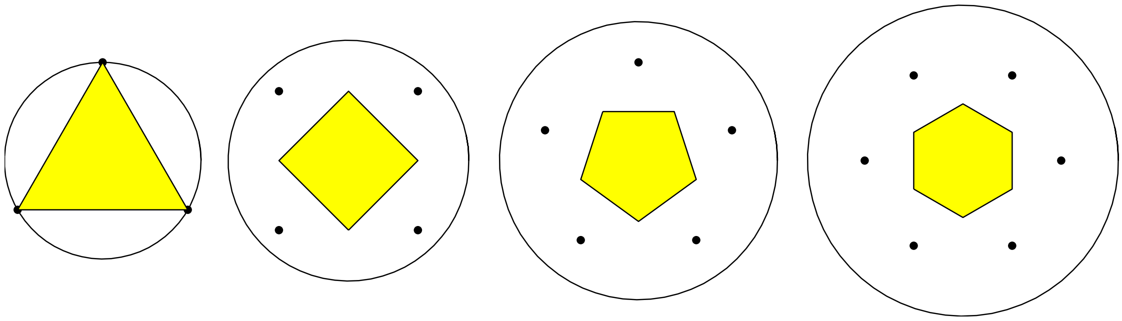

Figure 3: The $n = 3, 4, 5, 6$ equally spaced points P, with the ellipse E_P and region N_P.

Example 6. *Generalised Bernstein polynomials* (see Fig. 4). By the multinomial theorem,

$$(\lambda_1 + \cdots + \lambda_n)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} \lambda^\alpha = 1, \qquad \binom{k}{\alpha} := \frac{k!}{\alpha!}, \quad \lambda^\alpha = \prod_j \lambda_j^{\alpha_j},$$

where α is a multi-index, i.e., $\alpha \in \mathbb{Z}_+^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Thus the polynomials

$$B_\alpha := \binom{|\alpha|}{\alpha} \lambda^\alpha, \qquad |\alpha| = k,$$

(which span Π_k the polynomials on $\mathbb{R}^d$ of degree $\le k$) form a partition of unity which is nonnegative on the polytope N_P. They are linearly independent if and only if $n = d+1$, in which case they are the *multivariate Bernstein basis* for Π_k. The **multivariate Bernstein operator** of degree $k \ge 1$ (cf [2]) can be generalised via

$$B_{k,P} f(x) := \sum_{|\alpha|=k} f(v_\alpha) B_\alpha, \qquad v_\alpha := \sum_j \frac{\alpha_j}{|\alpha|} p_j.$$

This maps functions f which are nonnegative at the points $(v_\alpha)_{|\alpha|=k}$ (which are contained in the convex hull of the points in P) to polynomials in Π_k which are nonnegative on N_P. Further, $B_{k,P}$ reproduces the linear polynomials $\Pi_1 = \operatorname{span}\{\lambda_\ell\}$, since

$$B_{k,P}(\lambda_\ell) = \sum_{|\alpha|=k} \lambda_\ell(v_\alpha) B_\alpha = \sum_{|\alpha|=k} \lambda_\ell \Big(\sum_j \frac{\alpha_j}{k} p_j \Big) \frac{k!}{\alpha!} \lambda^\alpha = \sum_j \lambda_\ell(p_j) \lambda_j \sum_{\substack{|\alpha|=k \\ \alpha_j > 0}} \frac{\alpha_j}{k} \frac{k!}{\alpha!} \lambda^{\alpha - e_j}$$

$$= \sum_j \lambda_\ell(p_j) \lambda_j \sum_{|\beta|=k-1} \frac{(k-1)!}{\beta!} \lambda^\beta = \sum_j \lambda_\ell(p_j) \lambda_j = \lambda_\ell.$$

where e_j is the multiindex which is 1 at j and zero otherwise, and we let $\beta = \alpha - e_j$.

Similarly, the **multivariate Bernstein–Durrmeyer operator** can be generalised via

$$M_k f := \sum_{|\alpha|=k} \frac{\langle f, B_\alpha \rangle}{\langle 1, B_\alpha \rangle} B_\alpha, \qquad \langle f, g \rangle := \int f g \, d\mu,$$

where μ is a suitable Jacobi–type measure (cf [4]), which gives a self adjoint operator.

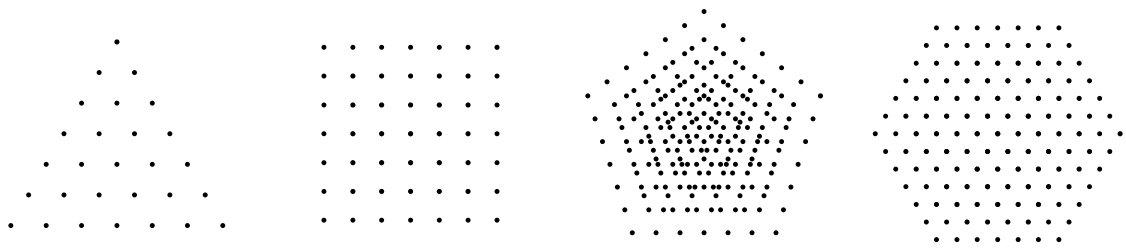

Figure 4: The $(v_\alpha)_{|\alpha|=k}$ for $B_{k,P}$, where P is $3, 4, 5, 6$ equally spaced points and $k = 6$.

Example 7. *Least squares method.* In view of (1.2), the map L_P of (2.8) is a projection onto the linear polynomials. It is similar, in spirit, to the global *least squares* (LS) method. The LS linear polynomial approximation to f from its values at a cloud of points (p_j) is the unique $g \in \Pi_1$ which minimises

$$\sum_j |g(p_j) - f(p_j)|^2.$$

Since each $g \in \Pi_1$ can be written in the form $g = \sum_j a_j \lambda_j$, this *LS* approximation is given by the least squares solution for $a = (a_j)$ to the linear system

$$\begin{pmatrix} g(p_1) \\ \vdots \\ g(p_n) \end{pmatrix} = \begin{pmatrix} \lambda_1(p_1) & \cdots & \lambda_n(p_1) \\ \vdots & & \vdots \\ \lambda_1(p_n) & \cdots & \lambda_n(p_n) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(p_1) \\ \vdots \\ f(p_n) \end{pmatrix}.$$

For this g to be equal to $L_P(f)$, we must have $a_j = f(p_j)$, i.e., the rank $d+1$ matrix above must be the identity. Thus the LS linear polynomial approximation and $L_P(f)$ are equal only when $n = d + 1$, in which they are Lagrange interpolation at the vertices of a simplex.

It is hoped these examples illustrate the geometric nature of our *affine generalised barycentric coordinates*, and hence their possible usefulness in applications. Further, since the coordinates λ_j are *linear polynomials*, they can be used to construct nice spanning sets for the polynomials of any degree defined on the convex hull of the points P (as per the generalised Bernstein polynomials of Example 6). The fact these spanning sets may not be a basis can be quite advantageous, as it allows the symmetries of the points to be retained, thereby allowing for simpler (albeit redundant) expansions (cf [9]).

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