Bi-orthogonal structures and averages of derivatives of characteristic polynomials over the classical groups $USp(2N)$, $SO(2N)$ and $O^{-}(2N)$

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- Moments of L-functions, of their derivatives, arising in number theory [KS 2001], [CRS 2006]
- Solution of the third Painlevé equation [FW 2006]
- Bi-orthogonal Polynomial Systems on Unit-Circle + von Mises weight
- averages over the groups $USp(2N)$, $SO(2N)$ and $O^{-}(2N)$ [ABP... 2014]
- New Bi-orthogonal Polynomial Systems on Unit-Circle
- Muttalib-Borodin ensembles [Mut 1995], [Bor 1999]

- Conrey and Ghosh 1998 reformulated an earlier conjecture that the $2k$-th moment of $|\zeta(1/2+it)|$ should grow like

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k} dt \sim \frac{a(k)g(k)}{k^2!}(\log T)^{k^2}, \quad k \in \mathbb{N}, \quad T \to \infty$$

$a(k)$ is an "arithmetic factor"
$g(k)$ is an integer

- Conjecture by Keating and Snaith, *Random matrix theory and* $|\zeta(1/2+it)|$, Commun. Math. Phys. **214**, 57-89 (2000)

$$\frac{g(k)}{k^2!} = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

- How? Average of characteristic polynomial $|\det(U - e^{i\theta} I)|$ raised to the $2k$-th power over the Unitary group $U(N)$: CUE$_N$

$$\mathbb{E}_{U(N)} |\det(U - e^{i\theta} I)|^{2k} = \prod_{j=0}^{N-1} \frac{j!(j+2k)!}{((j+k)!)^2}, \quad \text{by Selberg's Integral}$$

Identification of the rank N

$$N \to \frac{\log T}{2\pi}$$

Arithmetic factor

$$a(k) = \prod_{p,\text{primes}} (1-p^{-1})^{k^2} {}_2F_1(k,k;1;p^{-1})$$

where ${}_2F_1(a,b;c;t)$ is the Gauss hypergeometric function

B. Conrey, D. Farmer, J.P. Keating, M. Rubinstein and N. Snaith, *Integral moments of L-functions*, Proc. London Math. Soc. **91**, 33-104 (2005)

B. Conrey, D. Farmer, J.P. Keating, M. Rubinstein and N. Snaith, *Lower order terms in the full moment conjecture for the Riemann zeta function*, J. Num. Theory, **128**, 1516-1554 (2008).

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k} dt \sim \frac{1}{T}\int_0^T P_k\left(\log\frac{t}{2\pi}\right) dt$$

where $P_k(.)$ is an explicit polynomial of degree k^2 with leading coefficient $a(k)g(k)/k^2!$.

With A a Haar distributed element of the unitary group $U(N)$, and $e^{i\theta_1}, \dots, e^{i\theta_N}$ its eigenvalues, let

$$\Lambda_A(s) = \prod_{j=1}^{N} (1 - se^{-i\theta_j}),$$

and

$$\mathcal{Z}_A(s) = e^{-\pi i N/2} e^{i \sum_{n=1}^{N} \theta_n/2} s^{-N/2} \Lambda_A(s),$$

Note that $\mathcal{Z}_A(e^{i\theta})$ is real for θ real.

Two results from Conrey, Rubinstein and Snaith [2006] are

$$\langle |\Lambda_A'(1)|^{2k} \rangle_{A \in U(N)} \underset{N\to\infty}{\sim} b_k N^{k^2+2k},$$

where

$$b_k = (-1)^{k(k+1)/2} \sum_{h=0}^{k} \binom{k}{h} (k+h)! [x^{k+h}] \left(e^{-x} x^{-k^2/2} \det[I_{\alpha+\beta-1}(2\sqrt{x})]_{\alpha,\beta=1,\dots,k} \right),$$

and

$$\langle |\mathcal{Z}_A'(1)|^{2k} \rangle_{A \in U(N)} \underset{N\to\infty}{\sim} b_k' N^{k^2+2k},$$

where

$$b_k' = (-1)^{k(k+1)/2} (2k)! [x^{2k}] \left(e^{-x/2} x^{-k^2/2} \det[I_{\alpha+\beta-1}(2\sqrt{x})]_{\alpha,\beta=1,\dots,k} \right).$$

In the above the notation $[x^p] f(x)$ denotes the coefficient of x^p in $f(x)$.

This determinant can be identified in terms of $\tilde{E}_2^{\mathrm{HE}}$ - a *tau*-function
We had shown in a previous study [FW 2002] (see section 4.3, in particular Eq. (4.31)) that for $a \in \mathbb{Z}_{\geq 0}$

$$\tilde{E}_2^{\mathrm{HE}}(s; a, \mu; \xi = 1) = A(a, \mu)\left(\frac{2}{\sqrt{s}}\right)^{a\mu} e^{-s/4} \det[I_{\mu+\alpha-\beta}(\sqrt{s})]_{\alpha,\beta=1,\ldots,a}.$$

where

$$A(a, \mu) = a! \prod_{j=1}^{a} \frac{(j + \mu - 1)!}{j!}.$$

Interchanging row β by row $a - \beta + 1$ ($\beta = 1, \ldots, a$ in order) we see from this that

$$b_k = \frac{(-1)^k}{A(k,k)} \sum_{h=0}^{k} \binom{k}{h} (k + h)! [x^{k+h}] \tilde{E}_2^{\mathrm{HE}}(4x; k, k; \xi = 1)$$

$$b'_k = \frac{(-1)^k}{A(k,k)} (2k)! [x^{2k}] \left(e^{x/2} \tilde{E}_2^{\mathrm{HE}}(4x; k, k; \xi = 1)\right)$$

According to [FW 2002], [Okamoto 1987]

$$\tilde{E}_2^{\mathrm{HE}}(4x;k,k;\xi=1)=\exp\left(-\int_0^{4x}\frac{ds}{s}\,(\sigma_{\mathrm{III}'}(s)+k^2)\right),$$

where $\sigma_{\mathrm{III}'}(s)$ satisfies the particular σ-Painlevé III′ equation

$$(s\sigma''_{\mathrm{III}'})^2+\sigma'_{\mathrm{III}'}(4\sigma'_{\mathrm{III}'}-1)(\sigma_{\mathrm{III}'}-s\sigma'_{\mathrm{III}'})-\frac{k^2}{16}=0, \qquad (*)$$

subject to the boundary condition

$$\sigma_{\mathrm{III}'}(s)\underset{s\to 0}{\sim}-k^2+\frac{s}{8}+\mathrm{O}(s^2), \quad k\in\mathbb{N}.$$

Substituting

$$\sigma_{\mathrm{III}'}(s)=\eta(s)+\frac{s}{8},$$

(*) reads

$$(s\eta'')^2+4((\eta')^2-\frac{1}{64})(\eta-s\eta')-\frac{k^2}{4^2}=0.$$

We see immediately that $\eta(s)$ can be expanded in an even function of s about $s = 0$,

$$\eta(s) = \sum_{n=0}^{\infty} c_n s^{2n}, \qquad c_0 = -k^2, \quad k \in \mathbb{N}.$$

Proposition

The coefficients $c_1, c_2, \ldots$ can be computed by

$$c_1 = \frac{1}{64(4k^2 - 1)},$$

while for $p \geq 2$ by the recurrence relation

$$c_p = \frac{1}{2c_1 p(2p-1) + (2p-1)/64 - 8pk^2 c_1}$$
$$\times\left(4k^2 \sum_{l=1}^{p-2}(l+1)(p-l)c_{l+1}c_{p-l} - \sum_{l=1}^{p-2}(l+1)(p-l)(2l+1)(2p-2l-1)c_{l+1}c_{p-l} - 4\sum_{l=1}^{p-1}(1-2l)c_l A_{p-l-1}\right),$$

where

$$A_q = \sum_{l=0}^{q}(l+1)(q-l+1)c_{l+1}c_{q-l+1}.$$

Thus the 16th member of the sequences are

$$b_{16} = \frac{307 \cdot 23581 \cdot 92867 \cdot 760550281759}{2^{272} \cdot 3^{130} \cdot 5^{66} \cdot 7^{42} \cdot 11^{24} \cdot 13^{21} \cdot 17^{16} \cdot 19^{14} \cdot 23^{10} \cdot 29^{6} \cdot 31^{5} \cdot 37^{3} \cdot 41^{2} \cdot 43^{2} \cdot 47 \cdot 53 \cdot 59 \cdot 61},$$

$$b'_{16} = \frac{4148297603 \cdot 76230778088705861517484553692172135066713345305 97}{2^{264} \cdot 3^{133} \cdot 5^{66} \cdot 7^{42} \cdot 11^{25} \cdot 13^{21} \cdot 17^{16} \cdot 19^{14} \cdot 23^{11} \cdot 29^{7} \cdot 31^{6} \cdot 37^{3} \cdot 41^{2} \cdot 43^{2} \cdot 47 \cdot 53 \cdot 59 \cdot 61}.$$

What is a random unitary matrix?
Spectral Decomposition for CUE_N, aka Dyson Circular Ensemble

- $N \times N$ Unitary Matrices , $U = (u_{j,k})_{1\le j,k\le N}$, $UU^\dagger = I$
  Eigenvalue Analysis $z_1 = e^{i\theta_1}, \dots, z_N = e^{i\theta_N}$,

$$U = VZV^\dagger$$

  $Z = \mathrm{diag}(z_1, \dots, z_N)$, Unitary $V = (V_1, \dots, V_N)$

- Change of variables $\{u_{j,k}\} \mapsto \{z_j, \dots, z_N\}, \{V_1, \dots, V_N\}$
  Jacobian gives the Haar measure

$$(dU) = \frac{1}{(2\pi)^N N!} \prod_{1\le j<k\le N} \left|e^{i\theta_k} - e^{i\theta_j}\right|^2 \wedge_{j=1}^N d\theta_j$$

- Joint Eigenvalue Probability Density Function: Weyl integration formula for class functions, $f(\mathrm{Tr}U, \dots, \mathrm{Tr}U^N)$

$$\mathbb{E}_{U(N)} f := \frac{1}{(2\pi)^N N!} \int_{-\pi}^{\pi} d\theta_1 \dots \int_{-\pi}^{\pi} d\theta_N f(\dots, \sum_{l=1}^{N} e^{ik\theta_l}, \dots) \prod_{1\le j<k\le N} \left|e^{i\theta_k} - e^{i\theta_j}\right|^2$$

Consider a complex weight function $w(z)$, analytic in the cut complex z-plane. The latter specification means $w(z)$ possesses a Fourier expansion

$$w(z) = \sum_{k=-\infty}^{\infty} w_k z^k, \quad w_k = \int_{\mathbb{T}} \frac{dz}{2\pi i z} w(z) z^{-k},$$

$\mathbb{T}$ denotes the unit circle $|z| = 1$, appropriately deformed so not to cross the cut, and $z = e^{i\theta}, \theta \in (-\pi, \pi]$.
For $\epsilon = 0, \pm 1$ we define the Toeplitz determinants

$$I_N^{\epsilon}[w] := \det\left[\int_{\mathbb{T}} \frac{dz}{2\pi i z} w(z) z^{\epsilon-j+k}\right]_{0\le j<k\le N-1} = \det\left[w_{-\epsilon+j-k}\right]_{0\le j<k\le N-1}$$

NB. In certain circumstances the weight is real and positive, $\overline{w(z)} = w(z)$, and thus the Toeplitz matrix is Hermitian, $\bar{w}_k = w_{-k}$, but in general this will not be true.

If $I_N^0[w] \neq 0$ for each $N = 1, 2, \ldots$ then there exists a system of bi-orthogonal polynomials $\{\phi_n(z), \overline{\phi}_n(z)\}_{n=0,1,\ldots}$ with the orthonormality property

$$\int_{\mathbb{T}} \frac{dz}{2\pi i z} w(z) \phi_m(z) \overline{\phi}_n(z^{-1}) = \delta_{m,n}$$

Special notation for the various coefficients in $\phi_n(z)$ according to

$$\phi_n(z) = \kappa_n z^n + l_n z^{n-1} + m_n z^{n-2} + \ldots + \phi_n(0) = \sum_{j=0}^{n} c_{n,j} z^j$$

WLOG κ_n is chosen to be real and positive.
Define the reciprocal polynomial by

$$\phi_n^*(z) := z^n \bar{\phi}(1/z) = \sum_{j=0}^{n} \bar{c}_{n,j} z^{n-j}$$

The ratios $r_n = \phi_n(0)/\kappa_n, \bar{r}_n = \bar{\phi}_n(0)/\kappa_n$ are *reflection coefficients, Verblusky coefficients*
From the Szegö/Geronimus theory these coefficients are related to the above Toeplitz determinants

$$r_N = (-1)^N \frac{I_N^1[w]}{I_N^0[w]}, \quad \bar{r}_N = (-1)^N \frac{I_N^{-1}[w]}{I_N^0[w]}.$$

In the case that $w(z)$ is not real, $\bar{r}_N$ is not the complex conjugate of r_N but rather an independent variable.
Note - normalisation implies that $\kappa_0 = 1$ and thus $r_0 = \bar{r}_0 = 1$.
Knowledge of $\{r_N, \bar{r}_N\}_{N=0,1,\ldots}$ is sufficient to compute $\{I_N^0[w]\}_{N=0,1,\ldots}$.

$$\frac{I_{N+1}^0[w] I_{N-1}^0[w]}{(I_N^0[w])^2} = 1 - r_N \bar{r}_N.$$

$$
\begin{aligned}
I_N^{\epsilon}[w] &:= \det\left[\int_{\mathbb{T}} \frac{dz}{2\pi i z} w(z) z^{\epsilon-j+k}\right]_{0\le j<k\le N-1} \\
&= \frac{1}{N!}\int_{\mathbb{T}} \frac{dz_1}{2\pi i z_1} \cdots \int_{\mathbb{T}} \frac{dz_N}{2\pi i z_N} \prod_{l=1}^{N} z^{\epsilon} w(z_l) \prod_{1\le j<k\le N} \left|z_k - z_j\right|^2 \\
&= \mathbb{E}_{U(N)} \prod_{l=1}^{N} z_l^{\epsilon} w(z_l) \\
&= \frac{1}{N!}\int_{\mathbb{T}} \frac{dz_1}{2\pi i z_1} \cdots \int_{\mathbb{T}} \frac{dz_N}{2\pi i z_N} \prod_{l=1}^{N} z^{\epsilon} w(z_l) \prod_{1\le j<k\le N} \left(z_k - z_j\right)\left(z_k^{-1} - z_j^{-1}\right)
\end{aligned}
$$

Further formulae from the Szegö/Geronimus theory.
First, as a consequence of the orthogonality condition we have the mixed linear recurrence relations for ϕ_n and ϕ^*_n,

$$\kappa_n \phi_{n+1}(z) = \kappa_{n+1} z \phi_n(z) + \phi_{n+1}(0)\phi^*_n(z)$$
$$\kappa_n \phi^*_{n+1}(z) = \kappa_{n+1} \phi^*_n(z) + \bar{\phi}_{n+1}(0) z \phi_n(z)$$

as well as the three-term recurrences

$$\kappa_n \phi_n(0)\phi_{n+1}(z) + \kappa_{n-1}\phi_{n+1}(0) z \phi_{n-1}(z) = \left[\kappa_n \phi_{n+1}(0) + \kappa_{n+1}\phi_n(0) z\right]\phi_n(z)$$
$$\kappa_n \bar{\phi}_n(0)\phi^*_{n+1}(z) + \kappa_{n-1}\bar{\phi}_{n+1}(0) z \phi^*_{n-1}(z) = \left[\kappa_n \bar{\phi}_{n+1}(0) z + \kappa_{n+1}\bar{\phi}_n(0)\right]\phi^*_n(z)$$

[Maroni, Magnus, ... 1985 ->]

Semi-Classical weights defined by

$$\frac{1}{w(z)}\frac{d}{dz}w(z) = \frac{2V(z)}{W(z)}$$

with $V(z)$ and $W(z)$ irreducible polynomials in z.

For non-integer values of μ the generalised *von Mises* weight function

$$w(z) = z^{\mu} e^{\frac{1}{2}\sqrt{t}(z+z^{-1})}$$

has a branch point at $z = 0$.
Denoting a Hankel contour by $\mathcal{C}$, and noting that the integral representation of the Bessel function of pure imaginary argument gives

$$\int_{\mathcal{C}} \frac{dz}{2\pi i z} w(z) = I_{\mu}(\sqrt{t})$$

we see that

$$I^{\epsilon}_{N}[w] = \det[I_{\mu+\epsilon+j-k}(\sqrt{t})]_{j,k=1,\dots,N}$$

For general μ know from [FW 2002]

$$\tau^{\mathrm{III}'}[N](t;\mu) = \det[I_{\mu+j-k}(\sqrt{t})]_{j,k=1,\dots,N}$$

It follows from Adler & van Moerbeke 2002, eq. (0.0.17) that

$$\tfrac{1}{2}\sqrt{t}v_N(r_{N+1}+r_{N-1})+Nr_N=0$$
$$\tfrac{1}{2}\sqrt{t}v_N(\bar{r}_{N+1}+\bar{r}_{N-1})+N\bar{r}_N=0$$

where $v_N := 1-r_N\bar{r}_N$. The initial conditions are:

$$r_0=\bar{r}_0=1, \quad r_1=-\frac{I_{\mu+1}(\sqrt{t})}{I_\mu(\sqrt{t})}, \quad \bar{r}_1=-\frac{I_{\mu-1}(\sqrt{t})}{I_\mu(\sqrt{t})}$$

Discrete Painlevé equation associated with degeneration of the rational surfaces $D_6^{(1)} \to E_7^{(1)}$ aka (discrete Painlevé II)

The reflection coefficients also satisfy the coupled 1/1 order recurrences

$$\tfrac{1}{2}\sqrt{t}(r_{N+1}\bar{r}_N+r_N\bar{r}_{N-1})+N\frac{r_N\bar{r}_N}{1-r_N\bar{r}_N}-\mu=0$$
$$\tfrac{1}{2}\sqrt{t}(\bar{r}_{N+1}r_N+\bar{r}_Nr_{N-1})+N\frac{r_N\bar{r}_N}{1-r_N\bar{r}_N}+\mu=0$$

with the same initial conditions.

Altuğ, Bettin, Petrow, Rishikesh and Whitehead (2014): $USp(2N)$, $SO(2N)$ and $O^-(2N)$ averages

- Instead of the I-Bessel functions have hypergeometric functions,

$$g_m(u) = \frac{1}{2\pi i} \oint_{|w|=1} \frac{e^{w+\frac{u}{w^2}}}{w^{m+1}}\, dw$$
$$= \frac{1}{\Gamma(m+1)}\, {}_0F_2\left(; \frac{m}{2}+1, \frac{m+1}{2}; \frac{u}{4}\right),$$

  for $u \in \mathbb{C}$ and $m \in \mathbb{Z}$.

- The role of the τ-function is played y

$$\mathcal{T}_{k,\ell}(u) := \det_{k\times k}\left(g_{2i-j+\ell}(u)\right),$$

  for $k \geq 0$, $\ell \in \mathbb{Z}$ and $u \in \mathbb{C}$, where, here and in the following, the indices i and j of the matrix in the determinant range from 1 to k.
  In the context of the Theorem below ℓ takes values $0, -1, 0$ respectively.

Theorem

As $N \to \infty$ we have

$$M_k(\mathrm{USp}(2N),2) = b_k(\mathrm{USp}(2N),2)\cdot(2N)^{\frac{k^2+5k}{2}} + O(N^{\frac{k^2+3k}{2}})$$

$$M_k(\mathrm{SO}(2N),2) = b_k(\mathrm{SO}(2N),2)\cdot(2N)^{\frac{k^2+3k}{2}} + O(N^{\frac{k^2+k}{2}})$$

$$M_k(\mathrm{O}^-(2N),3) = b_k(\mathrm{O}^-(2N),3)\cdot(2N)^{\frac{k^2+5k}{2}} + O(N^{\frac{k^2+3k}{2}})$$

where

$$b_k(\mathrm{USp}(2N),2) = 2^{-\frac{k^2+5k}{2}}\frac{d^k}{du^k}\left(e^u\mathcal{T}_{k,0}(2u)\right)\Big|_{u=0}$$

$$b_k(\mathrm{SO}(2N),2) = 2^{-\frac{k^2+k}{2}}\frac{d^k}{du^k}\left(e^u\mathcal{T}_{k,-1}(2u)\right)\Big|_{u=0}$$

$$b_k(\mathrm{O}^-(2N),3) = 3\cdot 2^{-\frac{k^2+3k}{2}}\frac{d^k}{du^k}\left(e^u\mathcal{T}_{k,0}(2u)\right)\Big|_{u=0}$$

Note that we find above that $b_k(\mathrm{O}^-(2N),2) = 3\cdot 2^k\cdot b_k(\mathrm{USp}(2N),2)$.

We define $m_n = \int_{\mathbb{T}} \lambda^n w(\lambda) \frac{d\lambda}{2\pi i\lambda}$ Let D_n be the following determinant :

$$D_n = \det\begin{pmatrix} m_0 & m_{-1} & m_{-2} & \cdots & m_{-n+1} \\ m_2 & m_1 & m_0 & \cdots & m_{-n+3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{2n-2} & m_{2n-3} & m_{2n-4} & \cdots & m_{n-1} \end{pmatrix} \equiv \det\left(m_{2i-j}\right)_{0\le i,j\le n-1}$$

In addition define the following *generalized* determinants

$$D_{n,\ell} = \det\left(m_{2i-j+\ell}\right)_{0\le i,j\le n}$$

Two sequences of *monic* polynomials $p_1(z), p_2(z), \cdots$ and $q_1(z), q_2(z), \cdots$ ($\deg p_k(z) = \deg q_k(z) = k$) with the *bi-orthogonality condition :*

$$\int_{\mathbb{T}} \frac{d\mu(z)}{2\pi iz} q_n(z^{-2}) p_k(z) = h_n \delta_{kn}$$

where $d\mu(z) \equiv w(z)dz$ for some weight function $w(z)$
Equivalent to the following orthogonality relations, $0 \leq k \leq n-1$

$$\int_{\mathbb{T}} \frac{d\mu(z)}{2\pi iz} z^{-2k} p_n(z) = h_n \delta_{kn}$$

and

$$\int_{\mathbb{T}} \frac{d\mu(z)}{2\pi iz} q_n(z^{-2}) z^k = h_n \delta_{kn}$$

Relation

$$h_n = \frac{D_{n+1}}{D_n}$$

$$p_n(z) := \frac{1}{D_n} \det \begin{pmatrix} m_0 & m_{-1} & m_{-2} & \cdots & m_{-n} \\ m_2 & m_1 & m_0 & \cdots & m_{-n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{2n-2} & m_{2n-3} & m_{2n-4} & \cdots & m_{n-2} \\ 1 & z & z^2 & \cdots & z^n \end{pmatrix}$$

Also let us define the polynomials $Q_n(z)$ as

$$q_n(z) := \frac{1}{D_n} \det \begin{pmatrix} m_0 & m_{-1} & m_{-2} & \cdots & m_{-n+1} & 1 \\ m_2 & m_1 & m_0 & \cdots & m_{-n+3} & z \\ m_4 & m_3 & m_2 & \cdots & m_{-n+5} & z^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ m_{2n} & m_{2n-1} & m_{2n-2} & \cdots & m_{n+1} & z^n \end{pmatrix}$$

Note that

$$p_n(0) = (-1)^n \frac{D_{n,-1}}{D_{n,0}}$$

and

$$q_n(0) = (-1)^n \frac{D_{n,+2}}{D_{n,0}}$$

The Muttalib-Borodin model refers to an eigenvalue PDF proportional to

$$\prod_{l=1}^{N} w(x_l) \prod_{1\le j<k\le N} (x_j - x_k)(x_j^\theta - x_k^\theta)$$

where $w(x)$ is real analytic on some support.

Eg: with $w(x) = x^c e^{-x}$ on $x \in (0, \infty)$ and $\theta \in \mathbb{N}$ this PDF is realised with x_l the squared singular values of a product of θ matrices $X_1, \dots, X_\theta$ with $(X_l)_{i,j} \overset{\mathrm{d}}{\sim} N[0,1] + iN[0,1]$.

Konhauser 1965, 1967; Iserles and Nørsett 1989; Claeys and Romano 2014

$$\int_{\mathbb{R}_{>0}} dx\, w(x) p_n(x) q_m(x^\theta) = \delta_{m,n} h_n$$

Equilibrium measure problem with potential - see Bueckner 1966; Bierman 1971; Williams 1978; Ioakimidis 1984; Estrada and Kanwal 1987

$$V(x,y) = \log|x-y| + \log|x^\theta - y^\theta|$$

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