# Optimal splitting of normalized tight frames using Walsh matrices

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# Weaver, 2004

Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{C}^m$ satisfy $\|\boldsymbol{v}_i\|^2 \leq \alpha$ for each $i = 1, \dots, n$ and suppose that

$$\sum_{i=1}^{n} |\boldsymbol{v}_i^* \boldsymbol{x}|^2 = 1$$

for all $\boldsymbol{x} \in \mathbb{C}^m$ with $\|\boldsymbol{x}\| = 1$. Then we can partition $\mathcal{J} = \{1, \dots, n\}$ into two disjoint sets $\mathcal{J}_1, \mathcal{J}_2$ such that

$$\left| \sum_{i \in \mathcal{J}_k} |\boldsymbol{v}_i^* \boldsymbol{x}|^2 - \frac{1}{2} \right| \leq 5\sqrt{\alpha}$$

for each $k = 1, 2$ and all $\boldsymbol{x} \in \mathbb{C}^m$ with $\|\boldsymbol{x}\| = 1$.

# Marcus, Spielman, Srivistava, 2014

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{C}^m$ are such that $\|\boldsymbol{v}_j\|^2 \leq \alpha$ for all $j = 1, \ldots, n$ and

$$\sum_{j=1}^{n} \boldsymbol{v}_j \boldsymbol{v}_j^* = I$$

then there is a partition of the index set $\mathcal{J} = \{1, \ldots, n\} \subset \mathbb{N}$ into two disjoint subsets $\mathcal{J}_1$ and $\mathcal{J}_2$ such that

$$\left\| \sum_{j \in \mathcal{J}_k} \boldsymbol{v}_j \boldsymbol{v}_j^* \right\|_2 \leq \left( \frac{1}{\sqrt{2}} + \sqrt{\alpha} \right)^2$$

for each $k = 1, 2$. The norm used here is the 2-norm

# Walsh function $W_k : [0,1] \to \{-1,1\}$ for $k \in \mathbb{N} - 1$

Choose $m \in \mathbb{N}$
Let each $k < 2^m$ be represented in binary form as

$$k = k_m \dots k_1 \Leftrightarrow \sum_{s=1}^{m} k_s 2^{s-1} \Leftrightarrow \boldsymbol{k} = (k_1, \dots, k_m, 0, 0, \dots) \in \{0,1\}^\infty$$

Let each $x \in [0,1]$ be represented in binary form as

$$x = 0 \cdot x_1 x_2 \cdots \iff \cdots \iff \boldsymbol{x} = (x_1, x_2, \cdots) \in \{0,1\}^\infty$$

where no expansion is permitted with $x_s = 1$ for all $s \geq n$ for some $n = n(x) \in \mathbb{N}$ ,
Then for each $x \in [0,1]$ we have

$$W_k(\boldsymbol{x}) = (-1)^{p(\boldsymbol{k},\boldsymbol{x})}, \text{for each } k < 2^m \text{where } p(\boldsymbol{k}, \boldsymbol{x}) = \sum_{s=1}^{m} k_s x_s$$

# Walsh matrix $Y = Y_r$

Let $n = 2^r$ for some $r \in \mathbb{N}$. The *Walsh matrix* $Y = Y_r \in \mathbb{C}^{n \times n}$ is defined by the Sylvester construction which can be implemented using the Matlab algorithm

$Y = [1];$
**for** $i = 1 : r$
$Y = [Y \;\; Y; Y \;\; -Y];$
**end**

The Sylvester construction gives the *natural ordering* for the rows and columns. The *sequency ordering* is also used. The advantage of the sequency ordering is that it corresponds to the ordering used for the Walsh functions. The disadvantage is that there is no easy construction. We will use the natural ordering.

# Some properties of Walsh functions and Walsh matrices

- Walsh functions form a complete orthonormal set in a Hilbert space $L^2[0,1]$.
- The matrix $Y$ is real symmetric with $y_{ij} = \pm 1$ for all $i, j$ and $Y^* Y = nI$.
- The columns $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$ form an orthogonal basis for $\mathbb{C}^n$ with $\|\boldsymbol{y}_j\| = \sqrt{n}$ for all $j = 1, \ldots, n$.

# Walsh Matrices: Natural Ordering

We have

$$Y_1 = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & -1 \end{array}\right],$$

$$Y_2 = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right]$$

and

$$Y_3 = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ \hline 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{array}\right].$$

# Normalized tight frames defined by Walsh matrices

Let $r \in \mathbb{N}+1$ and define $n = 2^r$ and $m \leq n$.
$Y$ be the corresponding Walsh matrix.
Define $G = Y/\sqrt{n}$, Clearly it is a unitary matrix.
Set $G = [\boldsymbol{g}_1, \dots, \boldsymbol{g}_n]$ and $W \in \mathbb{C}^{n \times m} = [\boldsymbol{g}_1, \dots, \boldsymbol{g}_m]$.

Let $V = W^* \in \mathbb{C}^{m \times n} = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$

The column vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ form a normalized tight frame in $\mathbb{C}^m$ with $\|v_j\| = \sqrt{\frac{m}{n}}$ for all $j = 1, \dots, n$.

# Objective

To discuss the discrepancy of a special class of normalised tight frames defined by Walsh matrices, we show that if $m, n \in \mathbb{N}$ with $m \leq n = 2^r$ for some $r \in \mathbb{N}$, then there is a normalized tight frame defined by a pre-frame matrix operator $V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] \in \mathbb{C}^{m \times n}$ where

$\boldsymbol{v}_j \in \mathbb{C}^m$ with $v_{ij} = \pm 1/\sqrt{n}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$ and $\|v_j\| = \sqrt{\frac{m}{n}}$ for all $j = 1, \ldots, n$.

In particular, we have

- for $m \leq \frac{n}{2}$, these frames can be split into two identical tight frames with frame constant $c = \frac{1}{2}$.
- for $\frac{n}{2} < m < n$ we show that the frames can no longer be evenly split but we find explicit expression for the discrepancy in a best possible split.

# Splitting a Walsh frame: A simple example

Let $m = 3$ and $n = 8$.
Using the first three rows of the Walsh matrix $Y_3$, we define

$$V = \frac{1}{2\sqrt{2}} \left[ \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{array} \right] = \left[ \begin{array}{c|c} V_a & V_b \end{array} \right]$$

$$VV^* = I_3$$

.

The normalized tight frame can be split into two identical frames with $V_a = V_b$ and $V_a V_a^* = V_b V_b^* = I/2$.
Renormalizing, we get

$$V = \frac{1}{2}\left[\begin{array}{rr|rr} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{array}\right] = \left[\begin{array}{c|c} V_a & V_b \end{array}\right]$$

$$VV^* = I_3$$

.

If we split the new normalized tight frame into two parts $V_a$ and $V_b$ then the two parts are no longer identical.
So from above split, we have

$$\|V_a V_a{}^* - I_3/2\|_2 = \|V_b V_b{}^* - I_3/2\|_2 = 1/2$$

which is less than the general bound given by Weaver.

# Quadratic forms

We have

$$\begin{aligned} \boldsymbol{x}^* VV^* \boldsymbol{x} &= x_1^2 + x_2^2 + x_3^2 \\ &= (x_1^2/2 + x_2^2/2 + x_3^2/2 + x_1 x_3) + (x_1^2/2 + x_2^2/2 + x_3^2/2 - x_1 x_3) \\ &= (\boldsymbol{x}^* V_a {V_a}^* \boldsymbol{x} + \boldsymbol{x}^* V_b {V_b}^* \boldsymbol{x}) \end{aligned}$$

*Note*

- columns of $V_a$ and $V_b$ do not span $\mathbb{C}^3$ and so fail to define frames.
- For $n = 2^r$, we can split this normalized tight frame into two identical normalized tight sub-frames each having $2^{r-1}$ elements.

# Main Result - I

Let $n = 2^r$ for some $r \in \mathbb{N}$ and suppose $m \in \mathbb{N}$ with $m \leq n/2 = 2^{r-1}$.
$W = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m]/\sqrt{n} \in \mathbb{C}^{n \times m}$ be defined by the first $m$ columns of the Walsh matrix $Y_r \in \mathbb{C}^{n \times n}$ and
$V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] = W^* \in \mathbb{C}^{m \times n}$. Then,

- $VV^* = I$
- The normalized tight frame in $\mathbb{C}^m$ defined by the columns of the matrix $V$ can be split into two equal parts defined by the columns of the matrices $V_1 = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n/2}]$ and $V_2 = [\boldsymbol{v}_{n/2+1}, \ldots, \boldsymbol{v}_n]$
- $V_1 V_1{}^* = V_2 V_2{}^* = I/2$.

# Main Result - II

Let $n = 2^r$ for some $r \in \mathbb{N}$ and suppose $m \in \mathbb{N}$ with $2^{r-1} = n/2 \leq m < n = 2^r$.
$Y_{r-1} = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{n/2}] \in \mathbb{C}^{(n/2)\times(n/2)}$ be the Walsh matrix of order $r-1$
Define

$$Y = \left[\begin{array}{ccc|ccc} \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_{n/2} & \boldsymbol{y}_{k(1)} & \cdots & \boldsymbol{y}_{k(s)} \\ \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_{n/2} & -\boldsymbol{y}_{k(1)} & \cdots & -\boldsymbol{y}_{k(s)} \end{array}\right] = \left[\begin{array}{c} Y_a \\ \hline Y_b \end{array}\right] \in \mathbb{C}^{n\times m}$$

where $s = m - n/2$ and $1 \leq k(1) < \cdots < k(s) \leq n/2$ is an arbitrarily selected subset of $\{1, \ldots, n/2\}$.
Let $V = Y^*/\sqrt{n}$, $V_a = {Y_a}^*/\sqrt{n}$ and $V_b = {Y_b}^*/\sqrt{n}$.
Then, $VV^* = I$

# Result – II *contd..*

The split defined by $V = [V_a \mid V_b]$ gives

$$V_a V_a{}^* - I/2 = \left[\begin{array}{c|c} 0 & \Delta/2 \\ \Delta^*/2 & 0 \end{array}\right] \quad \text{and } V_b V_b{}^* - I/2 = \left[\begin{array}{c|c} 0 & -\Delta/2 \\ -\Delta^*/2 & 0 \end{array}\right]$$

where $\Delta = [\boldsymbol{e}_{k(1)} \cdots \boldsymbol{e}_{k(s)}] \in \mathbb{C}^{(n/2)\times s}$.

The best possible error is

$$\|V_a V_a{}^* - I/2\|_2 = \|V_b V_b{}^* - I/2\|_2 = 1/2$$

We also have $\|V_a V_a{}^*\|_2 = \|V_b V_b{}^*\|_2 = 1$.

An interesting question is whether this result is completely specific or whether similar results apply to other *normalized tight frames* constructed from equal length vectors.

# Lemma

Let $m \in \mathbb{N}$ and $n = 2^r$ for some $r \in \mathbb{N}$. If $m \leq n$ and $V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] \in \mathbb{C}^{m \times n}$ defines a *normalized tight frame* for $\mathbb{C}^m$ with $S = VV^* = I_m$ and $\|\boldsymbol{v}_j\|^2 = \alpha$ for each $j = 1, \ldots, m$ then $\alpha = m/n$.

# Key Idea

Let $m \in \mathbb{N}$ and $n = 2^r$ for some $r \in \mathbb{N}$ and suppose $m \leq n$.
Suppose also that $V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] \in \mathbb{C}^{m \times n}$ defines a *normalized tight frame* with $\|\boldsymbol{v}_j\| = \sqrt{m/n}$ for each $j = 1, \ldots, m$.

Define $W = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m] \in \mathbb{C}^{n \times m}$ by setting $W = V^*$.
Since $W^*W = VV^* = I_m$ and so $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\} \in \mathbb{C}^n$ is an orthonormal set.
Let us extend this set to an orthonormal basis $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\} \in \mathbb{C}^n$.

Define the orthogonal matrix

$$H = [H_1 \mid H_2] = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m \mid \boldsymbol{w}_{m+1}, \ldots, \boldsymbol{w}_n] \in \mathbb{C}^{n \times n}$$

where $H_1 = W \in \mathbb{C}^{n \times m}$ and $H_2 \in \mathbb{C}^{n \times (n-m)}$.

Define $G = H^* \in \mathbb{C}^{n \times n}$ and we can write

$$G = [G_1 \mid G_2] = [\boldsymbol{g}_1, \ldots, \boldsymbol{g}_m \mid \boldsymbol{g}_{m+1}, \ldots, \boldsymbol{g}_n]$$

where $G_1 \in \mathbb{C}^{n \times m}$ and $G_2 \in \mathbb{C}^{n \times (n-m)}$.

The matrix $G$ defines an orthonormal basis $\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_m \mid \boldsymbol{g}_{m+1}, \ldots, \boldsymbol{g}_n\}$ for $\mathbb{C}^n$.

Let $Y = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n] \in \mathbb{C}^{n\times n}$ be the Walsh matrix of order r defined by the Sylvester algorithm.
Define $F = [\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m \mid \boldsymbol{f}_{m+1}, \ldots, \boldsymbol{f}_n] \in \mathbb{C}^{n\times n}$ by setting $F = \frac{Y}{\sqrt{n}} \in \mathbb{C}^{n\times n}$. we have

$$\boldsymbol{f}_j = \frac{y_j}{\sqrt{n}}, j = 1, \ldots, n$$

Note that the normalised Walsh matrix $F$ is real symmetric and orthogonal.

Define an orthogonal matrix $P \in \mathbb{C}^{n\times n}$ by setting $P = FH$. Therefore, $PG = F$ and $P\boldsymbol{g}_i = \boldsymbol{f}_i$ for all $j = 1, \ldots, n$. We use this matrix P to change the coordinate representation for the embedded normalized tight frame defined by $G$ into representation defined by $F$.

*Thank You*!