Grassmannians as universal minimizers of frame potentials

Kasso Okoudjou
(joint work with X. Chen, E. Goodman, M. Ehler, and V. Gonzales)

Department of Mathematics
&
Norbert Wiener Center
University of Maryland, College Park

Workshop on Tight frames and Approximation
Taipa, Doubtless Bay, NZ
February 20–23, 2018

Outline

p^{th} frame potentials and their minimizers

Definition

Let $N, d \geq 2$ and $p \in (0, \infty]$. The p^{th} frame potential of $\Phi = \{\varphi_k\}_{k=1}^N \subset S^{d-1}$ is given by

$$\mathrm{FP}_{p,N,d}(\Phi) = \sum_{k,\ell=1}^{N} |\langle \varphi_k, \varphi_\ell \rangle|^p$$

when $p < \infty$ and

$$\mathrm{FP}_{\infty,N,d}(\Phi) = \max_{k \neq \ell} |\langle \varphi_k, \varphi_\ell \rangle|$$

Let $\mu_{p,N,d}(0, \infty] \to (0, \infty)$ be the function given by

$$\mu_p := \mu_{p,N,d} = \min_{\Phi \subset S^{d-1}} \mathrm{FP}_{N,d,p}(\Phi) = \min_{\Phi \subset S^{d-1}} \sum_{k,\ell=1}^{N} |\langle \varphi_k, \varphi_\ell \rangle|^p.$$

Questions

Remark

For $p \in (1, \infty)$, $\mathrm{FP}_{p,N,d}$ is induced by a conservative force.

Question

Given $d \geq 2$, $p \in (0, \infty]$, and $N \geq 2$
- *What are the optimal configurations $\{\varphi_k\}_{k=1}^N \subset S^{d-1}$, i.e., the minimizers of $\mu_{p,d,N}$?*
- *Find an explicit formula for the function $\mu_{p,d,N}$.*

Frames, FUNTFs, and ETFs

Definition

A set $\{\varphi_k\}_{k=1}^N \subset \mathbb{R}^d$ is a *finite frame* for $\mathbb{R}^d$ if there exist , $0 < A, B < \infty$, such that for all $x \in \mathbb{R}^d$,

$$A\|x\|^2 \leq \sum_{k=1}^{N} |\langle x, \varphi_k \rangle|^2 \leq B\|x\|^2.$$

If $A = B$, then the frame is *tight*, and if each $\|\varphi_k\| = 1$, then the frame is *unit-norm*. If, for a constant $c \geq 0$, $|\langle \varphi_k, \varphi_\ell \rangle| = c$ for every $k \neq \ell$, the frame is *equiangular*. An (unit-norm) equiangular tight frame is called an *ETF*.

A special case: $p = 2$

Remark

For $N \geq 2$ and $d \geq 2$,

$$\mu_{2,N,d} = \begin{cases} N^2/d & \textit{if} \quad N \geq d \\ N & \textit{if} \quad N \leq d \end{cases}$$

- *$N \geq d$ the minimizers of $\mu_{2,N,d}$ are FUNTFs.*
- *$N \leq d$ they are ON sets.*

Indeed, finding $\mu_{2,N,d}$ is a equivalent to minimizing

$$\mathit{tr}(G^2) = \sum_{k=1}^{d} \lambda_k^2$$

subject to

$$\mathit{tr}(G) = \sum_{k=1}^{d} \lambda_k = N.$$

Some properties of μ_p

Proposition (X. Chen, E. Goodman, V. Gonzales, K.O (2018))

Let $N \geq d \geq 2$ be given and $p \in (0, \infty)$. The the following statements holds.

(a) *If $\Phi = \{\varphi_k\}_{k=1}^{N} \subset S^{d-1}$ is a minimizer of $\mu_{p,N,d}$, then Φ is a frame for $\mathbb{R}^d$.*

(b) *μ_p is a continuous non-increasing function on $(0, \infty)$.*

Another special case: $p = \infty$

Remark

For $N \geq 2$ and $d \geq 2$, Welch's bound

$$\mu_{\infty,N,d} \geq \sqrt{\frac{N-d}{d(N-1)}}$$

with equality if and only if Φ is an ETF. Furthermore, equality can hold only when $N \leq \frac{d(d+1)}{2}$.

Grassmannian frames

Definition

Let $N \geq d$. A sequence $\Phi_d^N = \{\varphi_k\}_{k=1}^N \subset S^{d-1}$ is an (N, d) Grassmannian frame if it is a frame and if

$$\tilde{\mu}_{\infty,N,d} := \min \mathrm{FP}_{\infty,N,d}(\Phi) = \mathrm{FP}_{\infty,N,d}(\Phi_d^N)$$

where the minimum is taken over all N-elements unit-norm frames Φ in $\mathbb{R}^d$.

Remark

(a) $\mu_{\infty,N,d} \leq \tilde{\mu}_{\infty,N,d}$.

(b) *For $N > d(d+1)/2$, the Welch bound implies that Grassmannian frames cannot be ETFs.*

(c) *for $d = 2$ and $N \geq 3$, Benedetto and Kolesar characterized all Grassmannian frames: $\{e^{\pi i k/N}\}_{k=0}^{N-1} \subset \mathbb{R}^2$.*

Another special case: $N = 3, d = 2$

Theorem (M. Ehler, K.O. (2012))

$$\mu_{p,3,2} = \begin{cases} 5 & for\, p \in (0, \frac{\log(3)}{\log(2)}] \\ 3 + 6e^{-p\log 2} & for\, p \geq \frac{\log(3)}{\log(2)}. \end{cases}$$

Furthermore, on $(0, \frac{\log(3)}{\log(2)}]$ *the minimizers are ONB+ one repeated vector, and on* $[\frac{\log(3)}{\log(2)}, \infty]$ *the minimizers are ETFs. The graph of* $\mu_{p,3,2}$ *when* $p \in (0, \infty)$ *is given below*

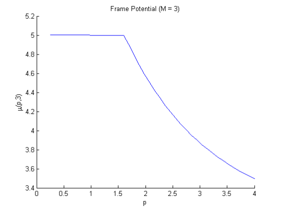

Figure: Graph of $\mu_{p,3,2}$ when $p \in (0, 4)$.

Sharp configurations

Definition

A finite subset of the unit sphere S^{d-1} is a sharp configuration if there are m inner products between distinct points in it and it is a spherical $(2m-1)$-design.

Example (Cohn and Kumar (2007))

Any N equally spaced points on S^1 is a sharp configuration.

Theorem (Cohn and Kumar (2007))

Let $f : (0,4] \to \mathbb{R}$ be completely monotonic (f is C^∞ and $(-1)^k f^{(k)}(x) \geq 0$), and let $\mathcal{C} \in S^{d-1}$ be a sharp configuration with $N = |\mathcal{C}|$. Then $\mathcal{C}$ is an optimal configuration (minimizer) for

$$\min_{\{x_k\}_{k=1}^N \subset S^{d-1}} \sum_{i,j} f(|x_i - x_j|^2)$$

Optimal configuration for the p^{th} frame potentials

Proposition

For $N \geq d \geq 2$ we have

- (a) $\mu_{p,2,3}$ *for $0 < p < \infty$, is minimized by either ONB+ (one repeated vector) or ETF (Ehler/Okoudjou (2012)).*
- (b) $\mu_{p,kd,d} = \frac{N^2}{d}$*, for $0 < p \leq 2$, is minimized by k copies of an ONB (Ehler/Okoudjou (2012)).*
- (c) $p > 2$*, $\mu_{p,N,d}$ is minimized by ETFs when they exist Ehler/Okoudjou (2012)).*
- (d) *New results: $\mu_{p,2k,2}$, for $p \in (2, \infty]$ and $k \geq 2$ (Chen, Goodman, Gonzales, K.O. (2018)).*

$\mu_{p,4,2}$ and its minimizers

Theorem (M. Ehler, K.O. (2012))

For $p \in (0, 2]$ $\mu_{p,4,2} = 8$ and the minimizers are 2 copies of an ONB. Furthermore, it was later conjecture that on $[2, \infty]$, $\mu_{p,4,2}$ should have the following graph and its minimizers must be equally spaced vectors on the upper half circle.

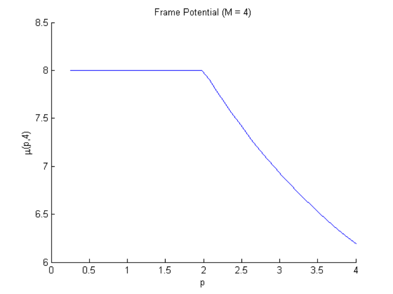

Figure: Graph of $\mu_{p,4,2}$ when $p \in (0, 4)$.

$\mu_{p,4,2}$ and its minimizers.

Theorem (Chen, Goodman, Gonzales, K.O. (2018))

$$\mu_{p,4,2} = \begin{cases} 8 & for\, p \in (0,2] \\ 4 + 2^{3-p/2} & for\, p \geq 2. \end{cases}$$

Furthermore, on $[2, \infty]$ *the (unique) minimizers are equally spaced points on the upper half circle.*

$\mu_{p,4,2}$ and its minimizers: Optimal configurations

Let $X_N = \{x_k\}_{k=1}^N$ be the subset of some sphere in dimension d. Let $K(x,y)$ be a nonnegative kernel and consider the energy

$$E_K(X_N) = \sum_{k<\ell} K(x_k, x_\ell).$$

Theorem (Chen, Goodman, Gonzales, K.O. (2018))

Let $f : (0, 4r^2] \to \mathbb{R}$ be a decreasing convex function defined at $t = 0$ by the (possibly infinite) value $\lim_{t\to 0^+} f(t)$. Any configuration X_4^* of 4 equally spaced points on any circle with radius r minimizes the discrete energy

$$E_K(X_4) = \sum_{k<\ell} K(x_k, x_\ell)$$

with the kernel $K(x,y) = f(\|x-y\|^2)$ when $N = 4$. If, in addition, f is strictly convex or strictly decreasing, then no other 4-point configuration is optimal.

$\mu_{p,4,2}$ and its minimizers: Optimal configurations

Remark

The last Theorem is the key to dealing with $\mu_{p,4,2}$. When we replace 4 by a number $N \geq 5$, then the argument breaks down.
The theorem is an extension of an old theorem of L. F. Tóth

Theorem (L. Fejes Tóth, (1959))

Let $f : (0, 2] \rightarrow \mathbb{R}$ be a decreasing convex function defined at $t = 0$ by the (possibly infinite) value $\lim_{t \to 0^+} f(t)$. Any configuration w_N^* of equally spaced points on S^1 minimizes the discrete energy E_K with the kernel $K(x, y) = f(\|x - y\|)$. If, in addition, f is strictly convex or strictly decreasing, then no other N-point configuration is optimal.

$\mu_{p,4,2}$ and its minimizers: A lifting trick

Let $M(2)$ denote the set of 2×2 matrices on $\mathbb{R}$. Define $P : S^1 \to M(2)$ by

$$P_x := P(x) = xx^* = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{bmatrix}$$

if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. P_x can be identified with $(x_1^2, \sqrt{2}x_1x_2, x_2^2) \in \mathbb{R}^3$

Remark

$P(S^1)$ is contained in the circle (in $\mathbb{R}^3$) centered at $(\frac{1}{2}, 0, \frac{1}{2})$ with radius $\frac{1}{\sqrt{2}}$. Furthermore, for any $x, y \in S^1$,

(a) $\langle P_x, P_y \rangle = \langle x, y \rangle^2$,

(b) $\|P_x - P_y\|^2 = 2 - 2\langle x, y \rangle^2$,

(c) $|\langle x, y \rangle|^p = \left(1 - \frac{\|P_x - P_y\|^2}{2}\right)^{p/2}$.

$\mu_{p,4,2}$ and its minimizers: Proof

The following Lemma is easy to check.

Lemma

If $\{P_{x_i} = P(x_i)\}_{i=1}^N$ are N equally spaced points on $P(S^1)$, then $\{x_i\}_{i=1}^N$ are equally spaced points on the half circle of S^1.

We can then prove that

Theorem (Chen, Goodman, Gonzales, and K.O. (2018))

$\mu_{p,4,2}$ is uniquely minimized by equally spaced points on the half circle. In other words, when $p > 2$, the optimal configuration for $\mu_{p,4,2}$ is $\{e^0, e^{\frac{\pi i}{4}}, e^{\frac{\pi i}{2}}, e^{\frac{3\pi i}{4}}\}$.

$\mu_{p,4,2}$ and its minimizers: Proof

Proof.

Let $X_4 = \{x_i\}_{i=1}^4$, $P_i = P_{x_i}$ and $\mathcal{P}_4 = \{P_i\}_{i=1}^4$.

$$F(X_4) = \sum_{i<j} |\langle x_i, x_j\rangle|^p = \sum_{i<j} \left(1 - \frac{\|P_i - P_j\|^2}{2}\right)^{p/2} := \sum_{i<j} f(\|P_i - P_j\|^2)$$

where $f(t) = (1 - t/2)^s$ with $s = p/2 > 1$. f is decreasing and strictly convex on $[0, 2)$. Consequently, $F(X_4)$ is minimized when and only when P_i are equally spaced points on the circle $P(S^1)$. □

$\mu_{p,N,2}$ and its minimizers: p even

Theorem (Chen, Goodman, Gonzales, and K.O. (2018))

Fix $N \geq 5$. If $p > 2$ is even, an optimal configuration for $\mu_{p,N,2}$ problem is

$$\{e^{i \cdot 0}, e^{i\frac{\pi}{N}}, e^{i\frac{2\pi}{N}}, \cdots, e^{i\frac{(N-1)\pi}{N}}\},$$

i.e., equally spaced points on the half circle.

Proof.

In this case f is completely monotone on $[0, 4]$ if p is an even integer. $\square$

Revisiting $\mu_{p,3,2}$

For $d = 2$ $N = 3$ we recall that the graph of $\mu_{p,3,2}$ when $p \in (0, \infty)$ is given below

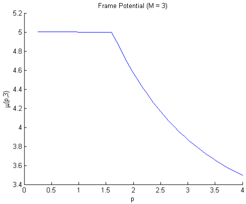

Figure: Graph of $\mu_{p,3,2}$ when $p \in (0, 4)$.

Furthermore, on $(0, \frac{\log(3)}{\log(2)}]$ the minimizers are ONB+ one repeated vector, and on $[\frac{\log(3)}{\log(2)}, \infty]$ the minimizers are ETFs.
What happen when $d \geq 3$?

ONB+, lifted ETFs

Definition

In $\mathbb{R}^d$:

• An ONB+ is a set of $d+1$ vectors consisting of an ONB plus one vector repeated.

• An ETF$_d$ is an ETF of $d+1$ vectors in $\mathbb{R}^d$.

• A lifted ETF from $\mathbb{R}^k$ to $\mathbb{R}^d$ is a frame containing $d+1$ vectors of the form

$$L_k^d = \begin{bmatrix} ETF_k & 0 \\ 0 & I_{d-k} \end{bmatrix}$$

where I_{d-k} is the identity matrix in $\mathbb{R}^{d-k}$.
By definition, ONB+ $= L_1^d$.

Examples of lifted ETFs

Example

$$L_2^2 = \begin{bmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & \sqrt{3/2} & -\sqrt{3/2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad L_2^4 = \begin{bmatrix} 1 & -1/2 & -1/2 & 0 & 0 \\ 0 & \sqrt{3/2} & -\sqrt{3/2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that these frames are neither tight nor equiangular, but there are two-distant frame, e.g., the frame operator and Gramian of L_2^4 are

$$S = \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & -1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 & 0 \\ -1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Conjecture on $\mu_{p,d+1,d}$

Conjecture

Let

$$p_k = \frac{\log(k+2) - \log k}{\log(k+1) - \log k}.$$

The following configurations minimize the p-frame potential $FP_{p,N+1,N}$:

- *when $p \in [0, p_1]$, the ONB+, or L_1^N, configuration,*
- *when $p \in [p_{k-1}, p_k]$ for $1 < k < N$, the L_k^N configuration, and*
- *when $p \in [p_{N-1}, \infty)$, the ETF$_N$, or L_N^N configuration.*

Conjecture on $\mu_{p,d+1,d}$ for $d \in \{3, \ldots, 7\}$

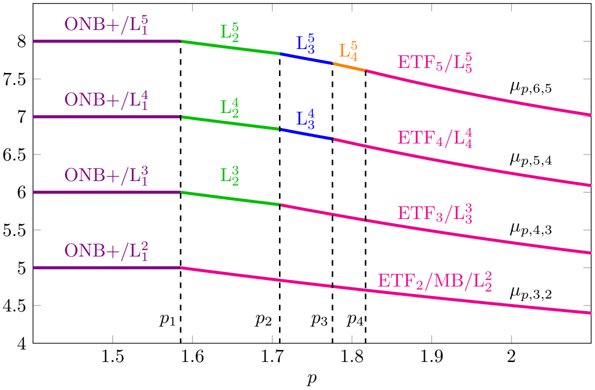

Figure: The conjectured minima of the p-frame potential, $\mu_{p,N+1,N}$, with the minimizing configurations for $N = 3, 4, 5$ compared to the known minima, $\mu_{p,3,2}$, for $N = 2$.

Thank You!
http://www2.math.umd.edu/ okoudjou