Open problems in finite frame theory: Packings in projective spaces

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Tight Frames and Approximation

February 20, 2018

Big picture

Finite frame theory: Application-driven arrangements of vectors

Recent problems solved with the help of AG:

- Phase retrieval injectivity threshold
- Finite-dimensional HRT conjecture
- Bilinear identifiability threshold
- Full spark unit norm tight frames

This talk: Can AG solve projective packing problems?

Conca, Edidin, Hering, Vinzant, Appl. Comput. Harmon. Anal., 2015

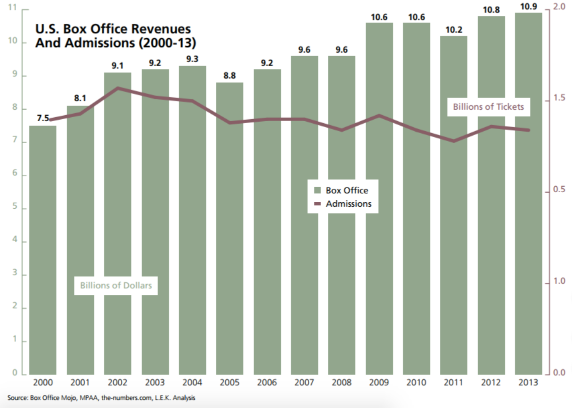
Vinzant, SampTA 2015

Malikiosis, Appl. Comput. Harmon. Anal., 2015

Kech, Krahmer, SIAM J. Appl. Algebra Geometry, 2017

Cahill, M., Strawn, SIAM J. Appl. Algebra Geometry, 2017

A motivating application



Why the decline? Better home theaters, **easy access online**

L.E.K., Box Office Trends, 2015

Stobing, How Do Movies Leak Before They Come Out on DVD and Blu-Ray?, 2016

A motivating application

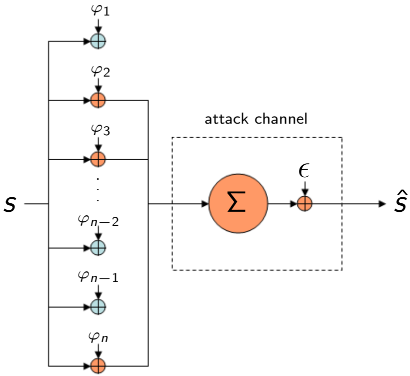
How to defeat media piracy?

Watermarks help, but these can be removed

Want: Robust personalized fingerprints to help identify culprits

image from `hollywoodandfine.com/the-screen-image-deteriorates/`

A motivating application



- Unit fingerprints $\{\varphi_i\}_{i\in[n]}$
- The ith user is given $s + \varphi_i$
- Users $K \subseteq [n]$ forge the signal:

$$\hat{s} = \sum_{k\in K} \alpha_k (s + \varphi_k) + \epsilon$$

- Interrogate $\operatorname{argmax}_{i\in[n]} |\langle \varphi_i, \hat{s} - s\rangle|$

Theorem

$$\max_{\substack{i,j\in[n]\\ i\neq j}} |\langle \varphi_i, \varphi_j\rangle| \text{ small}, \quad K, \epsilon \text{ small} \quad \Longrightarrow \quad \text{false positives unlikely}$$

M., Quinn, Kiyavash, Fickus, IEEE Trans. Inform. Theory, 2013

The problem

Find unit-norm vectors $\{\varphi_i\}_{i\in[n]} \subseteq \mathbb{F}^d$ that minimize **coherence**:

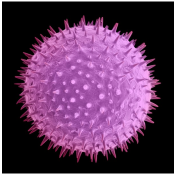
$$\mu(\{\varphi_i\}_{i\in[n]}) = \max_{\substack{i,j\in[n]\\ i\neq j}} |\langle \varphi_i, \varphi_j \rangle|$$

i.e., n points in $\mathbb{F}\mathbf{P}^{d-1}$ that maximize the minimum distance

Applications

- digital fingerprinting
- multiple description coding
- compressed sensing
- quantum state estimation

cf. Tammes problem:



Common mallow pollen grain

Strohmer, Heath, Appl. Comput. Harmon. Anal., 2003

image from wearedesignbureau.com/projects/weird-science/

Packing cheat sheet

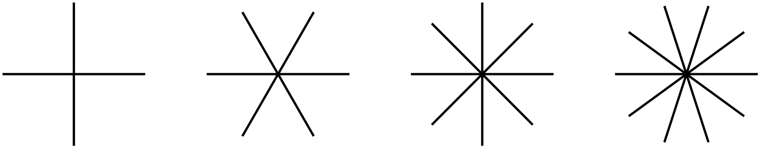
Step 1: Prove lower bound on coherence

- Isometric embedding
- Semidefinite programming
- Tarski–Seidenberg projection

Step 2: Construct packing that meets bound

- Group actions
- Combinatorial design
- Non-convex optimization

Packing in $\mathbb{R}\mathbf{P}^1$

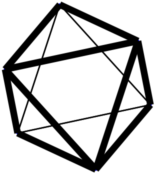


$n = 2$ $n = 3$ $n = 4$ $n = 5$

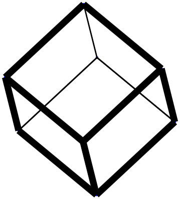
Easy proof of optimality:

- $\mathbb{R}\mathbf{P}^1$ and $\mathbb{S}^1$ are isometrically isomorphic
- pigeonhole $\Rightarrow$ equally spaced points are optimal

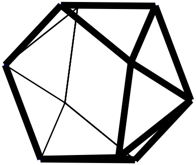
Packing in $\mathbb{R}\mathbf{P}^2$



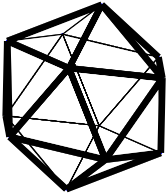
$n = 3$



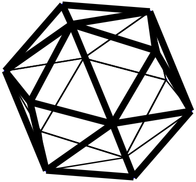
$n = 4$



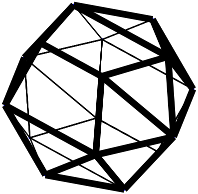
$n = 5$



$n = 6$



$n = 7$



$n = 8$

Case-by-case optimality proofs, **most cases are open**

Part I

The Welch bound

Theorem (Welch bound)

Suppose $n \geq d$. Then every $\{\varphi_i\}_{i\in[n]} \subseteq \mathbb{F}^d$ satisfies

$$\mu(\{\varphi_i\}_{i\in[n]}) \geq \sqrt{\frac{n-d}{d(n-1)}}.$$

Proof: Put $\Phi = [\varphi_1 \cdots \varphi_n] \in \mathbb{F}^{d\times n}$. Then

$$0 \leq \left\|\Phi\Phi^* - \tfrac{n}{d}I\right\|_F^2 = \|\Phi^*\Phi\|_F^2 - \tfrac{n^2}{d} \leq n + n(n-1)\mu(\Phi)^2 - \tfrac{n^2}{d}$$

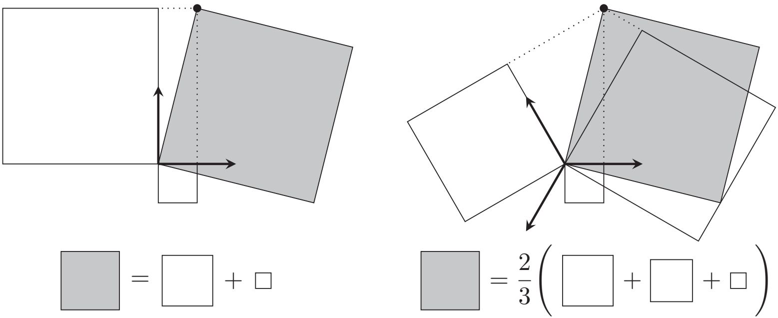
Equality if and only if

- $|\langle\varphi_i, \varphi_j\rangle| = \text{const for } i \neq j$ "equiangular"
- $\Phi\Phi^* = \frac{n}{d}I$ "tight frame"

Welch, IEEE Trans. Inform. Theory, 1974

Tight frames

Overcomplete generalization of orthonormal bases:



Offers painless solution to least-squares problem $y = \Phi^* x + \text{noise}$

aka "eutactic stars"

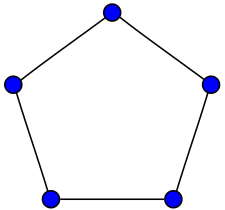
Daubechies, Grossmann, Meyer, J. Math. Phys., 1986

Schläfli, Theorie der vielfachen Kontinuität, 1901

The real case is "easy": Existence

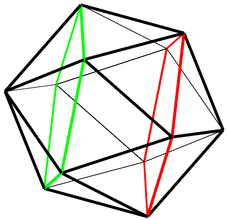
Strongly regular graph

- every vertex has k neighbors
- neighbors: λ common neighbors
- otherwise: μ common neighbors



Theorem

Every real equiangular tight frame comes from a strongly regular graph.



Real ETFs $\leftrightarrow$ Brouwer's table of SRGs

Caveat: Sometimes ETFs produce new SRGs (Tremain ETFs)

Waldron, Linear Algebra Appl., 2009

Brouwer, `www.win.tue.nl/~aeb/graphs/srg/srgtab.html`

Fickus, Jasper, M., Peterson, J. Combin. Theory A, to appear

The real case is "easy": Non-existence

Lemma

Given a symmetric matrix with integer entries, if the eigenvalues have distinct multiplicities, then they are integer.

$$\Lambda = \{0, \tfrac{n}{d}\} \quad \longrightarrow \quad \Phi^*\Phi = I + \mu S \quad \longleftarrow \quad \Lambda \subseteq 1 + \mu\mathbb{Z}$$

Corollary

Suppose $n \neq d, 2d$. There exists an n-vector ETF in $\mathbb{R}^d$ only if

$$\sqrt{\frac{(n-d)(n-1)}{d}}, \quad \sqrt{\frac{d(n-1)}{n-d}} \quad \in \quad \mathbb{Z}$$

Caveat: Not sufficient by computer-assisted proof (76 in $\mathbb{R}^{19}$)

Sustik, Tropp, Dhillon, Heath, Linear Algebra Appl., 2007

Azarija, Marc, arXiv:1509.05933

The complex case is hard

Existence: No analog to SRGs, so throw and see what sticks

- Group actions. abelian, Heisenberg–Weyl
- Generalize small examples. Steiner, Tremain, hyperovals
- Complexify real examples. DRACKNs, GQs, schemes
- Combinatorify algebraic examples. Kirkman

Non-existence: No analog to integrality conditions

The Fickus Conjecture (US$200/$100 prize for proof/disproof)

Consider d, $n-d$ and $n-1$. There exists an n-vector ETF in $\mathbb{C}^d$ only if one of these quantities divides the product of the other two.

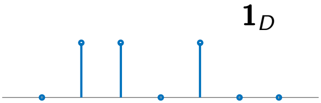
Holds for $(d,n)=(3,8)$ by Gröbner basis calculation

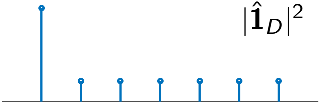
Fickus, M., arXiv:1504.00253

M., Short Fat Matrices, 2015

Szöllősi, arXiv:1402.6429

Example: Abelian group action





The following are equivalent:

- D is pseudorandom
- $|\hat{\mathbf{1}}_D|^2 = \text{spike} + \text{const}$
- D is a difference set

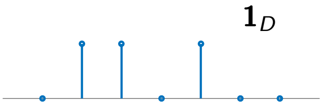
$-$	1	2	4
1	0	1	3
2	6	0	2
4	4	5	0

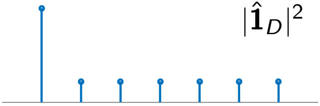
Turyn, Pacific J. Math, 1965

Strohmer, Heath, Appl. Comput. Harmon. Anal., 2003

Tao, Vu, Additive Combinatorics, 2006

Example: Abelian group action





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- D is pseudorandom
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4	4	5	0

$$\omega = e^{2\pi i/7}, \qquad h = \mathrm{diag}(\omega^1, \omega^2, \omega^4), \qquad \mathbb{Z}/7\mathbb{Z} \cong \langle h \rangle \leq U(3)$$

The orbit $\{g\mathbf{1}\}_{g \in \langle h \rangle}$ is an ETF with $\langle g^k \mathbf{1}, g^l \mathbf{1} \rangle = \hat{\mathbf{1}}_D(l-k)$

Turyn, Pacific J. Math, 1965

Strohmer, Heath, Appl. Comput. Harmon. Anal., 2003

Tao, Vu, Additive Combinatorics, 2006

Part II

Beyond the Welch bound

Welch revisited

Lift $L\colon \mathbb{F}\mathbf{P}^{d-1} \to \sqrt{1-\frac{1}{d}}\cdot \mathbb{S}^{D-1}$
$\varphi \mapsto \varphi\varphi^* - \frac{1}{d}I$

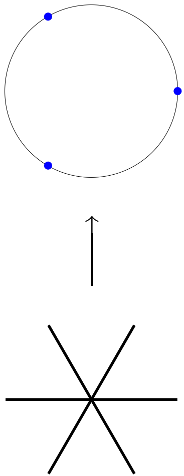
Then $\big\langle L(\varphi), L(\psi)\big\rangle = |\langle \varphi,\psi\rangle|^2 - \frac{1}{d}$

Theorem (Rankin's simplex bound)

If $n \leq D+1$, $\{x_i\}_{i\in[n]} \subseteq \mathbb{S}^{D-1}$ satisfies

$$\max_{\substack{i,j\in[n]\\ i\neq j}} \langle x_i, x_j\rangle \geq -\frac{1}{n-1}.$$

Pull back Rankin $\implies$ Welch bound



Rankin, Glasg. Math. J., 1955

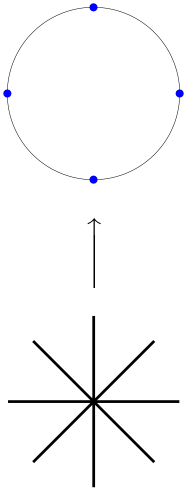
Conway, Hardin, Sloane, Experiment. Math., 1996

Bounds from lifting

Theorem (Rankin's orthoplex bound)

If $n > D+1$, $\{x_i\}_{i\in[n]} \subseteq \mathbb{S}^{D-1}$ satisfies

$$\max_{\substack{i,j\in[n]\\ i\neq j}} \langle x_i, x_j\rangle \geq 0.$$



Pull back Rankin:

Corollary

If $n > D+1$, $\{\varphi_i\}_{i\in[n]} \subseteq \mathbb{F}^d$ satisfies

$$\mu(\{\varphi_i\}_{i\in[n]}) \geq \frac{1}{\sqrt{d}}.$$

Equality: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$

Rankin, Glasg. Math. J., 1955

Conway, Hardin, Sloane, Experiment. Math., 1996

Bounds from lifting

Zonal kernel: Any function $f : \mathbb{R} \to \mathbb{R}$ with lifting L such that

$$\big\langle L(x), L(y) \big\rangle = f\big(|\langle x, y\rangle|^2\big) \qquad \forall x, y \in \mathbb{F}\mathbf{P}^{d-1}$$

e.g., $\quad f(t) = t - \frac{1}{d}, \quad L(\varphi) = \varphi\varphi^* - \frac{1}{d}I$

Lemma

Let f be a zonal kernel for $\mathbb{F}\mathbf{P}^{d-1}$ such that

$$f(1) = 1, \qquad\qquad f(t) < -\tfrac{1}{n-1} \quad \forall t \in [0, B).$$

Then every $\{\varphi_i\}_{i\in[n]} \subseteq \mathbb{F}^d$ satisfies $\mu(\{\varphi_i\}_{i\in[n]}) \geq \sqrt{B}$.

Proof: Otherwise, contradict Rankin's simplex bound. $\square$

$f \in \text{cone(special polynomials)} \implies$ **Delsarte's LP bound**

Delsarte, Goethals, Seidel, Philips Res. Rep.,1975

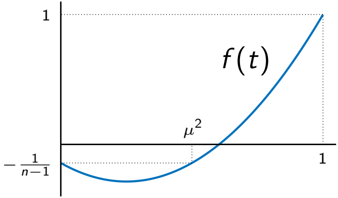
Haas, Hammen, M., in preparation

Bounds from lifting

Important instances of Delsarte:

$\deg(f) = 1$: Delsarte $\Rightarrow$ Welch

$\deg(f) = 2$: Delsarte $\Rightarrow$ **Levenstein**



Theorem (Levenstein bound)

Suppose $n \geq d$. Then every $\{\varphi_i\}_{i\in[n]} \subseteq \mathbb{F}^d$ satisfies

$$\mu(\{\varphi_i\}_{i\in[n]}) \geq \sqrt{\frac{n(m+1)-d(md+1)}{(n-d)(md+1)}}, \qquad 2m = [\mathbb{F} : \mathbb{R}].$$

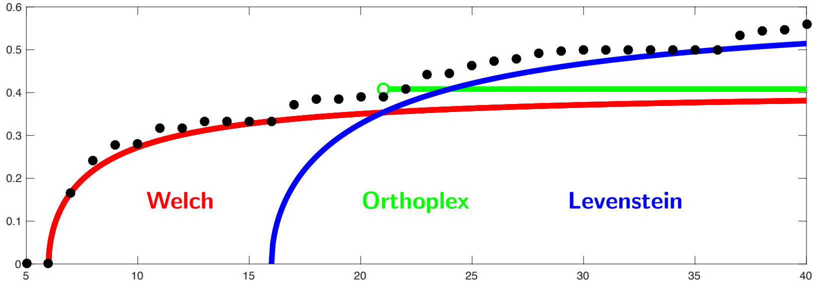
Equality: 2-designs with $|\langle\varphi_i, \varphi_j\rangle| \in \{0, \mu\}$ (cf. tight $\Leftrightarrow$ 1-design)

Few packings known, all exhibit interesting symmetries (e.g., E_8)

Levenstein, Soviet Math. Dokl., 1982

Haas, Hammen, M., in preparation

Bounds from lifting



How close? Welch is within a constant factor of optimal

How to remove the remaining gaps?

- Three-point generalization of Delsarte (still not tight)
- Tarski–Seidenberg projection (tight, but slow)

Sloane, Packings in Grassmannian Spaces, `neilsloane.com/grass/`

Cohn, Woo, J. Am. Math. Soc., 2012

Fickus, Jasper, M., `arXiv:1707.01858`

Tarski–Seidenberg projection

Tight bound $\quad\Leftrightarrow\quad$ minimizing over a **semialgebraic** set:

$$\left\{ (G, x) \ : \ \operatorname{rank} G = d,\ \operatorname{diag} G = \mathbf{1},\ G \succeq 0,\ \mu(G)^2 \leq x \right\}$$

Idea: Project onto x coordinate and minimize

Tarski–Seidenberg Theorem

The projection of a semialgebraic set is semialgebraic.

Algorithm: cylindrical algebraic decomposition (Mathematica)

Runtime is **double exponential** in number of variables

Bochnak, Coste, Roy, Real Algebraic Geometry, 1998

Tarski–Seidenberg projection

Half the variables in real case. Can we get fewer?

- **contact graph:** $i \leftrightarrow j$ whenever $|\langle \varphi_i, \varphi_j \rangle| = \mu$
- d-**secure graph:** There's no way to reach the empty graph by iteratively deleting vertices of degree $< d$

Lemma

The contact graph of an optimal packing is d-secure.

Proof:

- If not d-secure, reach the empty graph by deleting $\{j_k\}_{k \in [n]}$
- Slightly move each φ_{j_k} toward $(\{\varphi_i\}_{i \in N(j_k)})^\perp$
- Iterating through k decreases μ □

Fickus, Jasper, M., arXiv:1707.01858

Tarski–Seidenberg projection

d-secure says more when n is small, so take $n = d + 2$

Lemma

There are two minimal d-secure graphs of order $d + 2$:

- $K_{d+1} \cup v$
- complement of a maximum matching

Therefore, every optimal Gram matrix has one of two forms:

$$\begin{bmatrix} 1 & \pm\mu & \pm\mu & x_1 \\ \pm\mu & 1 & \pm\mu & x_2 \\ \pm\mu & \pm\mu & 1 & x_3 \\ x_1 & x_2 & x_3 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & x_1 & \pm\mu & \pm\mu \\ x_1 & 1 & \pm\mu & \pm\mu \\ \pm\mu & \pm\mu & 1 & x_2 \\ \pm\mu & \pm\mu & x_2 & 1 \end{bmatrix}$$

Given fixed sign pattern, each form has $\leq n$ variables

Fickus, Jasper, M., `arXiv:1707.01858`

Tarski–Seidenberg projection

Theorem

Every $\{\varphi_i\}_{i\in[6]} \subseteq \mathbb{R}^4$ satsfies

$$\mu(\{\varphi_i\}_{i\in[6]}) \geq \tfrac{1}{3},$$

and equality is achieved in Sloane's online database.

Proof:

- Apply CAD to project onto μ coordinate, take minimum
- Minimize over both forms and all sign patterns
- To avoid CAD queries, solve first form using its spectrum, reduce to 14 inequivalent sign patterns for the second form ☐

Open: $(d, n) = (5, 7)$. How to speed up CAD?

Fickus, Jasper, M., arXiv:1707.01858

Sloane, Packings in Grassmannian Spaces, neilsloane.com/grass/

Part III

Zauner's Conjecture

Maximal equiangular tight frames

Welch vs. Orthoplex: n-vector ETF in $\mathbb{C}^d$ requires $n \leq d^2$

ETF with $n = d^2$ is called **maximal** or **SIC-POVM**

Cornerstone object in theory of Quantum Bayesianism

Zauner's Conjecture

For each $d \geq 2$, $\mathbb{C}^d$ admits a maximal ETF (with very specific structure).

Known to hold for finitely many d

Fuchs, Schack, Found. Phys., 2011

Zauner, Ph.D. thesis, U. Vienna, 1999

solutions available at `www.physics.usyd.edu.au/~sflammia/SIC/`

Recent progress on Zauner's conjecture

Heisenberg–Weyl group H generated by $T, M \in \mathcal{U}(\mathbb{C}^{\mathbb{Z}/d\mathbb{Z}})$

$$T\delta_j = \delta_{j+1}, \qquad M\delta_j = e^{2\pi ij/d}\delta_j \qquad (j \in \mathbb{Z}/d\mathbb{Z})$$

Then $\{U\varphi\}_{U\in H} = d^2$ vectors $\times$ all dth roots of unity

Theorem

Every $\varphi \in \mathbb{C}^{\mathbb{Z}/d\mathbb{Z}}$ satisfies

$$\sum_{j,k\in\mathbb{Z}/d\mathbb{Z}} \left| \sum_{l\in\mathbb{Z}/d\mathbb{Z}} \varphi(l)\overline{\varphi(j+l)\varphi(k+l)}\varphi(j+k+l) \right|^2 \geq \frac{2}{d+1}$$

with equality precisely when $\{U\varphi\}_{U\in H}$ produces a maximal ETF.

Minimize LHS (non-convex!) $\Rightarrow$ numerical solutions for $d \leq 151$

Fickus, J. Fourier Anal. Appl., 2009

Fuchs, Hoang, Stacey, `arXiv:1703.07901`

solutions available at `www.physics.umb.edu/Research/QBism/solutions.html`

Recent progress on Zauner's conjecture

We want $\varphi \in \mathbb{C}^{\mathbb{Z}/d\mathbb{Z}}$ such that

$$\sum_{l \in \mathbb{Z}/d\mathbb{Z}} \varphi(l)\overline{\varphi(j+l)\varphi(k+l)}\varphi(j+k+l) = \frac{\delta_0(j)+\delta_0(k)}{d+1}$$

Compute Gröbner basis and find real solutions (provided d is small)

Observation/Conjecture

Entries of $\varphi\varphi^*$ lie in an abelian extension of $\mathbb{Q}(\sqrt{(d-3)(d+1)})$.

Chien's program to find larger seed vectors:

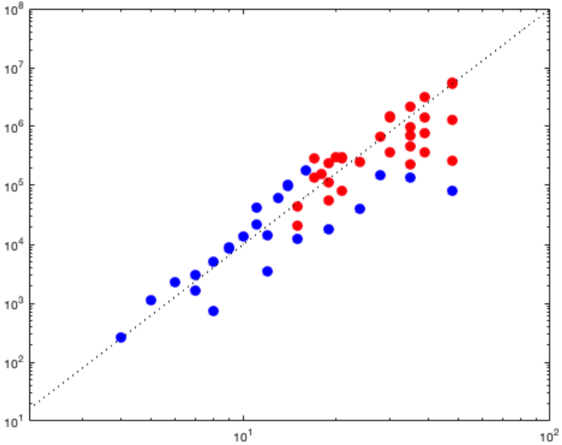
1. Take a numerically approximated ETF seed vector
2. Locally optimize to obtain $\sim 10^4$ digits of precision
3. Apply conjecture to guess analytic expression
4. Verify success by symbolic computation

Appleby, Yadsan-Appleby, Zauner, Quantum Inf. Comput., 2013

Appleby, Chien, Flammia, Waldron, arXiv:1703.05981

Recent progress on Zauner's conjecture

Coordinates are expressible by radicals, but not nicely:



Is there a shorter description?

M., Short Fat Matrices, 2017

Should we abandon Heisenberg–Weyl?

Hoggar's ETF: Spin $(-1+2i, 1, 1, 1, 1, 1, 1, 1)$ with HW over $(\mathbb{Z}/2\mathbb{Z})^3$

Theorem

HW over $(\mathbb{Z}/2\mathbb{Z})^k$ produces a maximal ETF only if $k \in \{1, 3\}$.

HW over other abelian groups? Numerics not promising

SmallGroups: When a group works, it gives a rotated HW ETF

Theorem

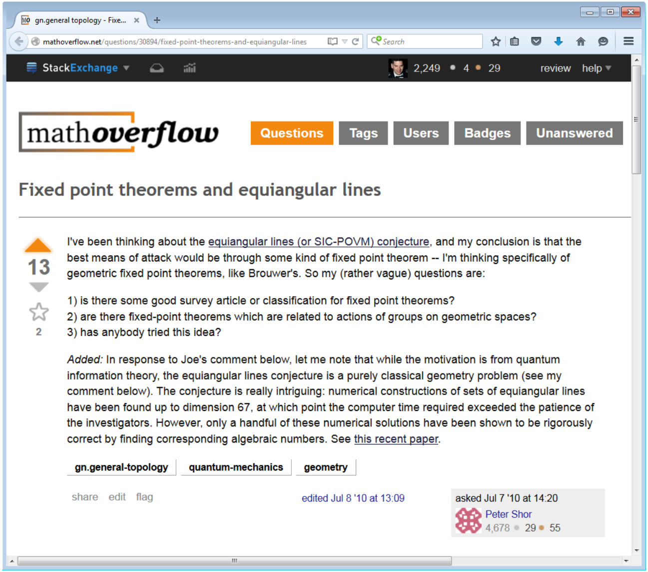
For $d > 3$ prime, if any group produces a maximal ETF in $\mathbb{C}^d$, then HW produces a rotated version of the same ETF.

Godsil, Roy, Eurpean J. Combin., 2009

Appleby, Flammia, Fuchs, J. Math. Phys., 2011

Zhu, J. Phys. A, 2010

How to avoid being explicit?



gn.general topology - Fixe...

mathoverflow.net/questions/30894/fixed-point-theorems-and-equiangular-lines

StackExchange 2,249 • 4 • 29 review help

mathoverflow Questions Tags Users Badges Unanswered

Fixed point theorems and equiangular lines

13

2

I've been thinking about the equiangular lines (or SIC-POVM) conjecture, and my conclusion is that the best means of attack would be through some kind of fixed point theorem -- I'm thinking specifically of geometric fixed point theorems, like Brouwer's. So my (rather vague) questions are:

1) is there some good survey article or classification for fixed point theorems?
2) are there fixed-point theorems which are related to actions of groups on geometric spaces?
3) has anybody tried this idea?

Added: In response to Joe's comment below, let me note that while the motivation is from quantum information theory, the equiangular lines conjecture is a purely classical geometry problem (see my comment below). The conjecture is really intriguing: numerical constructions of sets of equiangular lines have been found up to dimension 67, at which point the computer time required exceeded the patience of the investigators. However, only a handful of these numerical solutions have been shown to be rigorously correct by finding corresponding algebraic numbers. See this recent paper.

gn.general-topology quantum-mechanics geometry

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edited Jul 8 '10 at 13:09

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Peter Shor
4,678 • 29 • 55

Shor, mathoverflow.net/questions/30894/fixed-point-theorems-and-equiangular-lines

How to avoid being explicit?

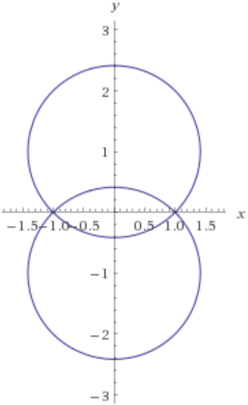
Relax to **biangular** tight frames

Consider $K = \langle T, e^{2\pi i/d} \cdot I \rangle \leq H$

Suppose there exist $\alpha, \beta \geq 0$ such that

$$|\langle \varphi, U\varphi \rangle| = \begin{cases} \alpha & \text{if } U \in K \setminus Z(H) \\ \beta & \text{if } U \in H \setminus K \end{cases}$$

Plot of all such $\varphi = (1, x + iy)$:



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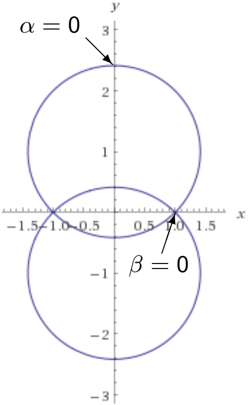

Surprise: An ETF exists by the intermediate value theorem!

Does this generalize?

Provable relaxations?

Relaxation 1

Does there exist $(u_0, \ldots, u_{d-1}), (v_0, \ldots, v_{d-1}) \in \mathbb{C}^{\mathbb{Z}/d\mathbb{Z}}$ such that

$$\sum_{l \in \mathbb{Z}/d\mathbb{Z}} u_l \; v_{j+l} \; v_{k+l} \; u_{j+k+l} = \tfrac{\delta_0(j)+\delta_0(k)}{d+1} \qquad \forall j, k \in \mathbb{Z}/d\mathbb{Z} \text{ ?}$$

Relaxation 2

What is the smallest r for which there exists $Z \succeq 0$ over $\mathbb{C}^{\mathbb{Z}/d\mathbb{Z}}$ of rank r such that

$$\sum_{l \in \mathbb{Z}/d\mathbb{Z}} Z_{l,j+l} \; Z_{j+k+l,k+l} = \tfrac{\delta_0(j)+\delta_0(k)}{d+1} \qquad \forall j, k \in \mathbb{Z}/d\mathbb{Z} \text{ ?}$$

Open problems

- The Fickus conjecture
- Better coherence bounds
- Fast Tarski–Seidenberg projection
- Zauner's conjecture
- Relaxations of Zauner's conjecture

Questions?

Tables of the existence of equiangular tight frames
M. Fickus, D. G. Mixon
arXiv:1504.00253

The Levenstein bound for packings in projective spaces
J. I. Haas IV, N. Hammen, D. G. Mixon
SPIE 2017, to appear

Packings in real projective spaces
M. Fickus, J. Jasper, D. G. Mixon
arXiv:1707.01858

Also, google **short fat matrices** for my research blog