Lines with few angles from association schemes

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Tight frames and Approximation
Doubtless Bay, Taipa NZ
February 22, 2018

VERNAM LAB
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Thank you!

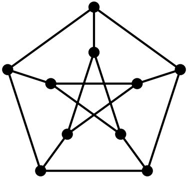
Thank you to the organizers, to Dan Hughes for helping me, and especially to Shayne Waldron for putting this thing together.

Thanks to my Co-Authors

- Brian Kodalen (PhD student at WPI)
- Jason Williford (U. Wyoming – former postoc at WPI)

Start with the Petersen Graph

Can we find a spherical 2-distance set X where distances depend only on adjacency in the Petersen graph?



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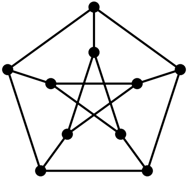

Eigenvalues: $[3]^1$ $[1]^5$ $[-2]^4$

Eigenvalues and Cosines – Petersen Graph

First and second eigenmatrices

$$P = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

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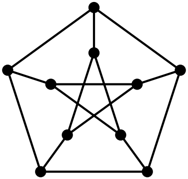
$$P = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$Q = |X|P^{-1} = \begin{bmatrix} 1 & 5 & 4 \\ 1 & 5/3 & -8/3 \\ 1 & -5/3 & 2/3 \end{bmatrix}$$

Amazing Fact: For any configuration of ten unit vectors in $\mathbb{R}^d$ with inner products only depending on adjacency / non-adjacency in the Petersen graph, the cosines in the Gram matrix must be some convex combination of the following three columns:

1	1	1
1	1/3	-2/3
1	-1/3	1/6

All possible Gram Matrices for the Petersen Graph



For cosines

$$
\begin{aligned}
1 &= w_0 + w_1 + w_2 \\
\alpha &= w_0 + \frac{1}{3} w_1 - \frac{2}{3} w_2 \\
\beta &= w_0 - \frac{1}{3} w_1 + \frac{1}{6} w_2
\end{aligned}
$$

where all $w_j \geqslant 0$, the rank of G is either 1, 4, 5, 5, 6, 9, 10, depending only on which w_j are non-zero. (E.g., $w_1 = 1$ gives 10 lines in $\mathbb{R}^5$, as does $w_0 = 1/3, w_2 = 2/3$.)

Graphs with few distinct eigenvalues

By the “eigenvalues” of a graph Γ, we mean the eigenvalues of its zero-one adjacency matrix $A = A(\Gamma)$.

A connected graph with just two eigenvalues is complete (and obviously regular):

$$A^2 = \alpha A + \beta I$$

implies Γ is regular with valency β and any vertex reachable in two steps is reachable in one step.

Strongly Regular Graphs (SRGs)

A *strongly regular graph* (SRG) is a regular connected graph with just three distinct eigenvalues:

$$(A - kI)(A - rI)(A - sI) = 0.$$

So

$$A^2 \in \text{span}\{I, A, J\}.$$

Put another way, the adjacency matrix of the complement graph is

$$J - I - A \in \text{span}\{I, A, A^2\}$$

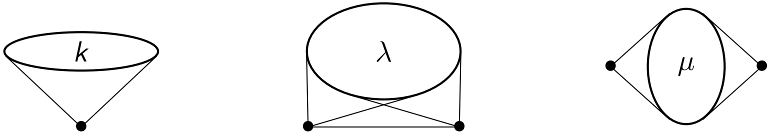
Strongly Regular Graphs (SRGs)

We write the first equation above as

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

The standard parameters for SRGs are

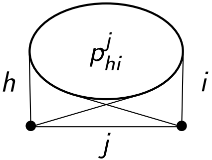
- v, the number of vertices
- k, the valency (number of neighbors of any vertex)
- λ, the number of triangles on any edge
- μ, the number of common neighbors of any two non-adjacent vertices



$|\Gamma(x)| = k$ $\quad$ $|\Gamma(x) \cap \Gamma(y)| = \lambda$ $\quad$ $|\Gamma(x) \cap \Gamma(y)| = \mu$

Strongly regular graphs are 2-class association schemes

We have shown that, for a regular graph Γ with just three eigenvalues, there exist constants p^j_{hi} $(h, i, j \in \{0, 1, 2\})$ such that, if x and y are at distance j, the number of vertices at distance h from x and i from y is this constant p^j_{hi}.



$$|\Gamma_h(x) \cap \Gamma_i(y)| = p^j_{hi}$$

Parameters of Strongly Regular Graphs (SRGs)

If the spectrum of Γ is

$$[k]^1 \ [r]^f \ [s]^g$$

with $1+f+g=v$, then we can solve for r,s,f,g in terms of v,k,λ,μ and generate tables such as this one from Andries Brouwer.

Parameters include v, k, λ, μ, $[r]^f$ and $[s]^g$

Andries Brouwer's Tables

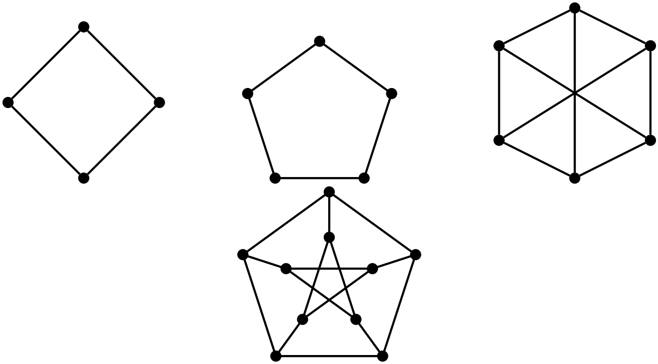
∃?	v	k	λ	μ	eigenvalues		
+	63	30	13	15	3^{35}	-5^{27}	intersection-8 graph of a quasisymmetric 2-(36,16,12) design with intersection numbers 6, 8; O(7,2) Sp(6,2); pg(6,4,3); 2-graph*
		32	16	16	4^{27}	-4^{35}	S(2,4,28); intersection-6 graph of a quasisymmetric 2-(28,12,11) design with intersection numbers 4, 6; NU(3,3); 2-graph*
!	64	14	6	2	6^{14}	-2^{49}	8^2; from a partial spread of 3-spaces: projective binary [14,6] code with weights 4, 8
		49	36	42	1^{49}	-7^{14}	OA(8,7)
167!	64	18	2	6	2^{45}	-6^{18}	complete enumeration by Haemers & Spence; GQ(3,5); from a hyperoval: projective 4-ary [6,3] code with weights 4, 6
		45	32	30	5^{18}	-3^{45}	
-	64	21	0	10	1^{56}	-11^{7}	Krein2; Absolute bound
		42	30	22	10^{7}	-2^{56}	Krein1; Absolute bound
+	64	21	8	6	5^{21}	-3^{42}	OA(8,3); $\mathrm{Bilin}_{2x3}(2)$; from a Baer subplane: projective 4-ary [7,3] code with weights 4, 6; Brouwer(q=2,d=2,e=3,+); from a partial spread of 3-spaces: projective binary [21,6] code with weights 8, 12
		42	26	30	2^{42}	-6^{21}	OA(8,6)
+	64	27	10	12	3^{36}	-5^{27}	Mesner; from a unital: projective 4-ary [9,3] code with weights 6, 8; $VO^{-}(6,2)$ affine polar graph; $RSHCD^{-}$; 2-graph
		36	20	20	4^{27}	-4^{36}	from 2-(8,2,1) with 1-factor Fickus et al.; 2-graph

Andries Brouwer's Tables

It gets more yellow as v gets large:

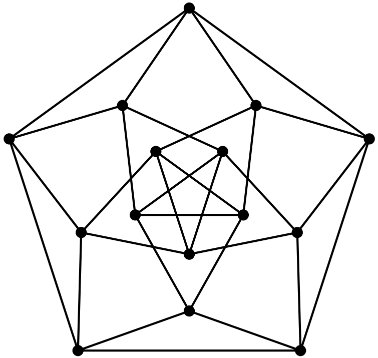
	v	k	λ	μ	r^f	s^g	comments
!	153	32	16	4	14^{17}	-2^{135}	Triangular graph T(18)
		120	91	105	1^{135}	-15^{17}	pg(8,14,7)
?	153	56	19	21	5^{84}	-7^{68}	pg(8,6,3)?
		96	60	60	6^{68}	-6^{84}	
?	153	76	37	38	5.685^{76}	-6.685^{76}	2-graph*?
?	154	48	12	16	4^{98}	-8^{55}	pg(6,7,2)?
		105	72	70	7^{55}	-5^{98}	
-	154	51	8	21	2^{132}	-15^{21}	Krein2
		102	71	60	14^{21}	-3^{132}	Krein1
?	154	72	26	40	2^{132}	-16^{21}	
		81	48	36	15^{21}	-3^{132}	
+	155	42	17	9	11^{30}	-3^{124}	S(2,3,31); lines in PG(4,2)
		112	78	88	2^{124}	-12^{30}	
+	156	30	4	6	4^{90}	-6^{65}	O(5,5) Sp(4,5); GQ(5,5)

The smallest SRGs



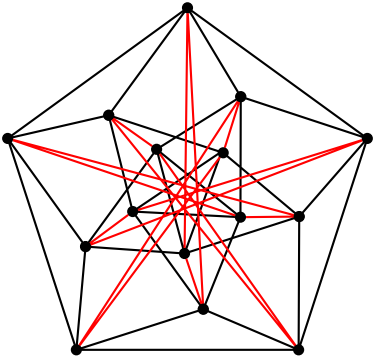
The Line Graph of the Petersen Graph is a 3-Class Association Scheme

One vertex for each of the fifteen edges of Petersen graph.
Diameter 3. Antipodal.

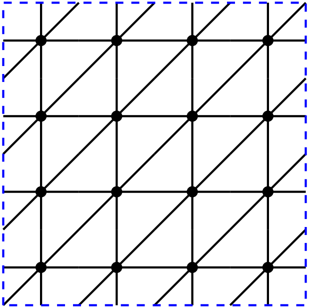


A Small Generalized Quadrangle

We may obtain the line graph of $GQ(2,2)$ by inserting a spread.



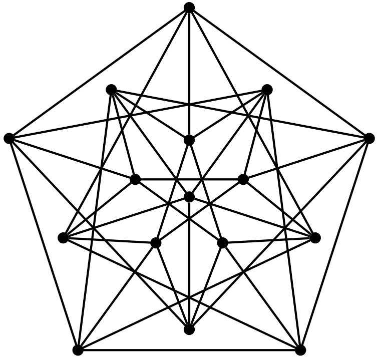
Smallest Cospectral Strongly Regular Graphs



The Shrikhande graph, with parameters $(16, 6; 2, 2)$, is cospectral with the grid graph $K_4 \times K_4$. This figure depicts a toroidal embedding with the usual cut-and-paste rules.

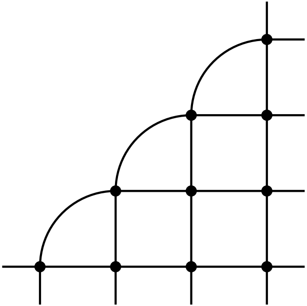
The automorphism group is not edge transitive in this case.

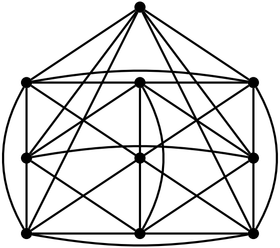
Clebsch Graph: Sixteen Lines in $\mathbb{R}^5$ with Two Angles



The folded 5-cube is an srg$(16, 5; 0, 2)$ and is one of the few known triangle-free strongly regular graphs that are not bipartite. The complement of this graph is the Clebsch graph, with parameters $(16, 10; 6, 6)$.

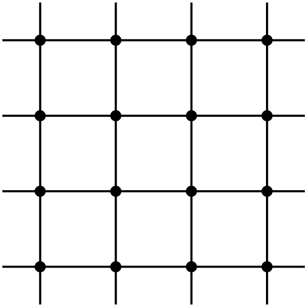
Triangular Graphs are the Diameter Two Johnson Graphs

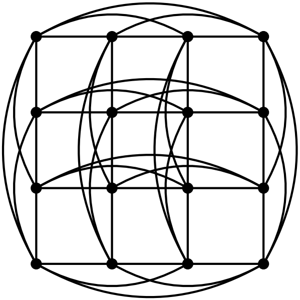




The triangular graph T_5 is the collinearity graph of a simple geometry.

The Grid Graphs are the Diameter Two Hamming Graphs





A grid and its collinearity graph. This is a dual thin generalized quadrangle.

Latin Square Graphs are Abundant and Most are Rigid

2	1	3
3	2	1
1	3	2

3	2	1
2	1	3
1	3	2

and here are three mutually orthogonal latin squares of order four (3 MOLS(4)):

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

1	3	4	2
2	4	3	1
3	1	2	4
4	2	1	3

(Symmetric, Commutative) Association Schemes

Algebraically, we study vector spaces of symmetric matrices closed under ordinary multiplication, under entrywise multiplication $\circ$ and containing the identities for both (I and J).

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A **symmetric association scheme** is an ordered pair $(X, \mathcal{R})$ where X is a finite set and $\mathcal{R} = \{R_0, \ldots, R_d\}$ is a partition of $X \times X$ into $d+1$ symmetric binary relations satisfying

- R_0 is the identity relation on X
- for $0 \leqslant h, i, j \leqslant d$, there exists $p_{hi}^{j} \in \mathbb{Z}$ such that

$$|R_h(a) \cap R_i(b)| = p_{hi}^{j}$$

whenever $(a, b) \in R_j$. (Here $R_h(a) = \{c \in X \mid (a, c) \in R_h\}$)

Association Schemes

Algebraically, we study vector spaces of symmetric matrices closed under ordinary multiplication, under entrywise multiplication $\circ$ and containing the identities for both (I and J).

A **symmetric association scheme** is an ordered pair $(X, \mathcal{A})$ where X is a finite set and $\mathcal{A} = \{A_0, \ldots, A_d\}$ is an ordered set of $|X| \times |X|$ symmetric 01-matrices satisfying $\sum_i A_i = J$ and

- $A_0 = I$, the identity matrix
- for $0 \leqslant h, i, j \leqslant d$, there exists $p_{hi}^j \in \mathbb{Z}$ such that

$$A_h A_i = \sum_{j=0}^{d} p_{hi}^j A_j$$

Properties of Association Schemes

- Association scheme $(X, \mathcal{R})$ is **imprimitive** if some graph (X, R_i) is disconnected $(i \neq 0)$ $(\sum_{i=0}^{e} A_i = I_w \otimes J_r,\ rw = |X|)$

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$$\langle A^0 = I,\ A^1,\ A^2, \ldots, A^i \rangle \quad 0 \leqslant i \leqslant d$$

- Association scheme $(X, \mathcal{A})$ is **cometric** (or "Q-polynomial") with respect to orthogonal projection $E \in \operatorname{span} \mathcal{A}$ if it admits matrix subalgebras of increasing dimension

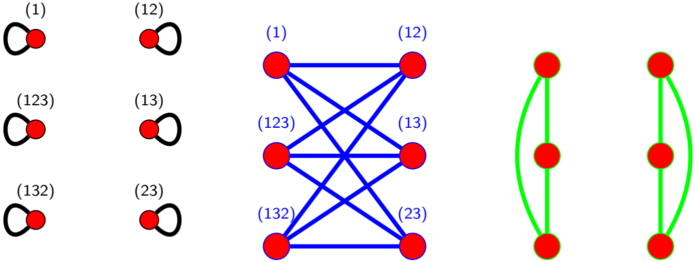
$$\langle J,\ E,\ E \circ E, \ldots, \underbrace{E \circ E \circ \cdots \circ E}_{i \text{ terms}} \rangle \quad 0 \leqslant i \leqslant d$$

Examples of Cometric Association Schemes

- conjugacy class schemes of all finite groups
- multiplicity-free permutation actions
- all Q-polynomial distance-regular graphs (P- and Q-polynomial schemes)
  - all strongly regular graphs are Q-polynomial
  - all distance-regular graphs with classical parameters
  - e.g., Hamming graphs, Johnson graphs, etc.
- shortest vectors of E_6, E_7, E_8 and Leech lattice (and some derived designs)
- block schemes of t-(v, k, λ) designs for $(t, v, k, \lambda) \in \{(4, 11, 5, 1),\ (5, 12, 6, 1),\ (5, 24, 8, 1),\ (5, 24, 12, 48),\ (4, 47, 11, 48)\}$
- more examples below

The Association Scheme of Symmetric Group $\mathfrak{S}_3$

$$X = \{ \ (1), \ (12), \ (13), \ (23), \ (123), \ (132) \ \}$$



One Cayley graph for each conjugacy class
$\mathcal{C}_0 = \{(1)\}, \mathcal{C}_1 = \{(12),(13),(23)\}, \mathcal{C}_2 = \{(123),(132)\}$

The Association Scheme of $\mathfrak{S}_3$

$$A_0 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$A_1 = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right], \quad A_2 = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Two Bases for Bose-Mesner Algebra

Schur idempotents $\{A_0, A_1, \ldots, A_d\}$ (adjacency matrices)

$$A_i \circ A_j = \delta_{i,j} A_i$$

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$$

matrix idempotents $\{E_0, E_1, \ldots, E_d\}$ (projections onto eigenspaces)

$$E_i E_j = \delta_{i,j} E_i$$

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{ij}^k E_k$$

where the structure constants q_{ij}^k are called *Krein parameters* and we know $q_{ij}^k \geqslant 0$.

Orthogonality Relations

Change of Basis Matrices P and Q

$$A_i = \sum_{j=0}^{d} P_{ji} E_j \qquad E_j = \frac{1}{|X|} \sum_{i=0}^{d} Q_{ij} A_i$$

So that the P_{ji} are the "eigenvalues" and the Q_{ij} are the "cosines" or "dual eigenvalues".

$$PQ = |X| I$$

Computing the the Hilbert-Schmidt inner product $\langle A_i, E_j \rangle$,

$$\mathrm{SUM}\left(\frac{Q_{ij}}{|X|} A_i\right) = \mathrm{SUM}(A_i \circ E_j) = \mathrm{tr} A_i E_j = \mathrm{tr} P_{ji} E_j$$

gives us

$$Q_{ij} \times (\text{valency of } R_i) = P_{ji} \times (\text{rank of } E_j).$$

Imprimitive Cometric Association Schemes

An association scheme is *imprimitive* if some graph (X, R_i) $(i \neq 0)$ is disconnected. The partition of X into connected components of this graph is called a **system of imprimitivity** and I call these components **fibres**.

An association scheme is Q-**bipartite** ("projective"?) if it is imprimitive cometric with complete subscheme $r = 2$ (columns of E_1 come in $\pm$ pairs, so only R_0 and R_d within fibres) . . .
THESE are systems of lines with $d/2$ angles!

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An association scheme is Q-**antipodal** ("linked"?) if it is imprimitive cometric with complete quotient scheme ($\lceil d/2 \rceil$ relations, R_i for i odd, between fibres, even R_i within fibres)

Cometric Imprimitivity Theorem

Theorem (Suzuki, 1998; Suzuki/Cerzo, '09; Tanaka/Tanaka, '10): If a cometric association scheme $(X, \mathcal{R})$ is imprimitive, and not a polygon, then it is either Q-bipartite, Q-antipodal, or both.

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Corollary: The following (when defined) are also cometric:

- (Q-antipodal): the induced scheme on any Q-antipodal fibre
- (Q-bipartite): the (index two) Q-bipartite quotient
- (Q-antipodal and Q-bipartite): the the Q-bipartite quotient of the induced scheme on any fibre

Moreover, for $m > 2$ and $(X, \mathcal{R})$ imprimitive, at least one of the above is a primitive cometric association scheme.

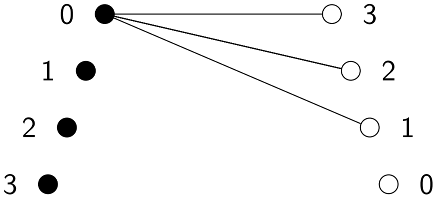
Imprimitive 3-Class Cometric Schemes

Theorem: (Seidel? 1973?) Three-class Q-bipartite association schemes are in one-to-one correspondence with regular two-graphs.

Theorem: (Van Dam, 1999) Three-class Q-antipodal association schemes are in one-to-one correspondence with (non-degenerate) linked systems of symmetric designs.

A Small Example

Here is a linked system of symmetric $(4, 3, 2)$-designs with two fibres (i.e., simply a symmetric design).



Difference set $D = \{1, 2, 3\}$ in group $\mathbb{Z}_4$

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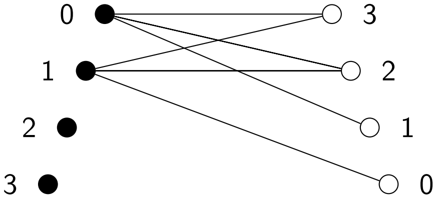

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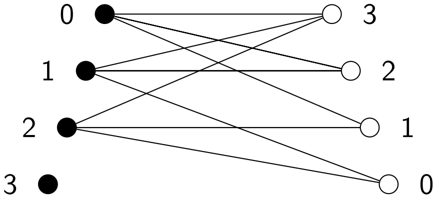

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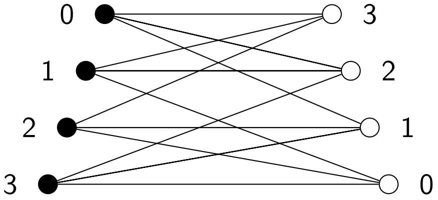

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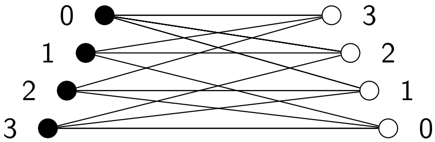
A Small Example

Next we will add more fibres . . .



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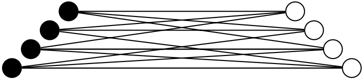
Here is a linked system of symmetric $(4, 3, 2)$-designs with five fibres



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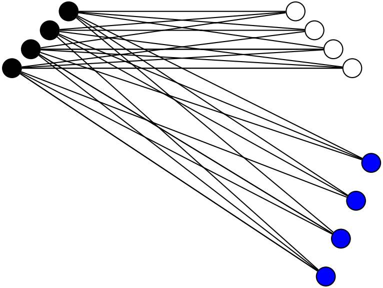

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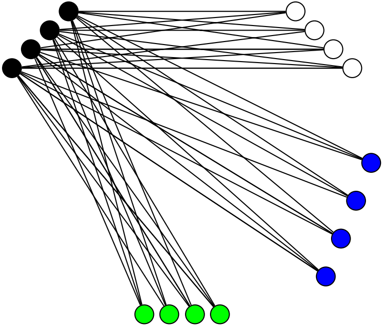

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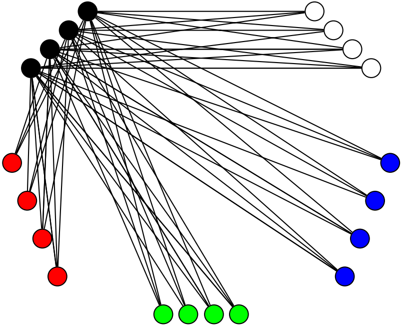

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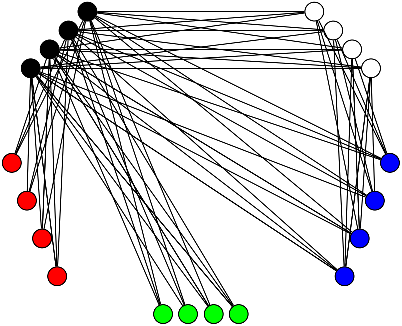

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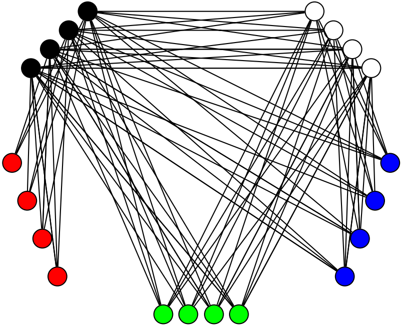

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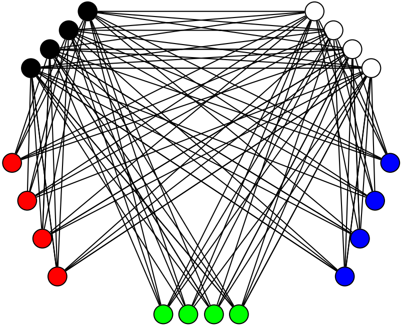

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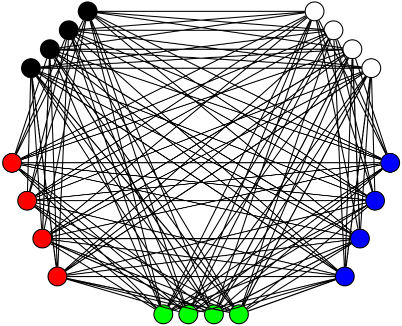
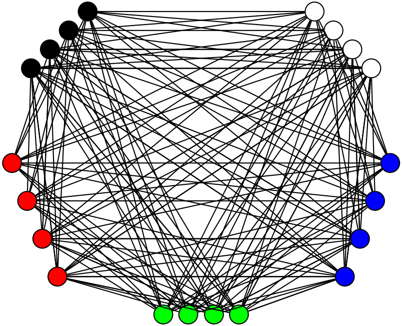


Definition: Linked Condition (Cameron, 1973)



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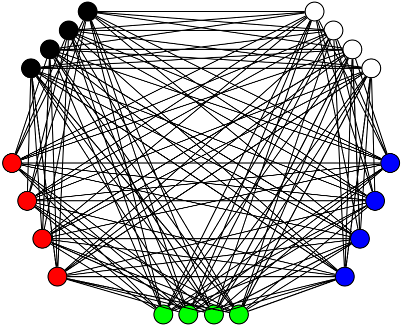

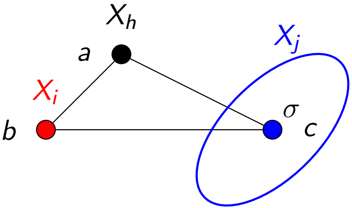

Number of common neighbours in fibre X_j of a in fibre X_h and b in fibre X_i (h, i, j distinct) is

$$|\Gamma(a) \cap \Gamma(b) \cap X_j| = \begin{cases} \sigma & \text{if } a \sim b; \\ \tau & \text{if } a \not\sim b. \end{cases}$$

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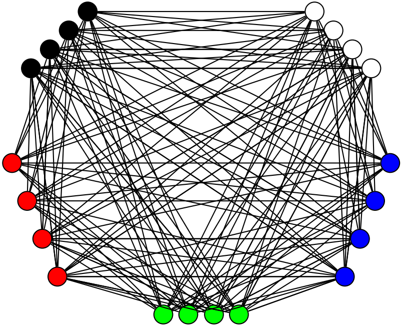

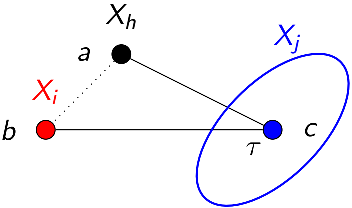

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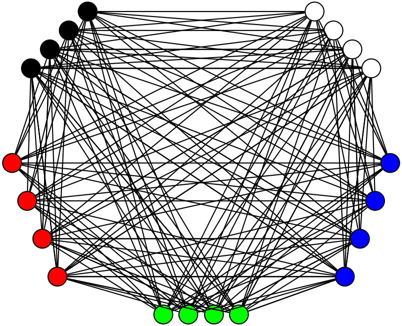

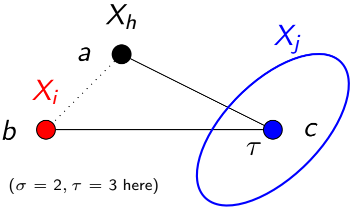

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Linked Systems of Symmetric Designs (Cameron)

Let Γ be a graph with vertex set X and adjacency relation $\sim$. We say Γ is a **linked system of symmetric** (v, k, λ) **designs (LSSD) with** w **fibres** if it is possible to partition X into w vertex subsets $X_1, \ldots, X_w$ such that

- no edge joins two vertices in the same fibre X_i (proper colouring)
- the subgraph induced between any X_i and X_j $(i \neq j)$ is the incidence graph of some symmetric (v, k, λ) design (so $|X_i| = v$ for all i)
- for distinct h, i, j, if $a \in X_h$ and $b \in X_i$,

$$|\Gamma(a) \cap \Gamma(b) \cap X_j| = \begin{cases} \sigma & \text{if } a \sim b; \\ \tau & \text{if } a \not\sim b. \end{cases}$$

Linked Systems Again

A 3-class Q-antipodal association scheme with w fibres has

- R_0 trivial (the identity relation, joins vertices in same fibre)

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- R_0 trivial (the identity relation, joins vertices in same fibre)
- R_1 joining pairs in distinct fibres $X_1, \ldots, X_w$
- R_2 is union of w complete graphs on the fibres $X_1, \ldots, X_w$
- R_3 joining the remaining pairs in distinct fibres

Note that we don't have such good upper bounds on w. Noda's bound (1974) was later shown by Mathon (1981) to follow from the Krein conditions.

Theorem (Van Dam, WJM, Muzychuk): $w \leqslant \frac{v-1}{2}$ unless Krein condition is tight, where $w \leqslant v/2$.

Linked Simplices

In $\mathbb{R}^d$, we seek full-dimensional simplices (consisting of unit vectors) such that vectors from distinct simplices form just two possible angles.

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Theorem (Kodalen, 2017): Every such configuration must come from a linked system of symmetric designs. Conversely, linked system of symmetric (v, k, λ) designs with w fibres yields (via E_1) w linked simplices in $\mathbb{R}^{v-1}$.

Real Mutually Unbiased Bases

Let $\mathcal{B}$ and $\mathcal{B}'$ be orthonormal bases for $\mathbb{R}^m$. We say that basis $\mathcal{B}$ is **unbiased** with respect to basis $\mathcal{B}'$ if $|\mathbf{b} \cdot \mathbf{b}'|$ is constant whenever $\mathbf{b}$ is chosen from $\mathcal{B}$ and $\mathbf{b}'$ is chosen from $\mathcal{B}'$.

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We say orthonormal bases $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_w$ for $\mathbb{R}^m$ are **mutually unbiased** if each $\mathcal{B}_i$ is unbiased with respect to each $\mathcal{B}_j$ $(j \neq i)$.
[w MUBs in $\mathbb{R}^m$]
Connection to association schemes (the famous "Cameron-Seidel scheme") discovered by Abdukhalikov, Bannai and Suda (2009).

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Theorem (LeCompte, WJM, Owens, 2010): Four-class Q-bipartite, Q-antipodal association schemes are in one-to-one(ish) correspondence with sets of real MUBs.

The Extended Q-Bipartite Double (WJM,Muzychuk,Williford)

There is a doubling construction for certain cometric association schemes that turns a d-class scheme into a $(d+1)$-class scheme on twice as many vertices. In the case of the Cameron-Seidel construction, this turns LSSDs into real MUBs.

When else does it happen?

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When else does it happen?
Theorem (Kodalen, 2016): LSSDs double to real MUBs if and only if (v, k, λ) are parameters of Menon type $(4u^2, 2u^2 - u, u^2 - u)$ or the complement.

Geometric view of the Transformation

The (scaled) first idempotent of the scheme of an LSSD has block form

$$\left[\begin{array}{cccc|cccc} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} & -\frac{3}{8} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{8} & \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} & -\frac{3}{8} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{3}{8} & -\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} \\ \hline \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} & -\frac{3}{8} & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{8} & \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} & -\frac{3}{8} & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \end{array}\right]$$

Geometric view of the Transformation

We push the dimension up by one

$$\left[\begin{array}{cccc|cccc}
1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} & -\frac{3}{8} \\
-\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{8} & \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} \\
-\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} & -\frac{3}{8} \\
-\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{3}{8} & -\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} \\
\hline
\frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} & -\frac{3}{8} & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{3}{8} & \frac{1}{8} & -\frac{3}{8} & -\frac{3}{8} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} & -\frac{3}{8} & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\
-\frac{3}{8} & -\frac{3}{8} & -\frac{3}{8} & \frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1
\end{array}\right] + \frac{1}{4} J$$

Geometric view of the Transformation

$$= \left[\begin{array}{cccc|cccc} \frac{5}{4} & 0 & 0 & 0 & \frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ 0 & \frac{5}{4} & 0 & 0 & -\frac{1}{8} & \frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} \\ 0 & 0 & \frac{5}{4} & 0 & -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} & -\frac{1}{8} \\ 0 & 0 & 0 & \frac{5}{4} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} \\ \hline \frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{5}{4} & 0 & 0 & 0 \\ -\frac{1}{8} & \frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} & 0 & \frac{5}{4} & 0 & 0 \\ -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} & -\frac{1}{8} & 0 & 0 & \frac{5}{4} & 0 \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} & 0 & 0 & 0 & \frac{5}{4} \end{array}\right]$$

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You can see that we failed to get MUBs. But if the two values in the off-diagonal blocks are "just right", we successfully convert LSSDs to MUBs.

Linking Systems (Davis, WJM, Polhill)

Let G be a finite group of size v, written multiplicatively. We seek

$$\{D_{i,j} | 1 \leqslant i, j \leqslant w, \ \ i \neq j\}$$

satisfying

- $D_{j,i} = D_{i,j}^{(-1)}$ for all $i \neq j$
- $D_{i,j} D_{j,i} = \lambda G + (k - \lambda) 1$
- $D_{h,i} D_{i,j} = \sigma D_{h,j} + \tau (G - D_{h,j})$ whenever h, i, j distinct

where these equations hold in the group algebra $\mathbb{C}[G]$.

Q-Bipartite 4-Class Association Schemes

Again, "Q-bipartite" means the spherical code is closed under $x \mapsto -x$.

- Equiangular lines come from 3-class Q-bipartite schemes
- Q-bipartite 4-class schemes correspond to lines with two angles, one of which is $\pi/2$
- Q-bipartite 5-class schemes correspond to lines with two angles, neither of which is $\pi/2$
- Jason Williford (Wyoming) now manages the tables: e.g. `http://www.uwyo.edu/jwilliford/data/qbip4_table.html`

Williford's Table of 4-class Q-bipartite Parameter Sets

Parameters	∃	v	m1	Krein Array	multiplicities	valencies	2nd Q	P	DRG	Quotient	Hyp	Comments
<42,6>	-	42	6	{6,5,27/7,12/5; 1,15/7,18/5,6}	1,6,14,15,6	1,10,20,10,1	-	01234	{10,6,3,1;1,3,6,10}	<21,10,3,6>		BCN Thm 4.4.11
<70,7>	!	70	7	{7,6,49/10,7/2; 1,21/10,7/2,7}	1,7,20,28,14	1,16,36,16,1	-	01234	{16,9,4,1;1,4,9,16}	<35,16,6,8>	FS	J(8,4)
<72,6>	+	72	6	{6,5,9/2,3; 1,3/2,3,6}	1,6,20,30,15	1,20,30,20,1	-	-		<36,15,6,6>		E6, Doubly Subtended Subquadrangles of GQ(3,9), Latin Square Type
<126,7>	+	126	7	{7,6,49/9,35/8; 1,14/9,21/8,7}	1,7,27,56,35	1,32,60,32,1	-	-		<63,30,13,15>		E7
<128,8>	!	128	8	{8,7,6,5; 1,2,3,8}	1,8,28,56,35	1,28,70,28,1	-	01234	{28,15,6,1;1,6,15,28}	<64,28,12,12>	FS	Halved 8-cube, Latin Square Type
<132,11>	+	132	11	{11,10,242/27,11/5; 1,55/27,44/5,11}	1,11,54,55,11	1,45,40,45,1	-	-		<66,20,10,4>	FS	Witt 5-(12,6,1)
<200,12>	?	200	12	{12,11,256/25,36/11; 1,44/25,96/11,12}	1,12,75,88,24	1,66,66,66,1	-	-		<100,33,14,9>		
<240,8>	+	240	8	{8,7,32/5,6; 1,8/5,2,8}	1,8,35,112,84	1,56,126,56,1	-	-		<120,56,28,24>		E8
<240,15>	+	240	15	{15,14,25/2,5; 1,5/2,10,15}	1,15,84,105,35	1,63,112,63,1	-	-		<120,56,28,24>	FS	NO+(8,2)
<240,18>	+	240	18	{18,17,72/5,6; 1,18/5,12,18}	1,18,85,102,34	1,51,136,51,1	-	-		<120,51,18,24>	FS	Doubly Subtended Subquadrangles of GQ(4,16)
<252,21>	-	252	21	{21,20,49/3,7; 1,14/3,14,21}	1,21,90,105,35	1,45,160,45,1	-	01234	{45,32,9,1;1,9,32,45}	<126,45,12,18>		Jurisic and Koolen
<260,13>	?	260	13	{13,12,169/15,13/3; 1,26/15,26/3,13}	1,13,90,117,39	1,81,96,81,1	-	-		<130,48,20,16>		
<308,28>	?	308	28	{28,27,245/11,14/3; 1,63/11,70/3,28}	1,28,132,126,21	1,72,162,72,1	-	-		<154,72,26,40>		
<324,36>	-	324	36	{36,35,27,6; 1,9,30,36}	1,36,140,126,21	1,56,210,56,1	-	01234	{56,45,12,1;1,12,45,56}	<162,56,10,24>		BCN, Thm. 11.4.6
<378,21>	?	378	21	{21,20,147/8,7/2; 1,21/8,35/2,21}	1,21,160,168,28	1,128,120,128,1	-	-		<189,60,27,15>		
<380,15>	?	380	15	{15,14,250/19,45/7; 1,35/19,60/7,15}	1,15,114,175,75	1,105,168,105,1	-	-		<190,84,38,36>		
<392,21>	?	392	21	{21,20,35/2,9; 1,7/2,12,21}	1,21,120,175,75	1,75,240,75,1	-	-		<196,75,26,30>		
<462,21>	?	462	21	{21,20,196/11,49/5; 1,35/11,56/5,21}	1,21,132,210,98	1,90,280,90,1	-	-		<231,90,33,36>		
<486,45>	?	486	45	{45,44,36,5; 1,9,40,45}	1,45,220,198,22	1,110,264,110,1	-	-		<243,110,37,60>		
<512,16>	+	512	16	{16,15,128/9,8; 1,16/9,8,16}	1,16,135,240,120	1,135,240,135,1	-	-		<256,120,56,56>		Lattice OBW16, Latin Square Type

Williford's Table of 4-class Q-bipartite Parameter Sets

```
<v, Krein array>
<240, [ 8, 7, 32/5, 6, 1, 8/5, 2, 8 ]>

P =
[   1   56 126   56    1]
[   1   28    0 -28   -1]
[   1    8  -18    8    1]
[   1   -2    0    2   -1]
[   1   -4    6   -4    1]

Q =
[    1     8    35   112    84]
[    1     4     5    -4    -6]
[    1     0    -5     0     4]
[    1    -4     5     4    -6]
[    1    -8    35  -112    84]

L =
[
    [1  0  0  0  0]
    [0  1  0  0  0]
    [0  0  1  0  0]
    [0  0  0  1  0]
    [0  0  0  0  1],

    [ 0 56   0   0   0]
    [ 1 27 27   1   0]
```

Spherical Designs

These schemes are best viewed as spherical codes. Spherical t-designs (avg over X is same as avg over sphere for polynomials of degree $\leqslant t$) yield Q-polynomial schemes. But most Q-polynomial schemes are only 2-designs or 3-designs.

A special sort of spherical code

We may instead view X as a subset of a unit sphere with relations R_i given by inner products.

X is a *(symmetric) association scheme* if there exists a function

$$\star : \mathbb{R}[t] \times \mathbb{R}[t] \rightarrow \mathbb{R}[t]$$

such that, for all $a, b \in X$ and all $f, g \in \mathbb{R}[t]$

$$\sum_{c \in X} f(\langle a, c \rangle) g(\langle b, c \rangle) = (f \star g)(\langle a, b \rangle)$$

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such that, for all $a, b \in X$ and all $f, g \in \mathbb{R}[t]$

$$\sum_{c \in X} f(\langle a, c \rangle) g(\langle b, c \rangle) = (f \star g)(\langle a, b \rangle)$$

X is *cometric* if $\star$ may be chosen so that $\deg f \star g \leqslant \min\{\deg f, \deg g\}$

Tight Designs and Dual Width

Definition: The *degree* of $D \subseteq X$ is $s = |\{\ i \neq 0\ :\ \chi_D^\top A_i \chi_D \neq 0\}|$.

Theorem (Delsarte, 1973): If $D \subseteq X$ is a $2s$-design in cometric scheme $(X, \mathcal{R})$ with Q-polynomial structure $E_0, E_1, \ldots, E_d$ (i.e. $E_j \chi_D = \mathbf{0}$ for $1 \leqslant j \leqslant 2s$), then

$$|D| \geqslant m_0 + m_1 + \cdots + m_s\ .$$

If equality holds, or if D has degree s, then D induces a cometric subscheme inside $(X, \mathcal{R})$.

Theorem (Brouwer, Godsil, Koolen, WJM, 2003): Let $(X, \mathcal{R})$ be a cometric scheme. If $D \subseteq X$ has degree s and *dual width* w^*, then $w^* + s \geqslant d$. If equality holds, then D induces a cometric subscheme.

A Bounty of New Examples

- linked systems of symmetric designs (Davis, WJM, Polhill)
- real mutually unbiased bases (Kharaghani, et al.)
- hemisystems in generalized quadrangles (Segre; Cossidente, Penttila; Bamberg, et al.)
- Sho Suda and Hadi Kharaghani have extended these ideas to linked systems of group divisible designs (cf. Suda's talk Friday)

Some New Examples

Penttila & Williford (2011)

- relative hemisystems in a generalized quadrangle with respect to a subquadrangle
- 3-class, primitive
- not P-polynomial, nor duals of P-polynomial schemes
- they first construct Q-bipartite schemes, some of which are the extended Q-bipartite doubles of these primitive schemes

Some New Examples

Moorhouse & Williford (2016)

- double covers of symplectic dual polar space graphs (Maslov index)
- unbounded class number, imprimitive (Q-bipartite)
- not P-polynomial, nor duals of P-polynomial schemes
- for field order q not a square, the scheme has irrational eigenvalues

Gavin King's Exceptional Schurian Cometric Schemes

Group G acts multiplicity-freely on the left cosets of subgroup H

d	$\|X\|$	struc	multiplicities	valencies	G , H
3	1288	P	$1, 22, 230, 1035$	1, 165, 330, 792	M_{23} , M_{11}
4	11178	P	$1, 23, 275, 2024, 8855$	1, 1100, 5600, 4125, 352	Co_3 , HS
		2Q	(02431)		
4	13056	P	$1, 135, 3400, 8925, 595$	1, 210, 1575, 5600, 5670	$\mathrm{Sp}(8,2)$, S_{10}
4	28431	P	$1, 260, 9450, 18200, 520$	1, 960, 3150, 22400, 1920	$O_8^+(3).2$, $O_8^+(2).2$
5	352	A	$1, 21, 154, 154, 21, 1$	1, 35, 105, 126, 70, 15	$M_{22}.2$, A_7
5	28160	A	$1, 429, 13650, 13650, 429, 1$	1, 364, 3159, 12636, 10920, 1080	$\mathrm{Fi}_{22}.2$, $O_7(3)$
5	104448	P	$1, 187, 7700, 56100, 39270, 1190$	1, 462, 5775, 30800, 62370, 5040	$\mathrm{PSO}^-(10,2)$, S_{12}
6	704	AB	$1, 22, 175, 308, 175, 22, 1$	1, 50, 175, 252, 175, 50, 1	$\mathrm{HS}.2$, $U_3(5)$
		2Q	(0523416)		
7	4050	A	$1, 22, 252, 1750, 1750, 252, 22, 1$	1, 176, 462, 1155, 1232, 672, 330, 22	$\mathrm{McL}.2$, M_{22}

Turning a Cometric Scheme into a Spherical Code

Let $(X, \mathcal{R})$ be a cometric association scheme with Q-polynomial structure

$$E_0, E_1, \ldots, E_d.$$

Turning a Cometric Scheme into a Spherical Code

Let $(X, \mathcal{R})$ be a cometric association scheme with Q-polynomial structure

$$E_0, E_1, \ldots, E_d.$$

We know that E_1 generates $\mathbb{A}$ under entrywise multiplication, so it has $d+1$ distinct entries.
We view this as the Gram matrix of a spherical code $X \subseteq \mathbb{R}^m$ ($m = m_1$). In fact, we will henceforth identify the vertices with these $|X|$ unit vectors.

Turning a Cometric Scheme into a Spherical Code

Write *Gram matrix*

$$G = \frac{|X|}{m} E_1,$$

Turning a Cometric Scheme into a Spherical Code

Write *Gram matrix*

$$G = \frac{|X|}{m} E_1, \qquad G = UU^\top .$$

We now identify X with the set of rows of U; this is a spherical code in $\mathbb{R}^m$ with pairwise inner products

$$1 = \omega_0 > \omega_1 > \cdots > \omega_d$$

where, with appropriate ordering of relations, $\omega_i = \frac{1}{m} Q i 1$.

Balanced Set Condition

Terwilliger: If $(X, \mathcal{R})$ is cometric with respect to E_1, then (with spherical code definitions as above):
for each $a, b \in X$ and each i, j with $0 \leqslant i, j \leqslant d$, the two vectors

$$b - a, \qquad \sum_{\substack{(a,c)\in R_i \\ (b,c)\in R_j}} c \;-\; \sum_{\substack{(a,c)\in R_j \\ (b,c)\in R_i}} c$$

are linearly dependent.

Thank You!

I welcome your questions and comments.

Doubtless Bay (Mangonui)
February 21, 2018

Truth River next??