NEGATIVE CLIQUES IN SETS OF EQUIANGULAR LINES

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Tight Frames and Approximation
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Outline

Equiangular Lines

Bounds on Real Equiangular Lines

Simplices Embedded in ETFs

OUTLINE

EQUIANGULAR LINES

BOUNDS ON REAL EQUIANGULAR LINES

SIMPLICES EMBEDDED IN ETFS

Orthonormal Bases

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How can we generalize these traits?

Equiangular tight frame = 1, 2A, 2B, 3 (up to scaling)
Equiangular lines = 1, 2B

EQUIANGULAR LINES

DEFINITION

Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. Let $\Phi = \{\varphi_j\}_{j=1}^n \subset \mathbb{F}^k$ with $\|\varphi_j\| = 1$ for all $j \in \{1, \ldots, n\}$. If there exists an α such that for all $j \neq \ell$, $|\langle \varphi_j, \varphi_\ell \rangle| = \alpha$, Φ is a set of equiangular lines.

If further for all $x \in \mathbb{F}^k$

$$\sum_{j=1}^{n} |\langle x, \varphi_j \rangle|^2 = \frac{n}{k}\|x\|^2 \quad \Leftrightarrow \quad x = \frac{n}{k}\sum_{j} \langle x, \varphi_j \rangle \varphi_j,$$

then Φ is an equiangular tight frame (ETF).

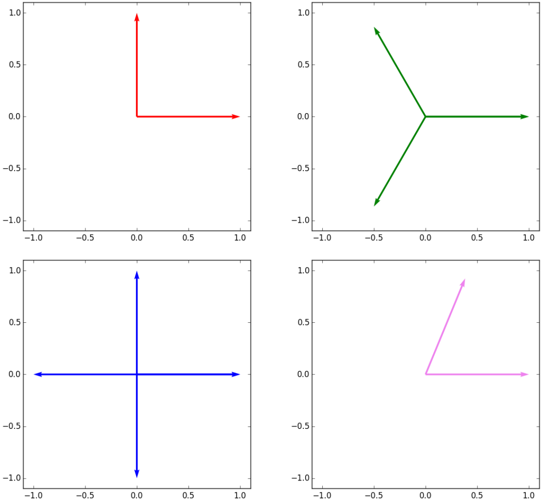
By slight abuse of notation,

- $\Phi = \begin{pmatrix} \varphi_1 & \varphi_2 & \ldots & \varphi_n \end{pmatrix}$, and
- α is the "angle."

THEOREM

(Goyal, Kovačević, Kelner 2001; Strohmer, Heath 2003; Benedetto, Kolesar 2006) ETFs are optimally robust against erasures and noise.

EXAMPLES IN $\mathbb{R}^2$



- ONB, ETF
- ETF, maximal set of equiangular lines
- FUNTF, worst case coherence
- Equiangular lines which are not an ETF

Research Questions

Q1 : Given d (and $0 < \alpha < 1$), what is the maximal size $s(d)$ (resp., $s_\alpha(d)$) of a set of equiangular lines (resp., with angle α) in $\mathbb{R}^d$?

Q2 : Given a specific ETF or class of ETFs, what is the structure of linear dependencies of the vectors?

Gram Matrices

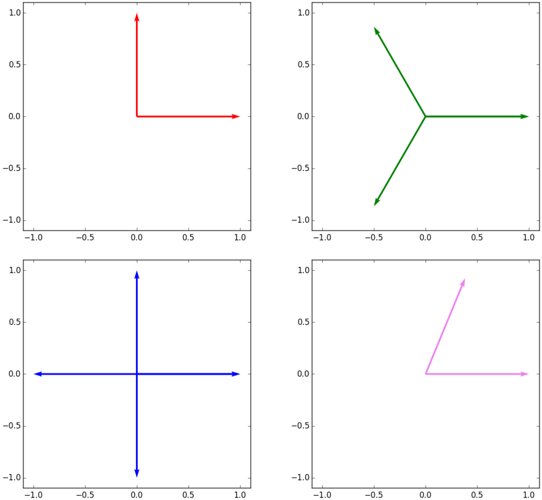
Instead of Φ, we will usually deal with the Gram matrix $G(\Phi) = \Phi^*\Phi$.

Let I_n be the $n \times n$ identity and J_n the $n \times n$ all-ones matrix (where we write I and J when clear from context).

Basic linear alg: If $G = (a-b)I_n + bJ_n$, then G has a simple eigenvalue $\lambda_1 = a + (n-1)b$ and an eigenvalue $\lambda_2 = a - b$ with multiplicity $n-1$.

$$\begin{pmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \dots & a \end{pmatrix}$$

Gram Matrices, II



- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 1 \end{pmatrix}$

Switching Equivalence

Basic linear algebra: $G(\Phi) = G(\Psi) \quad \Leftrightarrow \quad \Phi = U\Psi$ for some unitary U.

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Definition

Two sets of unit vectors Φ and Ψ in $\mathbb{F}^d$ are switching equivalent, denoted by $\Phi \cong \Psi$, if there exists a diagonal matrix B with unit norm diagonal entries and a permutation matrix C such that

$$(BC) \cdot G(\Phi) \cdot (BC)^{-1} \;=\; G(\Psi).$$

$\Phi \cong \Psi \Rightarrow$ there exists a unitary U, diagonal $(1, -1)$-matrix B with unit norm diagonal entries, and permutation matrix C such that

$$U\Phi(BC)^{-1} = \Psi.$$

=((Van Lindt & Seidel 1966 + generalization to $\mathbb{F}$) + permutations) = (Projective unitary equivalence + permutations)

Triple Products

Theorem

(Godsil and Royle 2001; Chien and Waldron 2016) Let $\Phi, \Psi \subset \mathbb{F}^d$ with $|\Phi| = |\Psi| = n$ be equiangular. Then $\Phi \cong \Psi$ if and only if there exists a $\sigma \in S_n$ such that for all $i \neq j \neq k \neq i$.

$$\langle \varphi_i, \varphi_j \rangle \langle \varphi_j, \varphi_k \rangle \langle \varphi_k, \varphi_i \rangle = \langle \psi_{\sigma(i)}, \psi_{\sigma(j)} \rangle \langle \psi_{\sigma(j)}, \psi_{\sigma(k)} \rangle \langle \psi_{\sigma(k)}, \psi_{\sigma(i)} \rangle.$$

When $\mathbb{F} = \mathbb{R}$ and ignoring permutations, this gives precisely the structure of the two-graph which represents equivalence classes of switching equivalences. (With permutations, get isomorphisms of the two-graphs.)

Negative Cliques

Definition

Let Φ be a set of equiangular lines with angle α. If $X \subset \Phi$ is such that $X \cong Y$ with $G(Y) = (1+\alpha)I - \alpha J$, then we call X a negative clique.

- (K, Tang 2016) Let X be a maximal negative clique in a given Φ. X is called a K-base.
- (Fickus, Jasper, K, Mixon 2017) If X is a negative clique of size $1 + (1/\alpha)$, we call X a $1/\alpha$-regular simplex.

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Negative cliques have size $\leq 1 + 1/\alpha$.
When the bound is saturated, they form a tight frame for their span (Fickus, Jasper, K, Mixon 2017);
otherwise they are linearly independent (e.g., Lemmens & Seidel 1973).

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Pillar Decomposition

Definition

(Lemmens & Seidel 1973) Let $\Phi \in \mathbb{R}^d$ be equiangular with K-base X. Let Ξ denote the subspace spanned by X. Elements of Φ which lie in the same coset of $\Xi^\perp$ are called pillars.

Proposition

(Lemmens & Seidel 1973; K, Tang 2016) Let $\varphi \in \Phi \backslash X$. If any K-base is of size $1 + 1/\alpha$, then the norm of $P_{\Xi^\perp}\varphi$ is equal to α. If any K-base is of size $< 1 + 1/\alpha$, then the norm of $P_{\Xi^\perp}\varphi$ depends on the number of negative inner products $\langle \varphi, x_i \rangle$, $x_i \in X$.

Basic Procedure of K, Tang 2016

- (Lemmens & Seidel 1973) We only need to compute upper bounds on $s_\alpha(d)$ for α the reciprocal of an odd integer between 5 and a $\sqrt{2d} + 1$. (3 solved.)

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- For each K-base size, we partition $\Phi \backslash X$ into equivalence classes based on the number of negative inner products with X and analyze these (using combinatorics, graph theory, and linear algebra).

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- For each K-base size, we partition $\Phi\backslash X$ into equivalence classes based on the number of negative inner products with X and analyze these (using combinatorics, graph theory, and linear algebra).
- We further split the above equivalence classes into classes based on with which elements of X the elements have a negative inner product and analyze these. (This will sometimes involve a bound of a size of particular spherical two-distance sets.)

Spherical Two-Distance Sets

Definition

Let $\Phi = \{\varphi_1, \ldots, \varphi_n\} \subset \mathbb{R}^d$ be a set of unit normed vectors and let $\alpha, \beta \in \mathbb{R}$. The Φ is a spherical two-distance set if $\langle \varphi_i, \varphi_j \rangle \in \{\alpha, \beta\}$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$. We denote by $s(d, \alpha, \beta)$ the largest size of a spherical two-distance set with the given parameters.

Note: An equiangular set of lines is a spherical two-distance set w.r.t $\alpha, -\alpha$, and thus $s_\alpha(d) = s(d, \alpha, -\alpha)$.

New Upper Bound on $s_{1/5}(d)$

Theorem

(K, Tang 2016) Let $\Phi \subset \mathbb{R}^d$ be an equiangular set with angle $1/5$. If $d > 60$, then

$$|\Phi| \;\leq\; 148 + 3 \cdot s\left(d, 1/13, -5/13\right) \;\leq\; 148 + \frac{648d(d+2)}{47d+169}.$$

A TASTE OF THE CASES

THEOREM

(K, Tang 2016) For $n = 1, \ldots, \lfloor K/2 \rfloor$ and for each equivalence class $\overline{x} \subset X(K, n)$, we have the following upper bounds on $|\overline{x}|$.

<u>(1)</u> *If $n = 1$, then*

$$|\overline{x}| \;\leq\; \begin{cases} r - K, & 1 \geq K - \frac{(1/\alpha)+1}{2} \\ \frac{1-\alpha}{l(K,1)-\alpha}, & 1 < K - \frac{(1/\alpha)+1}{2}. \end{cases}$$

<u>(2)</u> *If $1 < n < K - \frac{(1/\alpha)+1}{2}$, then $|\overline{x}| \;\leq\; r + 1$.*

<u>(3)</u> *If $n = K - \frac{(1/\alpha)+1}{2}$, then $|\overline{x}| \;\leq\; r - K + \lfloor 2\alpha \frac{r-K}{1-\alpha} \rfloor$.*

<u>(4)</u> *If $K - \frac{(1/\alpha)+1}{2} < n < \lfloor \frac{K}{2} \rfloor$, then*

$$|\overline{x}| \;\leq\; s\left(r, \beta, \gamma\right),$$

where $\beta = \frac{\alpha - l(K,n)}{1 - l(K,n)}$ and $\gamma = \frac{-\alpha - l(K,n)}{1 - l(K,n)}$.

EQUIANGULAR LINES IN $\mathbb{R}^d$

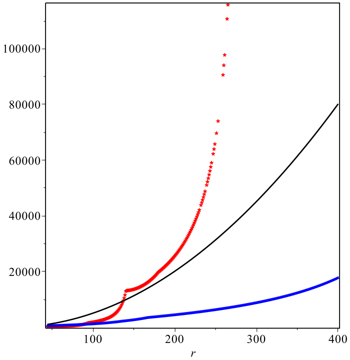
THEOREM (K, TANG 2016)
Let m be the largest positive integer such that $(2m+1)^2 \leq d+2$. Then

$$s(d) \;\leq\; \begin{cases} \frac{4d(m+1)(m+2)}{(2m+3)^2-d}, & d = 44, 45, 46, 76, 77, 78, 117, 118, 166, \\ & \qquad 222, 286, 358 \\ \frac{\left((2m+1)^2-2\right)\left((2m+1)^2-1\right)}{2}, & \text{other } k \text{ between } 44 \text{ and } 400 \end{cases} .$$

Applied the SDP approach of [Bachoc, Valentin 2008; Barg, Yu 2014] to bound the size of certain spherical two distance sets in the cases they arose.

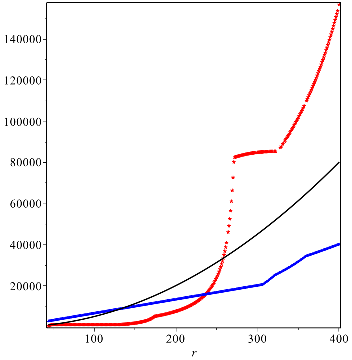
NEW BOUND VS. OLD

FIGURE: K, Tang 2016 and $\mathbf{sdp}(d, \frac{1}{5}, -\frac{1}{5})$



—: $\frac{d(d+1)}{2}$ ⋆⋆⋆ : $\mathbf{sdp}(d, \frac{1}{5}, -\frac{1}{5})$ +++ : K, Tang 2016

FIGURE: K, Tang 2016 and $\mathbf{sdp}(d, \frac{1}{7}, -\frac{1}{7})$



—: $\frac{k(k+1)}{2}$ ⋆⋆⋆ : $\mathbf{sdp}(d, \frac{1}{7}, -\frac{1}{7})$ +++ : K, Tang 2016

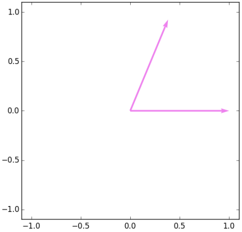
Positive Cliques

One may similarly define a positive clique.

For all $0 \leq \alpha < 1$ there exist at least d vectors In $\mathbb{F}^d$ with pairwise inner product α

Geometrically, one may think of "pushing" vectors in an onb together.

One may analyze projections onto orthogonal complements of positive cliques and use Ramsey theory to obtain asymptotic relative bounds. (Balla, Dräxler, Keevash, Sudakov 2018)



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Binders

Definition (Fickus, Jasper, K, Mixon 2017)
Let Φ be an ETF. The set of subsets of vectors (or the corresponding incidence matrix) which are $1/\alpha$-regular simplices is the binder.

These are the smallest sets of linearly dependent vectors in the ETF: For a general set of unit vectors Φ,

$$\text{size of the smallest set of linearly dependent vectors in } \Phi \;\geq\; 1 + \frac{1}{\mu(\Phi)}.$$

(Gerschgorin circle theorem applied to the Gram matrix. Donoho, Elad 2003)

BinderFinder

BinderFinder is a relatively short Matlab code that uses triple products and some clever combinatorial tricks to compute the binder of a given ETF.

(Could also be used on sets of equiangular lines.)

Code available for download at:
`http://www.math.uni-bremen.de/cda/`

Binders of ETFs in $\mathbb{C}^{3\times 9}$

Perhaps the first investigation of linear dependencies in equiangular Gabor frames (SIC-POVMs) was presented in a talk by Hughston in 2007 (cited in Dang, Blanchfield, Bengtsson, Appleby 2013), where the linear dependencies of certain SIC-POVMs in $\mathbb{C}^3$ were shown to be represented by the Hesse configuration.

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I came to the question via the construction of ETFs in (Jasper, Mixon, Fickus 2013). This construction involves a tensor-like construction of an incidence matrix of a BIBD with a $1/\alpha$-regular simplex. (Bad algebraic spread, but good geometric spread?!?!?)

Hesse Configuration

The Hesse configuration is the set of all lines in $\mathbb{F}_3^2$:

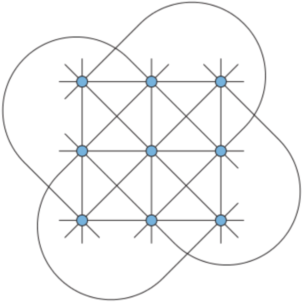


Figure: By David Eppstein - Own work, CC0, https://commons.wikimedia.org/w/index.php?curid=18920067

“Normal” Configuration

Let $\zeta = e^{2\pi i/3}$, $\theta \in [0, 2\pi/6] \backslash \{0, 2\pi/9\}$.
SIC-POVMs:(Hughston 2007; Dang, Blanchfield, Bengtsson, Appleby 2013)

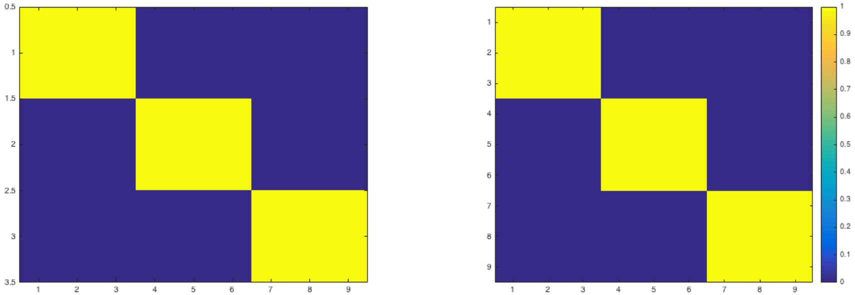
$$\begin{pmatrix} 0 & 0 & 0 & -e^{i\theta} & -e^{i\theta}\zeta & -e^{i\theta}\zeta^2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -e^{i\theta} & -e^{i\theta}\zeta & -e^{i\theta}\zeta^2 \\ -e^{i\theta} & -e^{i\theta}\zeta & -e^{i\theta}\zeta^2 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Kirkman ETFs (Fickus, Jasper, Mixon 2013):

$$\overset{\text{BIBD}(3,2,1)}{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}} \quad \& \quad \overset{\mathcal{D}_3}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix}} =: \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 \;\; 0 \;\; 0 & w_0 & w_1 \\ w_1 & 0 \;\; 0 \;\; 0 & w_0 \\ w_0 & w_1 & 0 \;\; 0 \;\; 0 \end{pmatrix}$$

“NORMAL” CONFIG. BINDER & GRAM OF BINDER



Left: Binder of a “normal” SIC-POVM in $\mathbb{C}^3$, Right: The Gram matrix of the binder.

Obtaining the Hesse Cofiguration

Let $\zeta = e^{2\pi i/3}$, $\theta \in \{0, 2\pi/9\}$.
SIC-POVMs:(Hughston 2007; Dang, Blanchfield, Bengtsson, Appleby 2013)

$$\begin{pmatrix} 0 & 0 & 0 & -e^{i\theta} & -e^{i\theta}\zeta & -e^{i\theta}\zeta^2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -e^{i\theta} & -e^{i\theta}\zeta & -e^{i\theta}\zeta^2 \\ -e^{i\theta} & -e^{i\theta}\zeta & -e^{i\theta}\zeta^2 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Polyphase BIBD ETFs (Fickus, Jasper, Mixon, Peterson, Watson 2017; Fickus, Jasper, K, Mixon 2017):

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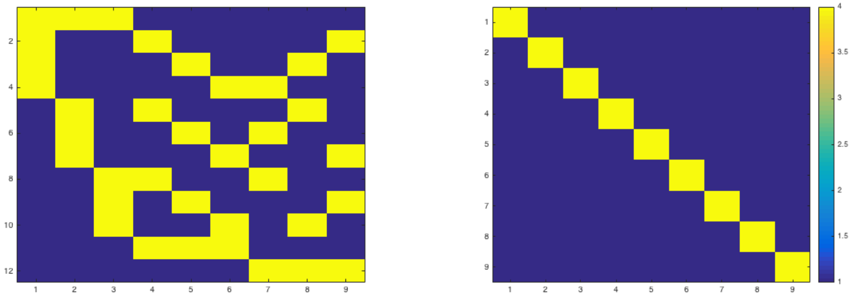
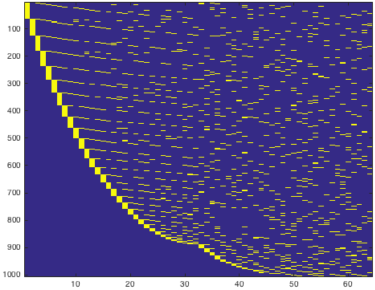
$$\Rightarrow \left(\begin{array}{ccc} 0 \;\; 0 \;\; 0 & w_0 & -w_1 \\ -w_1 & 0 \;\; 0 \;\; 0 & w_0 \\ w_0 & -w_1 & 0 \;\; 0 \;\; 0 \end{array}\right)$$

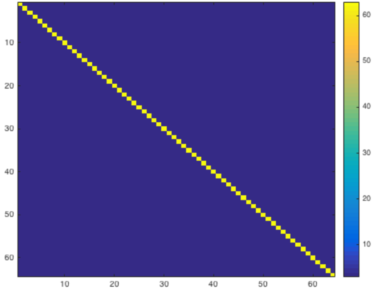
Hesse Configuration as a Binder



Left: Hesse Configuration binder of a SIC-POVM in $\mathbb{C}^3$, Right: The Gram matrix of the binder.

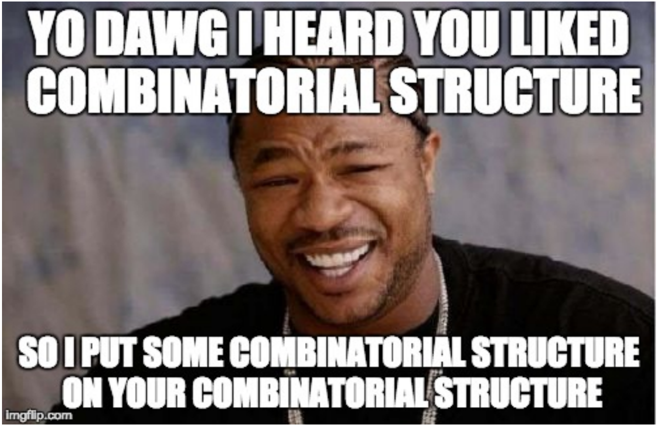
Binder of Hoggar's Lines





Left: Binder of Hoggar's lines (non-Gabor SIC POVM in $\mathbb{C}^8$ with 1008 simplices), Right: The Gram matrix of the binder.

Conclusion

- Fond memories (nightmares?) of using Sylow p-groups in the classification of finite simple groups.
- 

Missing Comrades

Thanks to Shayne for organizing/chauffeuring and to New Zealand for being so beautiful!

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http://www.math.uni-bremen.de/cda/

"New Upper Bounds for Equiangular Lines by Pillar Decomposition" on arXiv (the paper formerly known as "Computing Upper Bounds for Equiangular Lines in Euclidean Space")

"Equiangular tight frames that contain regular simplices" on arXiv