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Transfinite Diameter on Affine Algebraic Varieties

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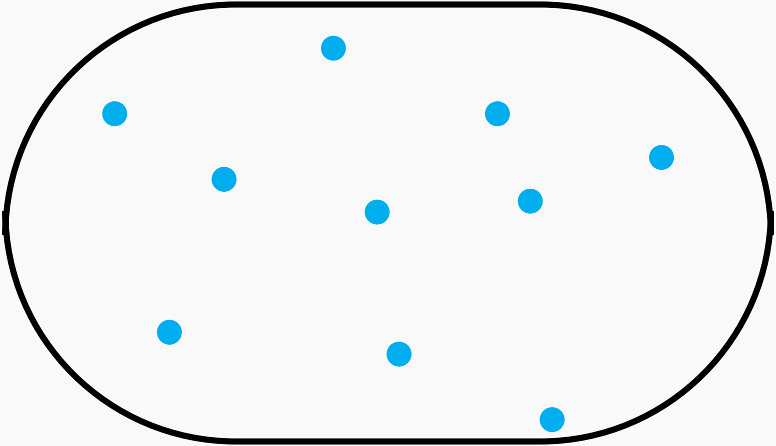
Content

Transfinite Diameter
Motivation



Transfinite Diameter

Motivation

- We observe that the charges spread themselves out across the capacitor in a way that minimises the overall energy of the system.
- This is equivalent to maximising the mutual distances between each of the charges.

Transfinite Diameter

Motivation

- We observe that the charges spread themselves out across the capacitor in a way that minimises the overall energy of the system.
- This is equivalent to maximising the mutual distances between each of the charges.
- To translate this to mathematics, let K be the body of the capacitor and each charge be represented by a point p_i. We can describe the maximum mutual distance as

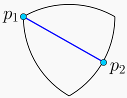
$$\sup_{p_1,\ldots,p_n \in K} \left(\prod_{1 \le j < i \le n} |p_i - p_j| \right)^{2/n(n+1)}$$

- We call this number the n-diameter for the set K, denoted $d_n(K)$.

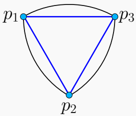
Transfinite Diameter

n-Diameters

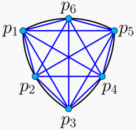
- Here are some *n*-diameters for the reuleaux triangle.



$$d_2(K) = |p_1 - p_2|$$

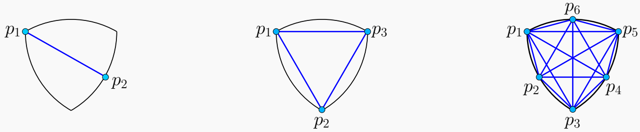


$$d_3(K) = \prod_{1\le j<i\le 3} |p_i - p_j|^{1/3}$$



$$d_6(K) = \prod_{1\le j<i\le 6} |p_i - p_j|^{1/21}$$

- Here are some n-diameters for the reuleaux triangle.



$$d_2(K) = |p_1 - p_2| \qquad d_3(K) = \prod_{1\le j<i\le 3} |p_i - p_j|^{1/3} \qquad d_6(K) = \prod_{1\le j<i\le 6} |p_i - p_j|^{1/21}$$

- It is well known that the product of the distances between n points is equal to a *Vandermonde determinant*.

$$\prod_{1\le j<i\le n} |p_i - p_j| = \left| \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_n \\ p_1^2 & p_2^2 & \dots & p_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ p_1^n & p_2^n & \dots & p_n^n \end{pmatrix} \right|$$

- n-diameters are of interest because the configurations of points give 'good' nodes for polynomial interpolation.

Transfinite Diameter
Definition

Definition (Fekete (1923), Szegö (1924))

Let $K \subset \mathbb{C}$ be a compact set. The *transfinite diameter* of K is defined to be

$$d(K) := \lim_{n \to \infty} d_n(K).$$

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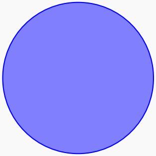
Theorem

Let $K \subset \mathbb{C}$ be a compact set. Then $\tau(K) = d(K)$ where $\tau(K)$ is the Chebyshev constant for K.

$$\tau(K) = \lim_{k\to\infty} \inf\{\|p\|_K^{1/k} : p(z) = z^k + \textit{lower order terms}\}.$$

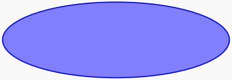
Transfinite Diameter
Examples

- All of the following sets have transfinite diameter equal to 1.

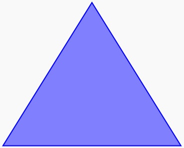


Circle radius 1

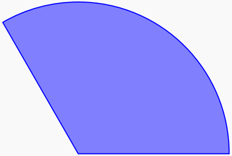
Line length 4



Ellipse, semi-axes 1.5 and 0.5



Equilaterial Triangle, side ~ 2.3711



Circular arc, radius 2, angle $\frac{2\pi}{3}$

- We lose the physical motivation for studying the transfinite diameter in $\mathbb{C}^N$ for $N \geq 2$ but because the transfinite diameter has interesting connections to approximation theory / potential theory it is still something that we are interested in studying.

Transfinite Diameter
Definition in Higher Dimensions

- We lose the physical motivation for studying the transfinite diameter in $\mathbb{C}^N$ for $N \geq 2$ but because the transfinite diameter has interesting connections to approximation theory / potential theory it is still something that we are interested in studying.
- Following this, n-diameters are defined in terms of Vandermonde determinants. If $\mathcal{P}_n = \{e_1(z), e_2(z), \ldots, e_{m(n)}(z)\}$ is a monomial basis for the polynomials of degree at most n in $\mathbb{C}^N$ and $l(n)$ the sum of the degrees of the monomials of degree at most n then

$$d_n(K) = \sup_{\zeta_1,\ldots,\zeta_{m(n)} \in K} \left| \det \begin{pmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \ldots & e_1(\zeta_{m(n)}) \\ e_2(\zeta_1) & e_2(\zeta_2) & \ldots & e_2(\zeta_{m(n)}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{m(n)}(\zeta_1) & e_{m(n)}(\zeta_2) & \ldots & e_{m(n)}(\zeta_{m(n)}) \end{pmatrix} \right|^{1/l(n)}$$

- We define $d(K) := \lim_{n\to\infty} d_n(K)$ as before. Equality with a generalised $\tau(K)$ can also be obtained [Leja 1959, Zakharyuta, 1975].

- An affine algebraic variety $\mathcal{V}$ is the common zero set of a collection of polynomials in $\mathbb{C}^N$. Precisely,

$$\mathcal{V} := \{z \in \mathbb{C}^N \,|\, p_1(z) = \ldots = p_s(z) = 0\}.$$

- By construction there are non-trivial zero polynomials on $\mathcal{V}$. A consequence of this is that the monomials in $\mathbb{C}^N$ are not linearly independent on $\mathcal{V}$.

Generalisation to Affine Algebraic Varieties

Algebraic Varieties

- An affine algebraic variety $\mathcal{V}$ is the common zero set of a collection of polynomials in $\mathbb{C}^N$. Precisely,

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- By construction there are non-trivial zero polynomials on $\mathcal{V}$. A consequence of this is that the monomials in $\mathbb{C}^N$ are not linearly independent on $\mathcal{V}$.
- For large n, we can do row operations on a Vandermonde determinant defining an n-diameter to obtain a row of zeroes.

$$\det \begin{pmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \ldots & e_1(\zeta_m) \\ e_2(\zeta_1) & e_2(\zeta_2) & \ldots & e_2(\zeta_m) \\ \vdots & \vdots & \ddots & \vdots \\ e_m(\zeta_1) & e_m(\zeta_2) & \ldots & e_m(\zeta_m) \end{pmatrix} \xrightarrow{\text{row ops.}} \det \begin{pmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \ldots & e_1(\zeta_m) \\ e_2(\zeta_1) & e_2(\zeta_2) & \ldots & e_2(\zeta_m) \\ \vdots & \vdots & \vdots & \vdots \\ p_j(\zeta_1) & p_j(\zeta_2) & \ldots & p_j(\zeta_m) \\ \vdots & \vdots & \vdots & \vdots \\ e_m(\zeta_1) & e_m(\zeta_2) & \ldots & e_m(\zeta_m) \end{pmatrix} = 0$$

- With the $\mathbb{C}^N$ definition $d(K) = 0$ for any $K \subset \mathcal{V}$.

- To get a useful transfinite diameter a natural approach is to modify monomial basis used in the $\mathbb{C}^N$ definition of a n-diameter.

Generalisation to Affine Algebraic Varieties

A Useful Transfinite Diameter

- To get a useful transfinite diameter a natural approach is to modify monomial basis used in the $\mathbb{C}^N$ definition of a n-diameter.
- The Cox-Ma‘u approach is an *algebraic approach*.

  We list the monomials with increasing degree and remove monomials which are linearly dependent to monomials that precede them. We call this set the *reduced monomials* for $\mathcal{V}$ and use these for our n-diameter. We use the notation $d^{cm}(K)$ to indicate the Cox-Ma‘u transfinite diameter.

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- The Berman-Boucksom approach is an *analytic approach*.

  Fix a probability measure μ. Take the monomials and perform the Gram-Schmidt on them. This produces an L^2_μ-orthonormal basis for the polynomials on $\mathcal{V}$, which we can use in the definition of our n-diameter. We use the notation $d^{bb}_\mu(K)$ to indicate the Berman-Boucksom transfinite diameter.

- We will now construct the reduced monomials for $\mathcal{V} = \{z_2^2 - z_1^2 - 1 = 0\}$.
- The $\mathbb{C}^2$ monomials ordered by *grevlex*[1] are

$$1, \quad z_1, \quad z_2, \quad z_1^2, \quad z_1 z_2, \quad z_2^2, \quad z_1^3, \quad z_1^2 z_2, \quad z_1 z_2^2, \quad z_2^3, \quad z_1^4, \ldots$$

[1] Ordered first by total degree then letting z_1 precede z_2 within monomials of the same degree.

Generalisation to Algebraic Varieties

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- On $\mathcal{V}$, we have that $z_2^2 = z_1^2 + 1$. So z_2^2 is a linear combination of monomials that precede it. The same is true of any monomial with a power of z_2^2.

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- Removing these terms gives us the reduced monomials

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- In general the reduced monomials can be found by looking at the quotient $\mathbb{C}[\mathcal{V}] := \mathbb{C}[z]/I(\mathcal{V})$ where $I(\mathcal{V})$ is the ideal associated to $\mathcal{V}$.

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Generalisation to Algebraic Varieties

Berman-Boucksom Transfinite Diameter

- We will now construct a L^2_μ-orthonormal polynomial basis for $\mathcal{V} = \{z_2^2 - z_1^2 - 1 = 0\}$.
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- We choose μ to be normalised Lebesgue measure on

$$T_{\mathcal{V}} = \{z \in \mathcal{V} : |z_1| = 1\}.$$

- A sample normalisation calculation in the Gram-Schmidt process is

$$\langle z_2, z_2\rangle_\mu^{1/2} = \left(\int_{T_{\mathcal{V}}} z_2\bar{z}_2\, d\mu\right)^{1/2} = \left(\int_{|z_1|=1} |z_1^2+1|\, d\mu\right)^{1/2} = \left(\int_0^{2\pi} \left|e^{2it}+1\right| \frac{dt}{2\pi}\right)^{1/2} = \frac{2}{\sqrt{\pi}}.$$

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- One can check that monomials involving z_2^2 are removed via Gram-Schmidt. This means an L^2_μ-orthonormal basis for the polynomials on $\mathcal{V}$ is

$$1, \quad z_1, \quad \frac{\sqrt{\pi}}{2} z_2, \quad z_1^2, \quad \frac{\sqrt{\pi}}{2} z_1 z_2, \quad z_1^3, \quad \frac{\sqrt{\pi}}{2} z_1^2 z_2, \quad z_1^4, \quad \frac{\sqrt{\pi}}{2} z_1^3 z_2, \quad z_1^5, \ldots$$

Equality of Transfinite Diameters

Overview

- In 2014 the question was asked of how $d^{cm}(K)$ and $d_\mu^{bb}(K)$ were related. In 2017 we showed that, under mild hypotheses, that

$$d^{cm}(K) = d_\mu^{bb}(K).$$

- The first hypothesis was that $\mathbb{C}[\mathcal{V}]$ was a *Noether normalisation*.
- The second hypothesis was that μ was normalised Lebesgue measure on $T_{\mathcal{V}}$.

Equality of Transfinite Diameters

Noether Normalisation

- $\mathbb{C}[\mathcal{V}]$ being a *Noether normalisation* means there are coordinates $z = (x, y) \in \mathbb{C}^M \times \mathbb{C}^{N-M}$ such that there are finitely many multi-indices α_i which allow the decomposition

$$\mathbb{C}[\mathcal{V}] = \bigoplus_i y^{\alpha_i} \mathbb{C}[x]$$

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- e.g. if $\mathcal{V} = \{z_2^2 - z_1^2 - 1 = 0\}$ then setting $z_2 = y$ and $z_1 = x$ we have $\mathbb{C}[\mathcal{V}] = \mathbb{C}[x] \oplus y\mathbb{C}[x]$ which can be seen directly:

monomials in $\mathbb{C}[\mathcal{V}] =$	$1,$	$x,$	$y,$	$x^2,$	$xy,$	$x^3,$	$x^2y,$	$\dots$
monomials in $\mathbb{C}[x] =$	$1,$	$x,$		$x^2,$		$x^3,$		$\dots$
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\end{array}$$

- This property is not always true, consider $\mathcal{V}' = \{xy - 1 = 0\}$ then the reduced monomials are

$$1,\ x,\ y,\ x^2,\ y^2,\ x^3,\ y^3,\ x^4,\ \dots$$

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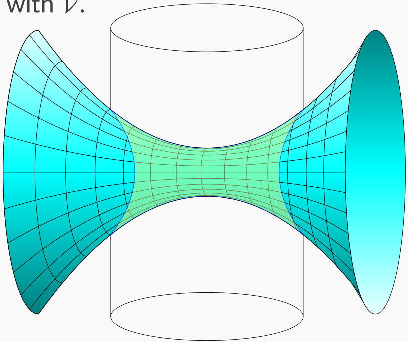
$$1,\ x,\ y,\ x^2,\ y^2,\ x^3,\ y^3,\ x^4,\ \ldots$$

- One can always make a linear change of variables to ensure this property holds.

Equality of Transfinite Diameters

The Measure μ on $T_{\mathcal{V}}$

- Given a Noether normalisation $z = (x, y)$, we define $T_{\mathcal{V}} := \{(x, y) \in \mathcal{V} : |x_i| = 1\}$. This is the intersection of a unit x-polycylinder with $\mathcal{V}$.



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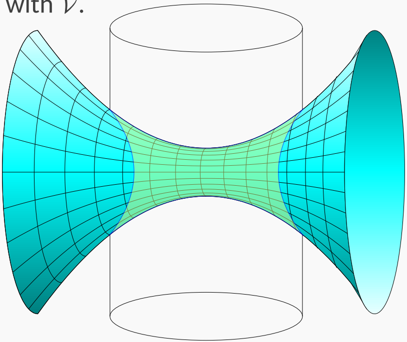

- μ is chosen to be normalised Lebesgue measure on $T_{\mathcal{V}}$. This choice ensures that the x-monomials are already orthonormal.
- The Noether normalisation hypothesis ensures that $T_{\mathcal{V}}$ is *bounded* in the y directions.

Equality of Transfinite Diameters

Main Idea

- The hypotheses ensure that the y-dependent scale factor between $d_n^{cm}(K)$ and $d_{n,\mu}^{bb}(K)$ has slower growth than the $l(n)$-root, so it tends to 1 as $n \to \infty$.

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- e.g. compare the reduced monomial/L^2_μ-orthonormal polynomial basis for $\mathcal{V} = \{z_2^2 - z_1^2 - 1 = 0\}$.

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- The hypotheses ensure that the y-dependent scale factor between $d_n^{cm}(K)$ and $d_{n,\mu}^{bb}(K)$ has slower growth than the $l(n)$-root, so it tends to 1 as $n \to \infty$.
- e.g. compare the reduced monomial/L^2_μ-orthonormal polynomial basis for $\mathcal{V} = \{z_2^2 - z_1^2 - 1 = 0\}$.

$$\begin{array}{cccccccccccc} 1, & z_1, & z_2, & z_1^2, & z_1 z_2, & z_1^3, & z_1^2 z_2, & z_1^4, & z_1^3 z_2, & z_1^5, & \dots \\ 1, & z_1, & \frac{\sqrt{\pi}}{2} z_2, & z_1^2, & \frac{\sqrt{\pi}}{2} z_1 z_2, & z_1^3, & \frac{\sqrt{\pi}}{2} z_1^2 z_2, & z_1^4, & \frac{\sqrt{\pi}}{2} z_1^3 z_2, & z_1^5, & \dots \end{array}$$

- We can show that
$$\left(\frac{\sqrt{\pi}}{2}\right)^{n/l(n)} d_n^{cm}(K) = d_{n,\mu}^{bb}(K)$$
Recalling that $l(n)$ is the sum of the degree of the monomials of at most degree n, the power can be simplified to
$$\frac{n}{l(n)} = \frac{n}{n(n+1)} = \frac{1}{n+1}.$$
This approaches 0 as $n \to \infty$ meaning the scale factor tends to 1 in the limit which gives the desired equality.

Generalisations

Different bases?

- Method generalises to two bases $\mathcal{B}$ and $\mathcal{C}$ if

$$\mathcal{B} = \bigcup_{1 \le i \le k} \{f_i(z)p(z) : p \in \mathcal{M}\} \qquad \mathcal{C} = \bigcup_{1 \le i \le k} \{g_i(z)p(z) : p \in \mathcal{M}\}$$

  where the f_i's and g_i's are elements in $\mathbb{C}[\mathcal{V}]$ and $\mathcal{M}$ is a countable subset of $\mathbb{C}[\mathcal{V}]$.
- Usually $\mathcal{M}$ is taken to be $\mathbb{C}[x]$ while the finite elements are linear combinations of elements in $\mathbb{C}[\mathcal{V}]$ which 'span' the y^{α_i} elements.

Generalisations

Can the normalisation root of $l(n)$ be replaced?

- Only by roots proportional to $l(n)$ e.g. $nm(n)$ where $m(n)$ is the number of monomials of degree at most n. We can calculate

$$\lim_{n\to\infty} \frac{nm(n)}{l(n)} = \frac{M+1}{M}.$$

Where M is the dimension of the variety. So transfinite diameters using a $nm(n)$ root are $M/M+1$ times smaller than those using a $l(n)$ root.

- The work of Cox-Ma‘u is motivated by classical methods from the $\mathbb{C}^N$ case which is advantageous from the point of view of studying analogues of things like Chebyshev constants.
- The work of Berman-Boucksom is rich and far reaching, but is inconvenient for studying something like Chebyshev constants on an algebraic variety.
- The equality of transfinite diameters allows the Cox-Ma‘u theory to tap into the results of Berman-Boucksom and obtain analogues of classical results on an algebraic variety.
- As an illustration we’ll look at the convergence of Fekete polynomials.

Applications

Fekete Polynomials

Let $K \subset \mathcal{V}$ be a compact set and let $\mathcal{M}[\mathcal{V}]$ be the reduced monomials from $\mathbb{C}[\mathcal{V}]$. We define the *n*th Fekete polynomial to be

$$F_n(z) = \frac{VDM_{\mathcal{M}}(\zeta_1, \ldots, \zeta_n, z)}{VDM_{\mathcal{M}}(\zeta_1, \ldots, \zeta_n)}$$

where $VDM_{\mathcal{M}}$ is a Vandermonde determinant over the given points over the first n elements of the reduced monomial basis $\mathcal{M}[\mathcal{V}]$ and $(\zeta_1, \ldots, \zeta_n)$ are an n-Fekete set (points where the Vandermonde determinant attains its maximum).

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Bloom (2001) showed that when $K \subset \mathbb{C}^N$ is compact, polynomially convex and regular then the regularised limit $\left[\limsup_{n\to\infty} \frac{1}{\deg(F_n)} \log|F_n(z)|\right]^*$ converges to the logarithmic extremal function $V_K(z)$.

$$V_K(z) = \sup\left\{\frac{1}{\deg p} \log|p(z)| : p \in \mathbb{C}[z], \|p\|_K \leq 0\right\}^*$$

Theorem

If $K \subset \mathcal{V}$ is compact, polynomially convex and regular then

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Main idea: Study the log-asymptotics of each function. These induce functions ρ_K and ρ_F defined on the hyperplane at ∞. One can show that the sub-zero sets of these functions have the same transfinite diameter.

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Lemma

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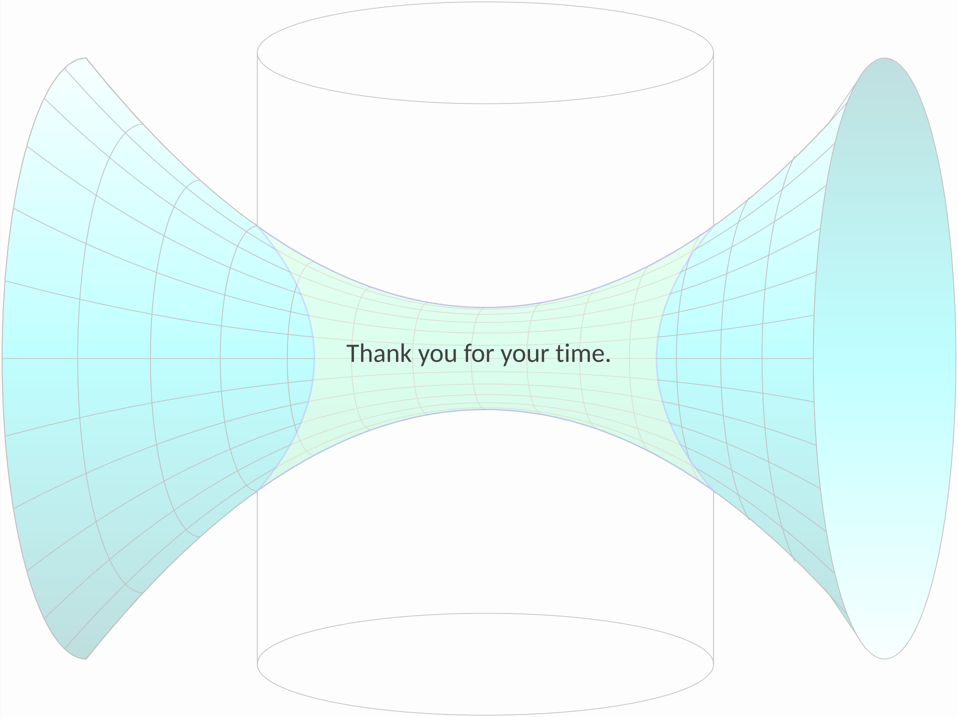
Conclusion: The log-asymptotics of the RHS can only arise from an extremal function, with a little more work, the only possibility is V_K.


Thank you for your time.