Some restrictions on the characteristic polynomial of a Seidel matrix and equiangular lines in $\mathbb{R}^{17}$

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Plan

- Equiangular line systems and Seidel matrices
- Systems almost achieving the relative bound
- Restricting the coefficients of the characteristic polynomial
- An application to dimension 17
- Concluding remarks

Equiangular line systems

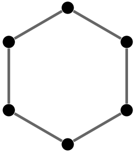
- Let $\mathcal{L}$ be a system of n lines spanned by $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^d$ with $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$.
- $\mathcal{L}$ is **equiangular** if $|\langle \mathbf{v}_i, \mathbf{v}_j \rangle| = \alpha$; ("*common angle* α").
- **Problem:** given d, what is the largest possible size $N(d)$ of an equiangular line system in $\mathbb{R}^d$?

Example

- An orthonormal basis: $n = d$ and $\alpha = 0$.
- $N(d) \geqslant d$.

Examples

3 lines in $\mathbb{R}^2$:



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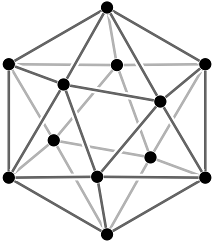

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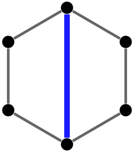

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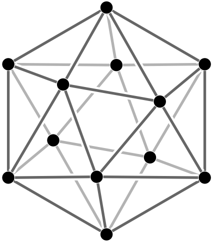

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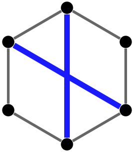

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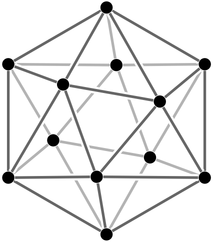

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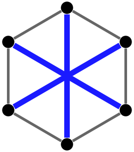

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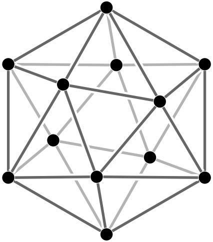

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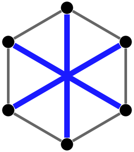

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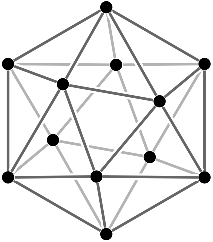

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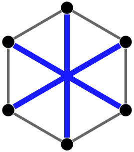

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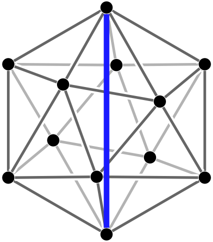

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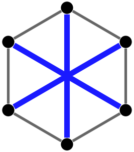

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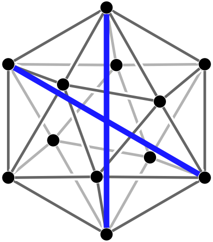

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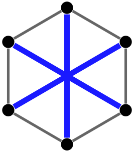

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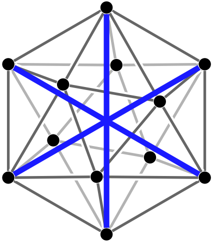

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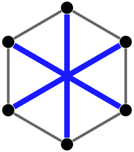

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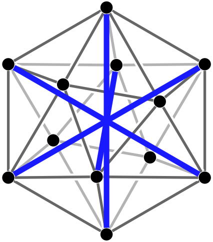

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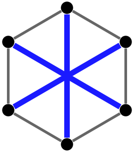

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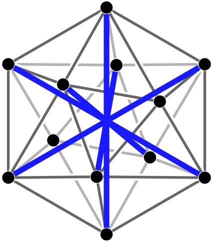

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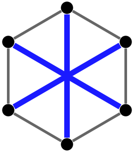

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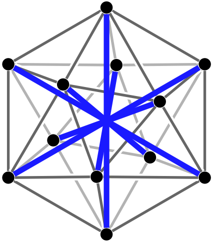

Bounds for small dimensions

Below is a table with upper and lower bounds for $N(d)$ for $d \leqslant 20$.

d	2	3	4	5	6	7 – 13	14	15	16	17	18	19	20
$N(d)$	3	6	6	10	16	28	28 29	36	40 41	48 49	54 60	72 75	90 95

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Seidel matrices

Equiangular lines $l_1, \ldots, l_n$ — common angle $\alpha > 0$

$\Updownarrow$

Unit spanning vectors $\mathbf{v}_i \ : \ l_i = \langle \mathbf{v}_i \rangle$ — $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \pm\alpha$

$\Updownarrow$

Gram matrix $M = (\langle \mathbf{v}_i, \mathbf{v}_j \rangle)_{ij}$ — $\begin{pmatrix} 1 & \pm\alpha & \pm\alpha \\ \pm\alpha & 1 & \pm\alpha \\ \pm\alpha & \pm\alpha & 1 \end{pmatrix}$

$\Updownarrow$

Seidel matrix $S = \dfrac{(M - I)}{\alpha}$ — $\begin{pmatrix} 0 & \pm1 & \pm1 \\ \pm1 & 0 & \pm1 \\ \pm1 & \pm1 & 0 \end{pmatrix}$

Multiplicity of the smallest eigenvalue

Unit vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathbb{R}^d$

n vectors

$\Updownarrow$

$$B = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix}$$

rank $= d$

$\Updownarrow$

Gram matrix $M = B^\top B$

smallest eigenvalue $[0]^{n-d}$

$\Updownarrow$

Seidel matrix $S = \dfrac{(M - I)}{\alpha}$

smallest eigenvalue $\left[\dfrac{-1}{\alpha}\right]^{n-d}$

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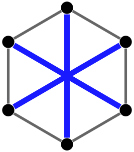

$$S = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Spectrum: $\{[-2]^1, [1]^2\}$;

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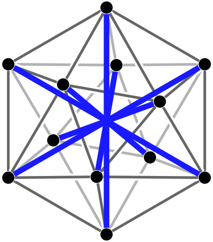

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Spectrum: $\{[-\sqrt{5}]^3, [\sqrt{5}]^3\}$.

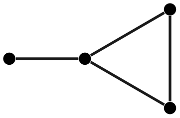
Adjacency matrices for graphs

- Start with a Seidel matrix, e.g., $S = \begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$

- Then $A = (J - I - S)/2$ is a **graph-adjacency** matrix:
$$A = \frac{1}{2}\left(\begin{pmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{pmatrix} - \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix}\right)$$

- So the Seidel matrix S corresponds to the following graph

$$A = \begin{pmatrix} 0&1&0&0 \\ 1&0&1&1 \\ 0&1&0&1 \\ 0&1&1&0 \end{pmatrix}$$



- Or equivalently $S = J - I - 2A$.

Line systems almost achieving the relative bound

The relative bound

Theorem (Relative bound)

Let $\mathcal{L}$ be an equiangular line system of n lines in $\mathbb{R}^d$ whose Seidel matrix has smallest eigenvalue λ_0 and suppose $\lambda_0^2 \geqslant d+2$.

$$n \leqslant \frac{d(\lambda_0^2 - 1)}{\lambda_0^2 - d}.$$

Equality implies that S has 2 distinct eigenvalues.

- In the case of equality, S has spectrum

$$\left\{ [\lambda_0]^{n-d}, \left[\frac{(d-n)\lambda_0}{d} \right]^d \right\}.$$

Relative bound in low dimensions

GG, Koolen, Munemasa, Szöllősi (2016): "Spectrum is determined for systems close to the relative bound"

d	λ_0	$\frac{d(\lambda_0^2-1)}{\lambda_0^2-d}$	$\left\lfloor \frac{d(\lambda_0^2-1)}{\lambda_0^2-d} \right\rfloor$	Spectrum
14	-5	≈ 30	30	$\{[-5]^{16}, [5]^9, [7]^5\}$
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Equiangular lines in $\mathbb{R}^{14}$

- Suppose there is $n > 2 \cdot 14$ equiangular lines in $\mathbb{R}^{14}$.
- Lemmens and Seidel (1973): $\implies \lambda_0 = -5$.
- Relative bound: $n \leqslant 30.54\cdots \notin \mathbb{N}$.
- Suppose we have $n = 30$ $(d = 14)$, with corresponding Seidel matrix S having eigenvalues

$$-5 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{14}.$$

Equiangular lines in $\mathbb{R}^{14}$

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- Suppose we have $n = 30$ ($d = 14$), with corresponding Seidel matrix S having eigenvalues

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- Observe that

$$\operatorname{tr} S = (n-d)\lambda_0 + \sum_{i=1}^{d} \lambda_i = 0;$$

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Hence $(\lambda_i - 6) \in \{\pm 1\}$.

Case study: equiangular lines in $\mathbb{R}^{17}$

- Suppose there is $n > 2 \cdot 17$ equiangular lines in $\mathbb{R}^{17}$.
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Note $(\lambda_i - 10)^2$ are +ve algebraic integers with sum 25

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- Good News: a similar computation has been done by McKee and Smyth (2005).
- Bad News: there are hundreds of candidate polynomials. (E.g., there are 686 irreducible, totally positive, monic, integer polynomials of degree 9 and trace 17.)

A modular characterisation of the characteristic polynomial of a Seidel matrix

Characteristic polynomial modulo 2^k

Let $S = J - I - 2A$ be a Seidel matrix of order n **even**.

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Characteristic polynomial modulo 2^k

Let $S = J - I - 2A$ be a Seidel matrix of order n **odd**.

- Haemers: $\chi_S(x) \equiv x(x-1)^{n-1} \mod 2\mathbb{Z}[x]$.

- GG, Koolen, Munemasa, Szöllősi (2016):
  $\det S \equiv n - 1 \mod 4$.

- GG and Yatsyna (2018+):

$$\chi_S(x) \equiv \chi_{J-I}(x) \mod 4\mathbb{Z}[x].$$

  - for $\chi_S(x)$ modulo $8\mathbb{Z}[x]$: $\leqslant 2$ possibilities;
  - for $\chi_S(x)$ modulo $16\mathbb{Z}[x]$: $\leqslant 4$ possibilities;
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- Instead of $S = J - I - 2A$, consider $T = J - 2A$.
- We have $\chi_{J-2A}(x) = \chi_{-2A}(x) - \mathbf{1}^\top \operatorname{adj}(xI + 2A)\mathbf{1}$.
- Further, $\operatorname{adj}(xI + 2A) = \sum_{i=0}^{n-1} (-2A)^{n-1-i} \sum_{j=0}^{i} x^{i-j} c_j$, where $\chi_{-2A}(x) = \sum_{i=0}^{n} c_i x^{n-i}$.

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Write $\chi_{J-2A}(x) = \sum_{i=0}^{n} a_i x^{n-i}$ *and* $\chi_A(x) = \sum_{i=0}^{n} b_i x^{n-i}$. *Then*

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Write $\chi_{J-2A}(x) = \sum_{i=0}^{n} a_i x^{n-i}$ *with* n ***even***. *Then* 2^r *divides* a_r.

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- $\chi_{J-2A}(x) \equiv x^n - nx^{n-1} + a_3 x^{n-3} + a_4 x^{n-4} \mod 32\mathbb{Z}[x]$

A note for the case when n is odd

Lemma

Write $\chi_{J-2A}(x) = \sum_{i=0}^{n} a_i x^{n-i}$ *and* $\chi_A(x) = \sum_{i=0}^{n} b_i x^{n-i}$*. Then*

$$a_r = (-2)^r \left(b_r + \frac{1}{2} \sum_{i=1}^{r} b_{r-i} \mathbf{1}^\top A^{i-1} \mathbf{1} \right).$$

- Observe that 2^{r-1} divides a_r for r odd; and 2^r divides a_r for r even.

Lemma (key lemma)

For $l \geqslant 2$*, we have*

$$\sum_{d \mid 2l} \varphi(2l/d) \operatorname{tr}(A^d) + l \mathbf{1}^\top A^l \mathbf{1} \equiv 0 \pmod{4l}.$$

Combining the two restrictions

Back to 50 lines in $\mathbb{R}^{17}$

- Suppose we have $n = 50$ ($d = 17$), with corresponding Seidel matrix S having eigenvalues

$$-5 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{17}.$$

Using Haemers' observation, we find that:

- $\chi_S(x) = (x+5)^{33} \prod_{i=1}^{17}(x - \lambda_i) \equiv (x+1)^{50} \mod 2\mathbb{Z}[x]$

- $F(x) = \prod_{i=1}^{17}(x - (\lambda_i - 10)^2) \equiv (x+1)^{17} \mod 2\mathbb{Z}[x]$

- So now we want to compute all totally positive, monic, integer polynomials with trace $= 25$ and degree $= 17$ congruent to $(x+1)^{17}$ modulo $2\mathbb{Z}$.

Finding all such totally positive algebraic integers

- There are 55 totally positive, monic, integer polynomials with trace $= 25$ and degree $= 17$ congruent to $(x+1)^{17}$ modulo $2\mathbb{Z}[x]$.
- Now convert each polynomial into putative characteristic polynomials for a Seidel matrix.
- Only two of these polynomials are congruent to $\chi_{J-I}(x)$ modulo $8\mathbb{Z}[x]$:

$$(x+5)^{33}(x-7)(x-9)^9(x-11)^7$$
$$(x+5)^{33}(x-9)^{12}(x-11)^4(x-13)$$

Now we have our targets

To show that there does not exist 50 lines in $\mathbb{R}^{17}$, show that there does not exist a Seidel matrix with characteristic polynomial

$$(x+5)^{33}(x-7)(x-9)^9(x-11)^7 \text{ or}$$
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Theorem (GG and Yatsyna)

There does not exist a Seidel matrix with characteristic polynomial

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What about other dimensions?

d	2	3	4	5	6	7 – 13	14	15	16	17	18	19	20
$N(d)$	3	6	6	10	16	28	28 29	36	40 41	48 49	54 60	72 75	90 95

What we did:

1. Suppose we have $n = 50$ ($d = 17$), with corresponding Seidel matrix S having eigenvalues

$$-5 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_d;$$

2. List all candidates for $\{(\lambda_i - 10)^2\}$ (degree 17, trace 25);
3. Produce all corresponding candidate char polys for S;
4. Only two of these satisfy modulo $8\mathbb{Z}[x]$ condition;
5. No Seidel matrix has either of the two char polys.

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For dimension 18:

1. Suppose we have $n = 60$ ($d = 18$), with corresponding Seidel matrix S having eigenvalues

$$-5 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_d;$$

2. List all candidates for $\{(\lambda_i - 12)^2\}$ (degree 18, trace 42);

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2. List all candidates for $\{(\lambda_i - 12)^2\}$ (degree 18, trace 42); this is currently too computationally difficult.
3. But we can list those with integer roots:

$$(x-13)^9(x-11)^6(x-9)^3(x+5)^{42}$$
$$(x-15)(x-13)^6(x-11)^9(x-9)^2(x+5)^{42}$$
$$(x-15)^2(x-13)^3(x-11)^{12}(x-9)(x+5)^{42}$$
$$(x-15)^3(x-11)^{15}(x+5)^{42}$$

Thanks for listening!