Tight frames and Approximation

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Symmetries of Weyl-Heisenberg SIC-POVMs

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joint work in progress with
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Tomography of Quantum States

General Problem:

What is the best way to identify an arbitrary unknown quantum state ρ in a d-dimensional Hilbert space?

- ρ is a Hermitian matrix
  $\Longrightarrow d^2 - 1$ real parameters
- one von Neumann measurement provides $d - 1$ independent parameters
  $\Longrightarrow$ at least $d + 1$ different (projective) measurements
- general measurements (POVMs)
  $\Longrightarrow$ at least d^2 POVM elements
- goal:
  "maximal independence" of the measurement results
  $\Longrightarrow$ optimal statistics with no *a priori* knowledge for a non-adaptive scheme

SIC-POVMs: Equiangular Lines in Complex Space

The General Problem

Find m normalized vectors $\{\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(m)}\} \subset \mathbb{C}^d$ such that the modulus of the inner product between any pair of vectors is constant, i. e.

$$|\langle \boldsymbol{v}^{(j)} | \boldsymbol{v}^{(k)} \rangle|^2 = \left| \sum_{\ell=1}^{d} \overline{v_\ell^{(j)}} v_\ell^{(k)} \right|^2 = \begin{cases} 1 & \text{for } j = k, \\ c & \text{for } j \neq k \end{cases}$$

Special Case: SIC-POVMs

Find d^2 normalized vectors $\{\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(d^2)}\} \subset \mathbb{C}^d$ such that the modulus of the inner product between any pair of vectors is constant, i. e.

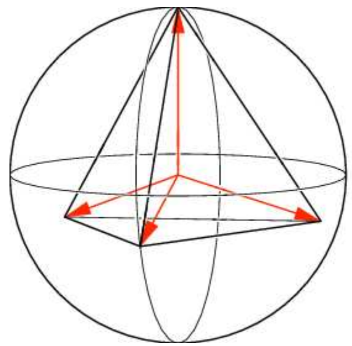
$$|\langle \boldsymbol{v}^{(j)} | \boldsymbol{v}^{(k)} \rangle|^2 = \begin{cases} 1 & \text{for } j = k, \\ 1/(d+1) & \text{for } j \neq k \end{cases}$$

Quantum Information: SIC-POVMs

- generalized quantum measurement (POVM) with d^2 rank-one elements $E_j = \Pi_j/d$ with $\Pi_j = |\boldsymbol{v}^{(j)}\rangle\langle\boldsymbol{v}^{(j)}|$
- The d^2 elements form a basis of $\mathbb{C}^{d\times d}$.
  $\Longrightarrow$ "informationally complete", i.e., reconstruction of a quantum state ρ is possible
- expectation values $p_j = \mathrm{tr}(\rho E_j)$ "maximally independent":

$$\mathrm{tr}\left(\Pi_j \Pi_k\right) = |\langle\boldsymbol{v}^{(j)}|\boldsymbol{v}^{(k)}\rangle|^2 = \frac{1}{d+1} \qquad \text{for } j \neq k,$$

  $\Longrightarrow$ "symmetric"



- applications in quantum cryptography as well

Related Problems

Complex Spherical 2-Designs

The integral of any degree-two polynomial over the complex sphere in $\mathbb{C}^d$ can be computed as finite average, i. e.

$$\frac{1}{\mu(\mathbb{C}S^{d-1})}\int_{g\in\mathbb{C}S^{d-1}} f(g)d\mu(g) = \frac{1}{m}\sum_{j=1}^{m} f(\boldsymbol{v}^{(j)})$$

if $m = d^2$ and the vectors $\boldsymbol{v}^{(i)}$ are equiangular lines.

Banach Spaces [König & Tomczak-Jaegermann 94]

The projection constant

$$\lambda(E) = \sup_{X\supseteq E}\inf_{P}\{\|P\| : P\colon X \to E \text{ is linear projection onto } E\}$$

of a complex d-dimensional normed space E is maximal iff a set of d^2 equiangular lines exists.

Ansatz: System of Polynomial Equations

use $2d$ real variables per vector

$$\boldsymbol{v}^{(j)} = \big(a_1^{(j)} + ib_1^{(j)}, \ldots, a_d^{(j)} + ib_d^{(j)}\big), \qquad |\langle \boldsymbol{v}^{(j)}|\boldsymbol{v}^{(k)}\rangle|^2 = \frac{1 + d\delta_{jk}}{1+d}$$

where $i^2 = -1$.

$d = 2$, $m = d^2 = 4$

$$\begin{aligned}
\boldsymbol{v}^{(1)} &= \big(a_1^{(1)} + ib_1^{(1)}, a_2^{(1)} + ib_2^{(1)}\big)\\
\boldsymbol{v}^{(2)} &= \big(a_1^{(2)} + ib_1^{(2)}, a_2^{(2)} + ib_2^{(2)}\big)\\
\boldsymbol{v}^{(3)} &= \big(a_1^{(3)} + ib_1^{(3)}, a_2^{(3)} + ib_2^{(3)}\big)\\
\boldsymbol{v}^{(4)} &= \big(a_1^{(4)} + ib_1^{(4)}, a_2^{(4)} + ib_2^{(4)}\big)
\end{aligned}$$

already rather complicated to solve for $d = 3$ and $m > 4$

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Symmetries of SIC-POVMs

SIC-POVM as set of rank-one projection operators

$$\mathcal{S} = \{P_1, \ldots, P_{d^2}\} \quad \text{where } P_i^2 = P_i,\ P_i = P_i^\dagger,\ \mathrm{tr}(P_i) = 1$$

unitary symmetry U acts on $\mathcal{S}$:

$$U P_i U^\dagger = P_{\pi(i)}$$

- permutation representation of the symmetry group $A(\mathcal{S})$

$$A(\mathcal{S}) \to S_{d^2}, U \mapsto \pi(U)$$

- for SIC-POVMs, the kernel corresponds to global phases
  $\Rightarrow$ projective representation of the permutation group

NB: $A(\mathcal{S})$ can be computed from $(T_{ij}) = \mathrm{tr}(P_1 P_i P_j)$

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Special Symmetries of SIC-POVMs

For $U \in A(\mathcal{S})$, the number $f(U)$ of fixed points i, i. e. $U P_i U^\dagger = P_i$ is given by

$$f(U) = |\operatorname{tr}(U)|^2.$$

[Zauner 99, Satz 2.34]

- **transitive symmetry group:**
  The SIC-POVM is a single orbit under $A(\mathcal{S})$, i. e. $P_i = U_i P_1 U_i^\dagger$.
- **regular symmetry (sub)group:**
  Up to phases, there is a unique element U_i with $P_i = U_i P_1 U_i^\dagger$.

candidates for regular symmetry groups are nice unitary error bases (UEBs)
[Klappenecker & Rötteler, quant-ph/0010082]

Weyl-Heisenberg Group

- generators: $$H_d := \langle X, Z\rangle$$

  where $X := \sum_{j=0}^{d-1} |j+1\rangle\langle j|$ and $Z := \sum_{j=0}^{d-1} \omega_d^j |j\rangle\langle j|$

  $$(\omega_d := \exp(2\pi i/d))$$

- relations:

$$\left(\omega_d^c X^a Z^b\right)\left(\omega_d^{c'} X^{a'} Z^{b'}\right) = \omega_d^{a'b-b'a}\left(\omega_d^{c'} X^{a'} Z^{b'}\right)\left(\omega_d^c X^a Z^b\right)$$

- basis:

$$H_d/\zeta(H_d) = \left\{X^a Z^b \colon a, b \in \{0,\ldots,d-1\}\right\} \cong \mathbb{Z}_d \times \mathbb{Z}_d$$

  trace-orthogonal basis of all $d \times d$ matrices

Constructing SIC-POVMs

Ansatz 1:

SIC-POVM that is the orbit under H_d, i. e.,

$$|\boldsymbol{v}^{(a,b)}\rangle := X^a Z^b |\boldsymbol{v}^{(0)}\rangle$$

$$|\langle \boldsymbol{v}^{(a,b)}|\boldsymbol{v}^{(a',b')}\rangle|^2 = \begin{cases} 1 & \text{for } (a,b) = (a',b'), \\ 1/(d+1) & \text{for } (a,b) \neq (a',b') \end{cases}$$

$$|\boldsymbol{v}^{(0)}\rangle = \sum_{j=0}^{d-1} (x_{2j} + i x_{2j+1}) |j\rangle,$$

($x_0, \ldots, x_{2d-1}$ are real variables, $x_1 = 0$)

$\Longrightarrow$ polynomial equations for $2d-1$ variables, but already quite complicated for $d = 6$

Jacobi Group (or Clifford Group)

- automorphism group of the Heisenberg group H_d, i. e.

$$\forall T \in J_d : T^\dagger H_d T = H_d$$

- the action of J_d on H_d modulo phases corresponds to the symplectic group $SL(2, \mathbb{Z}_d)$, i. e.

$$T^\dagger X^a Z^b T = \omega_d^c X^{a'} Z^{b'} \qquad \text{where } \begin{pmatrix} a' \\ b' \end{pmatrix} = \tilde{T} \begin{pmatrix} a \\ b \end{pmatrix}, \tilde{T} \in SL(2, \mathbb{Z}_d)$$

$\Longrightarrow$ homomorphism $J_d \to SL(2, \mathbb{Z}_d)$

- additionally: complex conjugation

$$X^a Z^b \mapsto X^a Z^{-b} \qquad \text{corresponding to } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Constructing SIC-POVMs (cntd.)

Ansatz 2:

SIC-POVM that is the orbit under H_d,

additionally:

$|\boldsymbol{v}^{(0)}\rangle$ lies in a (degenerate) ℓ-dimensional eigenspace of some $T \in J_d$

$$|\boldsymbol{v}^{(0)}\rangle = \sum_{j=0}^{\ell-1}(x_{2j} + ix_{2j+1})|b_j\rangle,$$

where $|b_j\rangle$, $j = 1, \ldots, \ell$ is the basis of that eigenspace

$\Longrightarrow$ reduced number of variables

$\Longrightarrow$ better chances to compute algebraic solutions

additionally: choose a "good" basis such that e.g. T resp. $|\boldsymbol{v}^{(0)}\rangle$ will be sparse

Fibonacci-Lucas SIC-POVMs

[Markus Grassl & Andrew J. Scott arXiv:1707.02944]

- (exact) symmetry analysis of a numerical solution for $d = 124$
  $\Longrightarrow$ symmetry group of order 30 (prescribed order 6)
- identified as part of a series of dimensions (related to Lucas numbers)
  $d = 4, 8, 19, 48, 124, 323, 844, 2208, 5779, 15128$
- symmetry group of order $6k$ related to Fibonacci numbers
- new exact solutions for $d = 124$ and $d = 323$ (previously $d = 48$)
- new numerical solution for $d = 844$ with 150 digits (previously $d = 323$)

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Fibonacci-Lucas SIC-POVMs

[Markus Grassl & Andrew J. Scott arXiv:1707.02944]

- Fibonacci numbers F_k with $F_0 = 0$, $F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$

$$F_k = \frac{\varphi^k - (-\varphi)^{-k}}{\sqrt{5}}, \qquad \varphi = \frac{1+\sqrt{5}}{2}$$

- Lucas numbers L_k with $L_0 = 2$, $L_1 = 1$, $L_{k+1} = L_k + L_{k-1}$

$$L_k = \varphi^k + (-\varphi)^{-k}$$

- prescribed anti-unitary symmetry related to the Fibonacci matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad A^k = \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix}$$

Fibonacci-Lucas SIC-POVMs

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$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad A^k = \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix}$$

- modulo $d_k = L_{2k} + 1$, the matrix A has order $6k$
- sequence of dimensions $d = 4, 8, 19, 48, 124, 323, 844, 2208, 5779, 15128$
- squarefree part D of $(d+1)(d-3)$ is always $D = 5$
  $\Longrightarrow$ ray class field over $\mathbb{Q}(\sqrt{5})$

Generalised Fibonacci-Lucas SIC-POVMs

- generalised Fibonacci numbers $F_{m,k}$ with
  $F_{m,0} = 0$, $F_{m,1} = 1$, $F_{m,k+1} = mF_{m,k} + F_{m,k-1}$
- generalised Lucas numbers $L_{m,k}$ with
  $L_{m,0} = 2$, $L_{m,1} = m$, $L_{m,k+1} = mL_{m,k} + L_{m,k-1}$
- prescribed anti-unitary symmetry related to the matrix

$$A_m = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix} \qquad A_m^k = \begin{pmatrix} F_{m,k-1} & F_{m,k} \\ F_{m,k} & F_{m,k+1} \end{pmatrix}$$

- modulo $d_{m,k} = L_{m,2k} + 1$, the matrix A has order $6k$
- squarefree part D of $(d+1)(d-3)$ equals the squarefree part of $m^2 + 4$
  $\Longrightarrow$ ray class field over $\mathbb{Q}(\sqrt{D})$

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Anti-Unitary Symmetries

	k	1	2	3	4	5	6	7	8
	ord(F)	6	12	18	24	30	36	42	48
m	D	$F_{e'}$	F_g						
1	5	4	8	19	48	124	323	844	2208
2	2	7	35	199	1155	6727	39203	228487	1331715
3	13	12	120	1299	14160	154452	1684803	18378372	200477280
4	5	19	323	5779	103683	1860499			
5	29	28	728	19603	528528	14250628			
6	10	39	1443	54759	2079363	78960999			
7	53	52	2600	132499	6754800				
8	17	67	4355	287299	18957315				
9	85	84	6888	571539	47430768				
10	26	103	10403	1060903					
11	5	124	15128	1860499					
12	37	147	21315	3111699					
13	173	172	29240	4999699					
14	2	199	39203	7761799					
15	229	228	51528						
16	65	259	66563						
17	293	292	84680						
18	82	327	106275						
19	365	364	131768						
20	101	403	161603						

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Families of SIC-POVMs with Unitary Symmetry

- prescribed unitary symmetry related to the matrix

$$B_m = \begin{pmatrix} 0 & 1 \\ -1 & m \end{pmatrix}$$

- similar recurrence relations for the entries of B_m^k and the corresponding dimension
- order of the symmetry is $3k$

Unitary Symmetries

	k	1	2	3	4	5	6	7	8	9
	ord(F)	3	6	9	12	15	18	21	24	27
m	D	F_z	F_b	F_d						
3	5	*4*	*8*	*19*	*48*	124	323	844	2208	5779
4	3	5	15	53	195	725	2703	10085	37635	140453
5	21	6	24	111	528	2526	12099	57966	277728	1330671
6	2	7	35	199	1155	6727	39203	228487	1331715	7761799
7	5	8	48	323	2208	15128	103683	710648	4870848	33385283
8	15	9	63	489	3843	30249	238143	1874889	14760963	116212809
9	77	10	80	703	6240	55450	492803	4379770	38925120	345946303
10	6	11	99	971	9603	95051	940899	9313931	92198403	912670091
11	13	12	120	1299	14160	154452				
12	35	13	143	1693	20163	240253				
13	165	14	168	2159	27888	360374				
14	3	15	195	2703	37635	524175				
15	221	16	224	3331	49728	742576				
16	7	17	255	4049	64515					
17	285	18	288	4863	82368					
18	5	19	323	5779	103683					
19	357	20	360	6803	128880					
20	11	21	399	7941	158403					

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Symmetries and Ray Class Fields

[Appleby, Chien, Flammia & Waldron arXiv:1703.05981]

Ray class field conjecture

nested tower of fields (for the minimal field)

$$\mathbb{Q} \lhd \mathbb{E}_c = \mathbb{Q}(\sqrt{D}) \lhd \mathbb{E}_0 \lhd \mathbb{E}_1 \lhd \mathbb{E} = \mathbb{E}_1(i\sqrt{d'}).$$

- $\mathbb{E}$ is the ray class field over $\mathbb{Q}(\sqrt{D})$ with conductor d' with ramification at both infinite places
- $\mathbb{E}_1$ is the ray class field with ramification only allowed at the infinite place taking $\sqrt{D}$ to a positive real number
- $\mathbb{E}_0$ is the Hilbert class field over $\mathbb{Q}(\sqrt{D})$, in particular $[\mathbb{E}_0 : \mathbb{Q}(\sqrt{D})]$ equals the class number of $\mathbb{Q}(\sqrt{D})$

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$$\mathbb{Q} \lhd \mathbb{E}_c = \mathbb{Q}(\sqrt{D}) \lhd \mathbb{E}_0 \lhd \mathbb{E}_1 \lhd \mathbb{E} = \mathbb{E}_1(i\sqrt{d'}).$$

- for $\mathcal{M}$ a certain maximal Abelian subgroup of $\mathrm{GL}(2, \mathbb{Z}/d'\mathbb{Z})$ and (essentially) the symmetry group $S(\Pi)$ of the SIC-POVM:

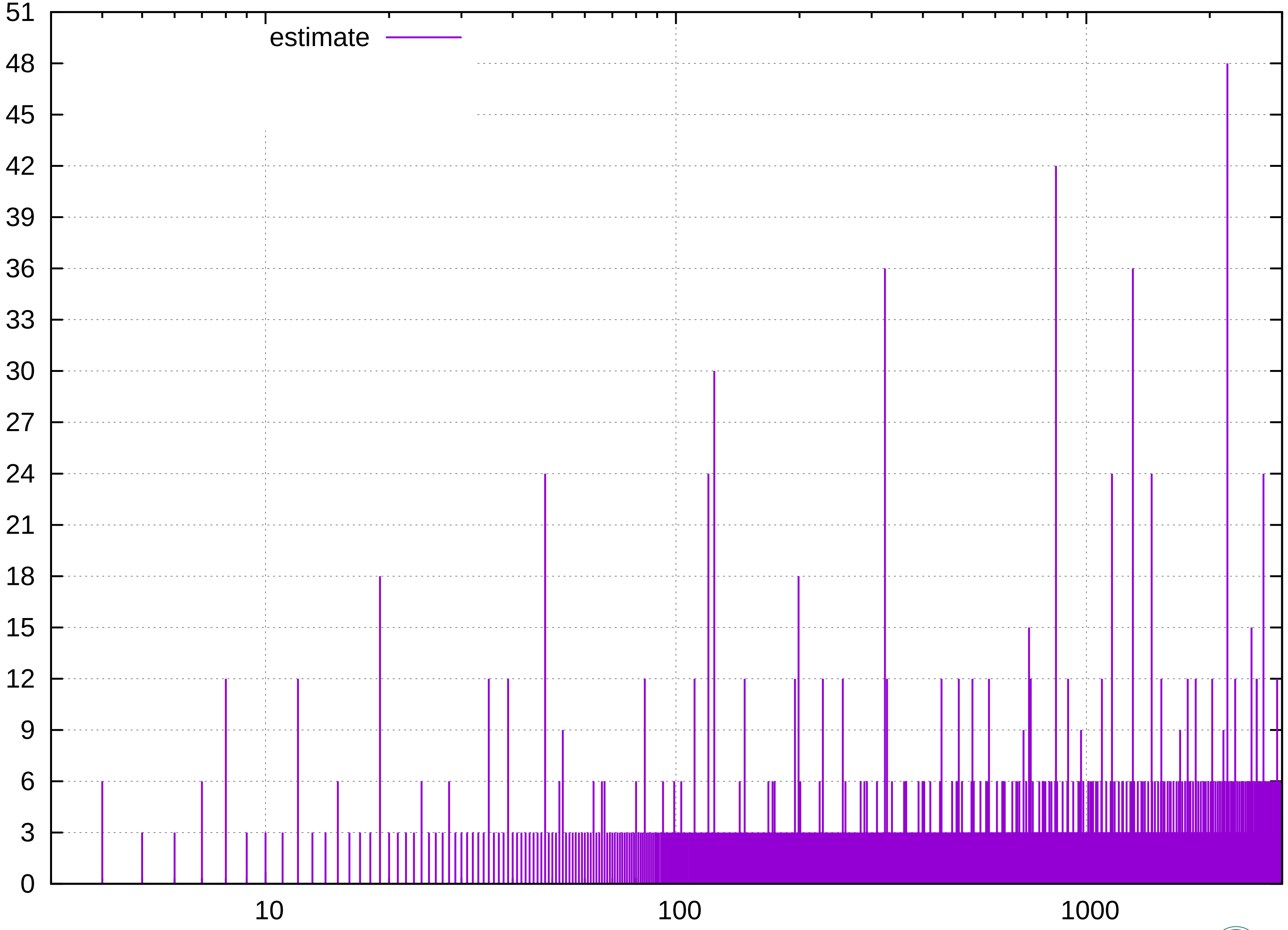
$$\mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0) \cong \mathcal{M}/S(\Pi)$$

- estimate for the group order:

$$|S(\Pi)| = \frac{|\mathcal{M}|}{|\mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_0)|} = \frac{|\mathcal{M}| \times |\mathrm{Gal}(\mathbb{E}_0/\mathbb{E}_c)|}{|\mathrm{Gal}(\mathbb{E}_1/\mathbb{E}_c)|}$$

- 1 or 4 cases for $|\mathcal{M}|$, but $|S(\Pi)|$ must be integral

Symmetries of SIC-POVMs



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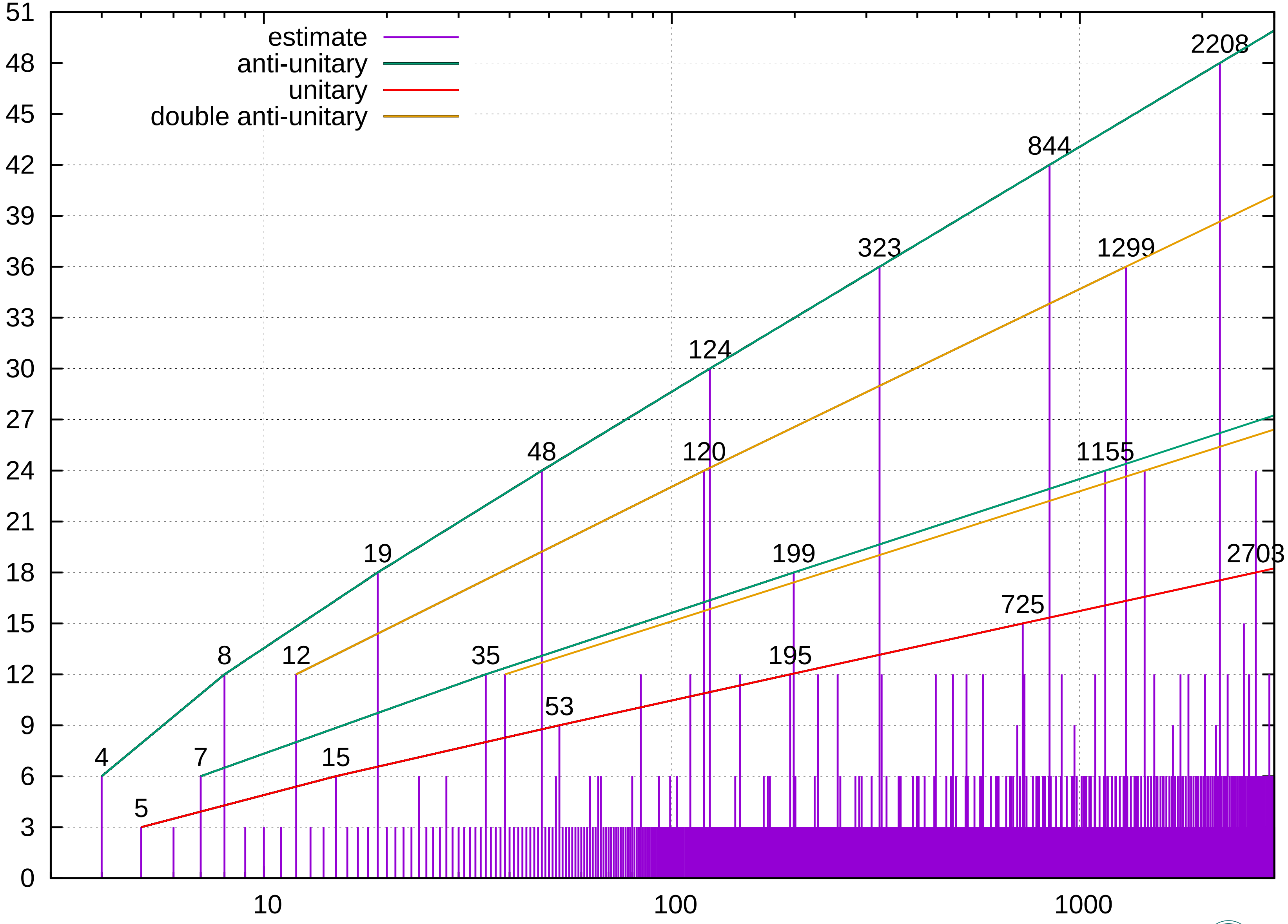

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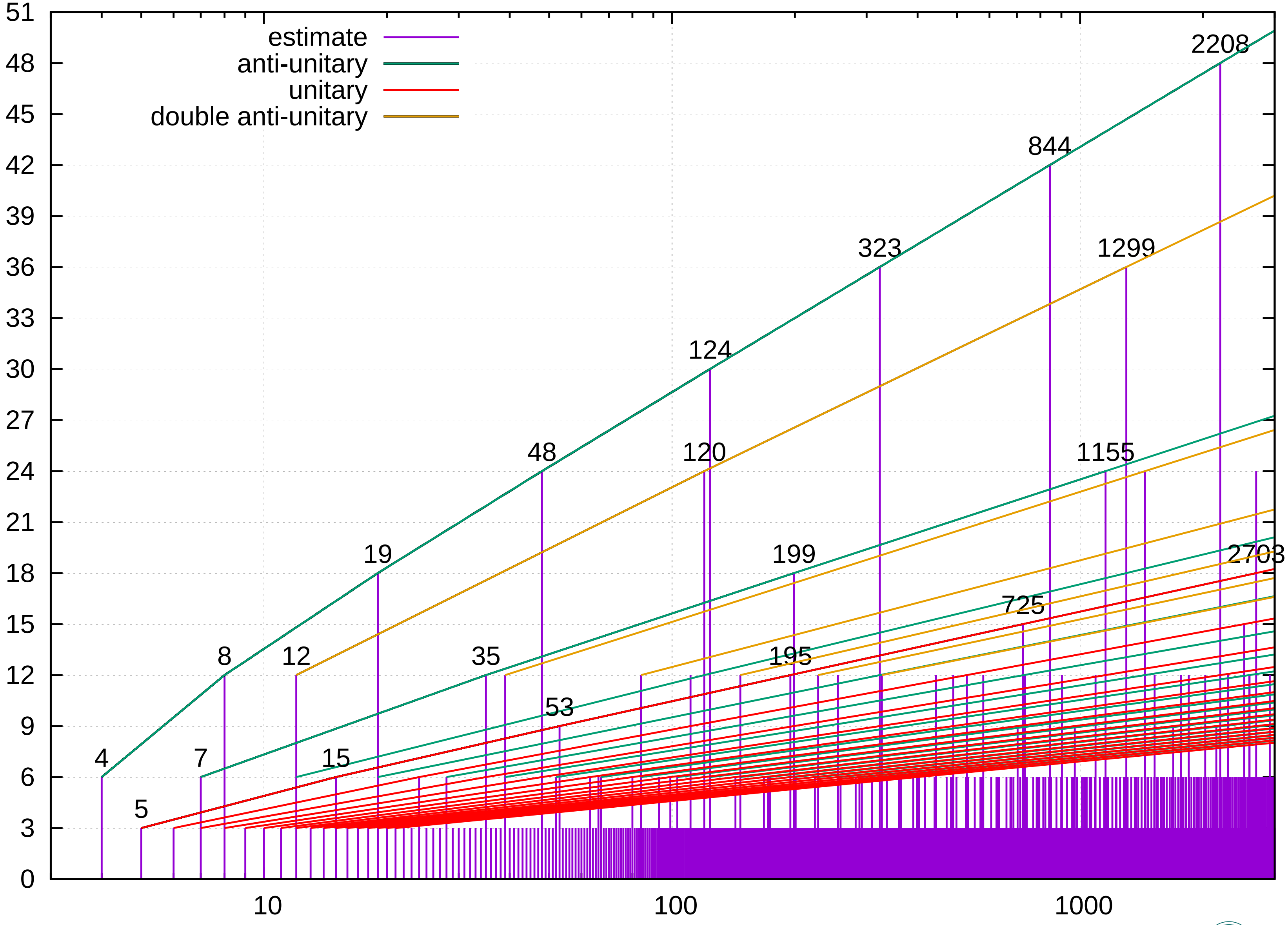

Symmetries of SIC-POVMs