Approximation by Polynomial Spaces

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Honeymoon Valley, February 22, 2018

Co-authors

Joint work with T. Bloom, N. Levenberg, S. Ma'u and F. Piazzon

Observation of Trefethen

- For a multivariate polynomial $p(x) = \sum_{\alpha} a_\alpha x^\alpha$
- Usual degree: $\deg(p) := \max_{a_\alpha \neq 0} \sum_{i=1}^{d} \alpha_i$
- $= \max_{a_\alpha \neq 0} \|\alpha\|_1$
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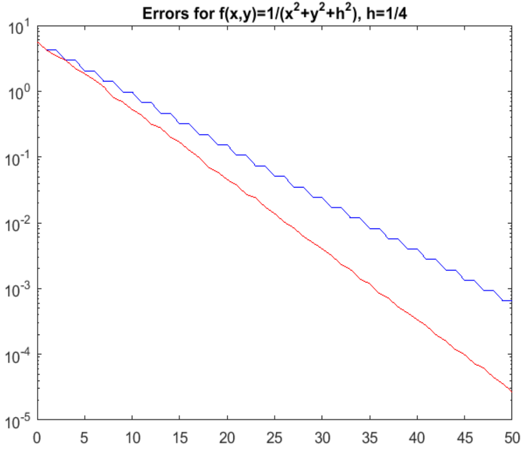
Errors in Approximation



but

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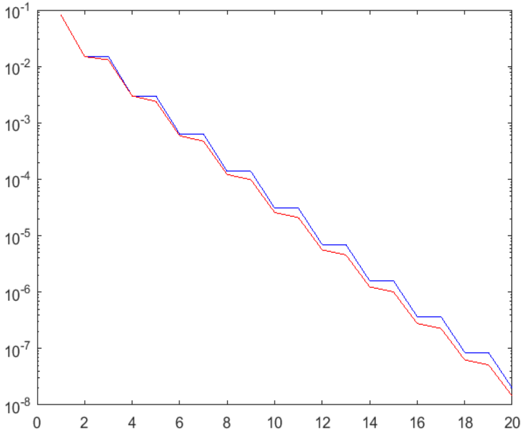
Error in Approximation to $f_1 = \frac{1}{1-z_1/2} + \frac{1}{1-z_2/2}$



Classical Bernstein-Walsh-Siciak

Theorem

Let $K \subset \mathbb{C}^d$ be compact and nonpluripolar with V_K continuous. Let $R > 1$, and let $\Omega_R := \{z : V_K(z) < \log R\}$. Let f be continuous on K. Then

$$\limsup_{n\to\infty} D_n(f, K)^{1/n} \leq 1/R$$

if and only if f is the restriction to K of a function holomorphic in Ω_R.

Siciak-Zaharjuta Extremal Function

- $$V_K(z) = \max[0, \sup\{\frac{1}{deg(p)} \log|p(z)| : ||p||_K := \max_{\zeta \in K} |p(\zeta)| \le 1\}]$$
- p is a nonconstant holomorphic polynomial
- $$D_n(f, K) := \inf\{||f - p_n||_K : p_n \in \mathcal{P}_n\}$$
- $\mathcal{P}_n$ is the space of holomorphic polynomials of degree at most n.

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Generalized Degree

- P a non-degenerate convex set
- e.g. $P_p := \{(x_1, ..., x_d) \in (\mathbb{R}^+)^d : (x_1^p + \cdots x_d^p)^{1/p} \le 1\}$
- For $p = 1$ we have $P_1 = \Sigma$ where

$$\Sigma := \{(x_1, ..., x_d) \in \mathbb{R}^d : x_1, ..., x_d \ge 0,\ x_1 + \cdots x_d \le 1\}.$$

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Polynomial Spaces and Generalized Extremal Function

- $$Poly(nP) := \{p(z) = \sum_{J \in nP \cap (\mathbb{Z}^+)^d} c_J z^J : c_J \in \mathbb{C}\}$$
- when $P = \Sigma$ we have $Poly(n\Sigma) = \mathcal{P}_n$, the usual space of holomorphic polynomials of degree at most n in $\mathbb{C}^d$.
- $$V_{P,K} = \lim_{n\to\infty} \frac{1}{n} \log \Phi_n$$

  where

  $$\Phi_n(z) := \sup\{|p_n(z)| : p_n \in Poly(nP),\ ||p_n||_K \le 1\}.$$

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- $D_n = D_n(f, K, P) \equiv \inf\{||f - p_n||_K \; : \; p_n \in Poly(nP)\}$.

Theorem

Let K be compact and PL−regular. Let $R > 1$, and let $\Omega_R = \Omega_{R(P,K)} := \{z : V_{P,K}(z) < \log R\}$. Let f be continuous on K.

1. *If f is the restriction to K of a function holomorphic in $\Omega_{R(P,K)}$, then $\limsup_{n\to\infty} D_n(f, P, K)^{1/n} \le 1/R$.*
2. *If $\limsup_{n\to\infty} D_n(f, P, K)^{1/n} \le 1/R$, then f is the restriction to K of a function holomorphic in $\Omega_{R(P,K)}$.*

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Order of Approximation

The order of approximation to a holomorphic function $f(z)$ by $Poly(nP)$ is given (essentially) by

$$D_n(f, K, P) = O(R^{-n})$$

where

$$\log(R) := \inf_{z \in S(f)} V_{P,K}(z)$$

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In terms of Orthogonal Polynomials

Let
$$\{p_\alpha(z) = p_\alpha(z,\mu)\ :\ \alpha \in (\mathbb{Z}^+)^d\}$$
be the family of orthonormal polynomials obtained by the Gram-Schmidt process with inner-product given by μ applied to the monomials $\{z^\alpha\ :\ \alpha \in (\mathbb{Z}^+)^d\}$.

Theorem

(Generalized Zeriahi) Under the above assumptions
$$V_{P,K}(z) = \limsup_{\alpha} \frac{1}{|\alpha|_P} \log|p_\alpha(z,\mu)| \quad \textit{for } z \in \mathbb{C}^d \setminus \hat{K}$$
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- $K = B_d := \{z \in \mathbb{C}^d \ : \ |z| \le 1\}$
- $p_\alpha(z) = c_\alpha z^\alpha, \ \alpha \in (\mathbb{Z}^+)^d$ with

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- Here $|\alpha| := \sum_{j=1}^d \alpha_j$ and $\alpha! := \prod_{j=1}^d (\alpha_j!)$.

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The Bloom Functional

We have, for

$$F_d(\theta; z) := \lim_{\substack{j\to\infty \\ \alpha(j)/|\alpha(j)|_P\to\theta}} \frac{1}{|\alpha(j)|_P} \log |c_{\alpha(j)} z^{\alpha(j)}|$$

Theorem

For $z \in \mathbb{C}^d \setminus K$

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Formula for the Bloom Function

Theorem

For K the complex ball in $\mathbb{C}^d$,

$$F_d(\theta) =$$

$$\frac{1}{2}\left\{\sum_{i\in I(z)} \theta_i \log(|z_i|^2) - \sum_{i\in I(z)} \theta_i \log(\theta_i) + \left(\sum_{i\in I(z)} \theta_i\right) \log\left(\sum_{i\in I(z)} \theta_i\right)\right\}$$

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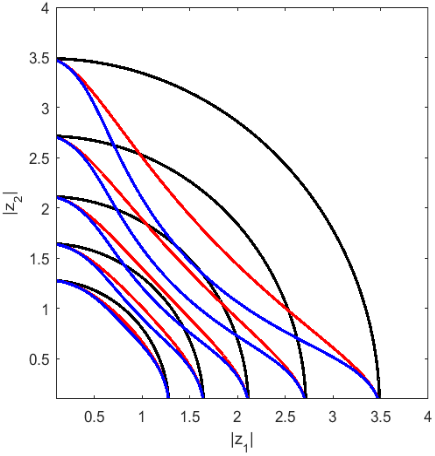
Level Sets of the Extremal Function



The $q = \infty$ (tensor-product) case

Theorem

Suppose that $d = 2$. Then for $|z| \geq 1$,

$$V_{P_\infty,B_2}(z) =$$

$$\begin{cases} \frac{1}{2}\left\{\log(|z_2|^2) - \log(1-|z_1|^2)\right\} & \text{if } |z_1|^2 \leq 1/2 \text{ and } |z_2|^2 \geq 1/2 \\ \frac{1}{2}\left\{\log(|z_1|^2) - \log(1-|z_2|^2)\right\} & \text{if } |z_1|^2 \geq 1/2 \text{ and } |z_2|^2 \leq 1/2 \\ \log(|z_1|) + \log(|z_2|) + \log(2) & \text{if } |z_1|^2 \geq 1/2 \text{ and } |z_2|^2 \geq 1/2 \end{cases}$$

Example One

Consider $f_1(z_1, z_2) := \dfrac{1}{1 - z_1/2} + \dfrac{1}{1 - z_2/2}$

Note that best L_2 approximations are equivalent to Taylor expansions.

$$f_1(z_1, z_2) = \sum_{k=0}^{n} \left(\frac{z_1^k}{2^k} + \frac{z_2^k}{2^k} \right) + \left\{ \frac{(z_1/2)^{n+1}}{1 - z_1/2} + \frac{(z_2/2)^{n+1}}{1 - z_2/2} \right\}$$

Note that for *any* $q \geq 1$, the degrees of z_1^k and z_2^k are both k. In particular, the best L_2 approximation for f_1 on K of degree n, for *any* $q \geq 1$ is

$$p_n(z_1, z_2) := \sum_{k=0}^{n} \left(\frac{z_1^k}{2^k} + \frac{z_2^k}{2^k} \right).$$

In other words there is no advantage in a higher value of q despite the fact that the spaces $Poly(nP_q)$ are of increasing dimension in q.

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$$p_n(z_1, z_2) := \sum_{k=0}^{n} \left(\frac{z_1^k}{2^k} + \frac{z_2^k}{2^k} \right).$$

In other words there is no advantage in a higher value of q despite the fact that the spaces $Poly(nP_q)$ are of increasing dimension in q.

Example One

Consider $f_1(z_1, z_2) := \dfrac{1}{1 - z_1/2} + \dfrac{1}{1 - z_2/2}$

Note that best L_2 approximations are equivalent to Taylor expansions.

$$f_1(z_1, z_2) = \sum_{k=0}^{n} \left(\frac{z_1^k}{2^k} + \frac{z_2^k}{2^k} \right) + \left\{ \frac{(z_1/2)^{n+1}}{1 - z_1/2} + \frac{(z_2/2)^{n+1}}{1 - z_2/2} \right\}$$

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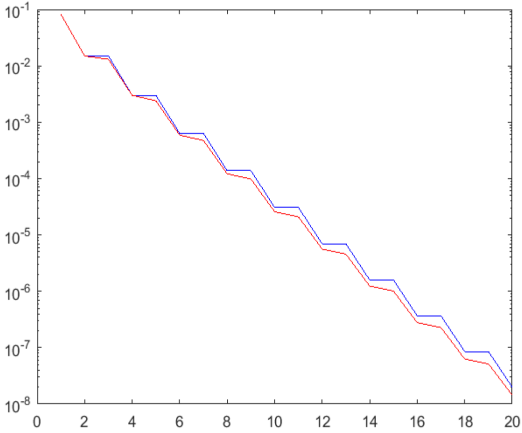
Error in Approximation to f_1



Example 2

Consider the bivariate Runge type function

$$f_2(z_1, z_2) := \frac{1}{a^2 + z_1^2 + z_2^2} \quad a > 1.$$

It's singular set is given by

$$S(f_2) := \{z \in \mathbb{C}^2 \ : \ z_1^2 + z_2^2 = -a^2\}.$$

Lemma

We have

$$\min_{z \in S(f_2)} V_{P_q,K}(z) = \log(a), \ q \geq 1,$$

attained at (among other points) $z_1 = ia, z_2 = 0$.

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Note that $|(ia, 0)|_{P_q} = |a|$ for all $q \geq 1$

In other words the rate of decay of the approximation errors to f_2 are also the same for all choices of $q \geq 1$; there is no approximation value added despite the fact that the dimensions of the spaces $Poly(nP_q)$ are increasing in q!

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Example 3

Take $f_3(z) = \dfrac{1}{1 - z_1 z_2}$

The best L_2 approximation is

$$f_3(z_1, z_2) = \frac{1}{1 - z_1 z_2} = \sum_{k=0}^{m} z_1^k z_2^k + \frac{(z_1 z_2)^{m+1}}{1 - z_1 z_2}.$$

The *uniform* norm of the error on K is easily bounded by

$$\max_{|z| \le 1} \left| \frac{(z_1 z_2)^{m+1}}{1 - z_1 z_2} \right| \le \frac{2^{-(m+1)}}{1 - 1/2} = 2^{-m}$$

For classical degree: $O(2^{-n/2})$

For euclidean degree $(q = 2)$: $O(2^{-n/\sqrt{2}})$

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Thank you for your attention.