Representational Fluency and the Newton–Raphson Method

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Abstract

An understanding of mathematics often requires one to develop representational fluency – the ability to think of concepts across a number of different representations. There is evidence that the graphic calculator with its intrinsic use of a number of representations may be employed to enhance this fluency for certain mathematical concepts. In this study we investigated the value of the TI–92 super–calculator for building understanding of some of the concepts associated with the Newton–Raphson method for finding zeros of functions, since this can easily become purely a symbolic algorithm with little understanding of its geometric basis. The results of this study with 16–17 year old students suggest that students developed an understanding of some concepts involved with the method, as well as a positive view of the role of the calculator.

Background

Many mathematics educators are becoming increasingly aware that student learning is often restricted by taking place within the limited confines of a single representation. This word representation has a number of different uses in the literature, but the idea presented by Kaput (1987, p. 23) proposes that "any concept of representation must involve two related but functionally separate entities. We call one entity the representing world and the other the represented world." In a later paper (Kaput, 1989, p. 169) he also refers to a representation system as a correspondence between two notation systems (for example equations, graphs and tables of ordered pairs) co-ordinating the "syntax of one notation system with the structure of another.", and later still (Kaput, 1998) used the terms representation system and notation system interchangeably. Of all the presentations and perspectives on representation, this is the one which resonates best with us. This concept of different representations of mathematical ideas introduces an important class of mathematical activity involving "translations between notation systems, including the coordination of action across notation systems." (Kaput, 1992, p. 524). This involves manipulation of mathematical processes and concepts both within and between these different representations. Lesh (2000, p. 74) has suggested the idea of representational fluency, or the ability to think across representational boundaries as being "at the heart of what it means to "understand" many of the more important underlying mathematical constructs". Such 'fluency' also includes the ability to interact with these representations, using them as conceptual tools (Thomas and Hong, 2001), but doing so, as Kaput (1998, p. 273) suggests being aware of the potential "inadequacy of linked representations and the strong need to provide experiential anchors for function representations." It appears that truly successful mathematics students are those who have not only the ability to perform mathematical procedures well but also have this representational fluency as part of their mathematical thinking. Among researchers expressing this opinion are Moshkovitch, Schoenfeld & Arcavi (1993, p. 97), who suggest that we should ask "Can the student move flexibly across representations and perspectives when the task warrants it?… Does any curriculum we propose make adequate connections across representations and perspectives? If not it had better be revised". The implications for curriculum development if we want our students to develop this fluency are clear.

Graphic calculators, including those with computer algebra systems (CAS) which are sometimes called super-calculators, provide an environment with dynamic, linked, interactive representations which may be used to enable students to interact with representations and access inter-representational thinking in the school and tertiary mathematics curriculums. They also provide an opportunity to promote investigation and discovery in mathematics classrooms through a multi–representational approach to problem solving (Hong, Thomas & Kwon, 2000), and have particular value in investigations of functions, since they can represent them as dynamically linked algebraic symbolic forms, ordered pairs, graphs and tables of values. A key advantage of graphic calculators over computers is that they are generally more accessible to students in many schools due to the price differential and low availability of computer rooms (Kissane, 1995; Thomas, 1996). There are, of course, still equity issues surrounding their use, which need to be addressed but given that such technology may be able to change the way students think about mathematics, and bring about opportunities for new content, new curricula, and new teaching techniques (Abramovich & Brown, 1999) it is worth pursuing. However, despite considerable effort to change classroom practice to emphasise problem solving strategies, visualisation, pattern recognition, and other more conceptually oriented techniques (Goldin, 1998), much of school mathematics is still devoted to the manipulation of formal notational systems. Even with the introduction of calculators into a teaching programme there may be very change and continued research is necessary (see e.g. Ruthven, 1990; Dunham & Dick, 1994; Penglase & Arnold, 1996; Graham & Thomas, 2000) to provide evidence of whether, and where and how, graphic calculators can usefully be integrated into the mathematics curriculum in order to improve understanding.

In Korea there has tended to be an emphasis on mathematics as a collection of procedural skills and calculators have not been used in the secondary school mathematics environment. Hence the present research presented an opportunity to consider the value of graphic calculators in a setting where their use has been minimal. The study adopted the approach of students using the super–calculator to learn the Newton–Raphson (NR - also called Newton's) method of approximating zeros of functions, since this a topic which has often been learned as a symbolic procedure with little presentation of the geometrical basis of the method and hence there was an opportunity to generate inter–representational thinking. Our module sought to do this by improving concept formation across representations, developing mathematical reasoning, and linking different mathematical ideas via investigation.

Method

The Form 6 students (age 16~17 years) involved in this research project comprised a class from a high school in Bundang, Korea. None of the students was experienced in using calculators in their mathematics learning since they were not used at all in the school, neither had they previously covered the NR method, since it is not in the curriculum of Korean schools, but is a tertiary topic. However the class had studied calculus prior to the research. For Korean students the university entrance examination is a major aim of their school study, not just because of its entry importance but also because the status of one's university will influence one's social position for the future. Since the super–calculators and the learning of NR method were both extra–curricular and had no direct influence on the students' entrance examination performance it was difficult to motivate them to make time for the research.

Instruments

A module of work, containing a description of the basic facilities of the TI-92 and addressing the concepts involved in the NR method, was prepared using a 'Press', 'See', and 'Explanation' format (see Figure 2). In the NR numerical method of solving equations, an initial value $x=x_1$ near a solution is chosen, the tangent to the graph at $(x_1, f(x_1))$ is drawn, and the next approximation is taken as the point where the tangent line intersects the x-axis. In the module two methods were presented: first a visual method using the TI-92 and based on the idea that we can often get nearer to a root α by drawing tangents at estimates; secondly, an algebraic one using the equivalence of the gradient of the tangent, $f'(x_1) = \dfrac{f(x_1)}{(x_1 - x_2)}$ to obtain a formula.

Two tests comprising questions on the NR method were compiled, based on standard school calculus textbooks. These tests were essentially parallel tests using different numerical values, divided into sections A and B, comprising questions highlighting process–oriented skills and conceptual understanding, respectively. In order to assess students' knowledge of the background pre–requisites for understanding Newton's method the tests also included questions on differentiation, limit and the use of the NR formula $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$. Section A in the tests comprised questions such as:

A2. Find: a) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{2x}$ b) $\lim\limits_{x\to 0}\dfrac{\tan 2x}{3}$ c) $\lim\limits_{x\to\infty}\dfrac{2x-1}{3x}$

A3. Calculate the second approximation x_2 to the root of $f(x) = 0$, using the Newton–Raphson method, when $f(x) = \sin x - x$, and $x_1 = \frac{\pi}{2}$.

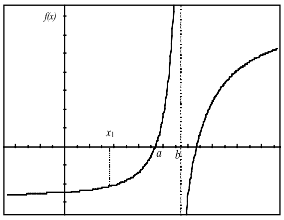
A4. What is the value of x at the point on the curve $y = x^2 + 7x - 8$ where the gradient is equal to 1?

In contrast, section B sought to address students' thinking about concepts when finding zeros or solving equations. We wanted to know if they knew how and why the method worked and could apply this understanding.

B3. If, in the Newton–Raphson method, for a function $y=f(x)$, $f(x)>0$, $x_1 = 2$ and $f'(x_1)>0$, where x_1 is the first approximation to the root, is $x_2>x_1$ or is $x_2<x_1$, where x_2 is the second approximation to the root? Explain your answer.

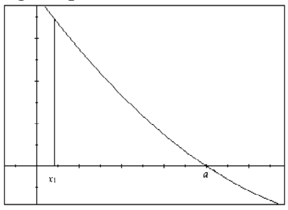
B4. a) Explain why x_1 in the diagram alongside is an unsatisfactory first estimate in the Newton-Raphson method for the root $x=a$ of $y=f(x)$.

b) When would x_1 be a satisfactory first estimate?

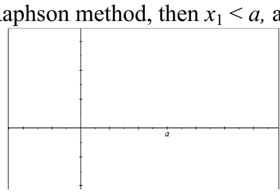


B7. How could you use the Newton–Raphson method to find the x-value of the intersection of the graphs of $y=2e^{-x}+\cos x$ and $y=2$? Explain your method clearly.

B5. For the function $f(x)$ shown below the 2nd approximation x_2 to the root $x = a$ is exactly 0.8 closer than the first approximation x_1. What is: (a) the relationship between $f'(x_1)$ and $f(x_1)$? (b) the gradient of the chord joining the points where $x = x_1$ and $x = x_2$?



B6.b) Draw a continuous function below where, if x_1 and x_2 are the 1st and 2nd approximations to the root $x=a$ using the Newton–Raphson method, then $x_1 < a$, and $x_2 > a$.



c) What determines whether x_2 and x_3 etc. are less than a or greater than a?

Figure 1. Some section B post–test questions on the Newton–Raphson method.

A selection of four of the section B questions on the Newton–Raphson method is shown in Figure 1. The ideas behind some of these questions included improving representational fluency through a geometric appreciation of:

- when the first estimate is likely to be unsatisfactory
- the relationship between the gradient of the function at the first estimate and the relative position of the second estimate
- the relationship of the sign of f and f' to the position of the estimates.

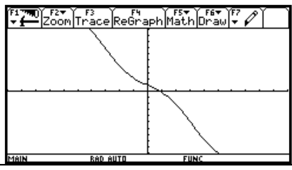
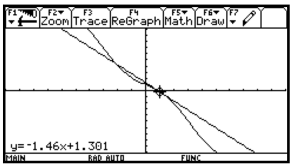
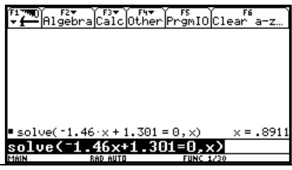
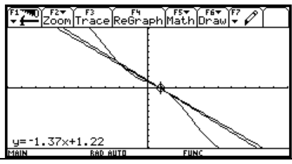
For example, in B3, B4 and B5 a general function *f(x)* rather than an explicit function was given discouraging students from immediately working through a procedure, but requiring an understanding and application of the relationship between the sign of *f(x)* and *f′(x)* and the relative position of the estimate of the zero.

Procedure

The first researcher met twice with the students' normal classroom teacher to answer his questions and to make sure that he felt comfortable with the TI–92 calculator and the material to be presented. Prior to the pre–test the class teacher gave a tutorial of 50 minutes duration to familiarise the students with differentiation of exponential and trigonometric functions, which they had not previously met, as well as an outline of the NR method. Following this the teacher taught the class for four hours, divided into two intensive sessions on two consecutive days, covering basic facilities of the calculators including graphs and tables, how to find limits and a gradient function and how to implement the NR method both visually and symbolically on the TI–92. The research project took place on November 18th and 19th, with the post–tests on November 20th. Each student had access to their own TI–92, which they kept with them for the whole of the time of the study, including their time at home. During lessons, which were observed by a researcher, the class teacher stood at the front and the class sat in traditional rows of desks. He demonstrated each step, employing a calculator viewscreen and projecting the image on his calculator using an overhead projector, while the students followed in the module and copied his working onto their own calculator. Following this the students, working individually but discussing progress with others, attempting the questions and investigations in the module. One problem which arose was that the Korean students had some anxiety caused by the calculator commands, since these commands appeared only in English. As each one was projected it was necessary for the teacher to translate it into the Korean version for the students. Some confusion ensued until the students became more accustomed to the English commands. The psychological anxiety caused by this could have affected their self-confidence and hence their ability to build mathematical understanding (Kota & Thomas, 1998).

A section of the module illustrating the layout, the teaching approach used, and giving one of the two methods used for solving the equation $\sin x = 2x - 1$, is presented in Figure 2.

Example. Solve the equation sin$x$=2x–1 using the Newton-Raphson method. Give the answer to 4 d.p.
The first step is to define the function y = sin$x$ – 2x + 1 and sketch its graph:

Press	See	Explanation
Method 1. [♦] [Y=] [SIN] x [)] [−] 2x [+] 1 [ENTER] [♦] [GRAPH]		The function is defined using $y1$ = sin$x$ – 2x + 1
F5 A:Tangent [ENTER] Tangent at? 1 [ENTER]		The first tangent line starts at the point x_1= 1. The equation of the tangent is given as $y = –1.46x+1.301$. This will be used to find x_2 (which should be closer to the root than x_1), by seeing where it crosses the x-axis.
[♦] [HOME] F2 1 [ENTER] [(-)] 1.46 x [+] 1.301 = 0, x [)] [ENTER]		We can see that the next point x_2= 0.8911
Return to [GRAPH] F5 A:Tangent [ENTER] Tangent at? 0.8911. [ENTER]		The equation of the tangent is given as $y = –1.372x + 1.218$ at the point x_2=0.8911. This will be used to find x_3 (which should be closer to the root than x_2).

This was then repeated until x = 0.8882 was obtained.

Figure 2. A section of the module on the Newton–Raphson Method showing the layout.

After the teacher's explanation, the students spent the rest of the time working on the practice exercises and investigations while the teacher circulated and assisted with any problems. An example of the type of investigative questions the students worked on using the calculators during the module is given below.

A function $f(x)$ is such that: $f(x)$=0 has only 2 solutions $x=a$ and $x=b$. Starting the Newton–Raphson method with $x_0>b$ it converges to $x=a$, where $b>a$. Can you (i) Sketch such a function $f(x)$? (ii) Find a possible formula for such a graph?

In order to answer a question such as this one can use basic concepts and principles to investigate possible solution functions, visualising their graphs. This requires linking data represented algebraically to a graphical representation, including tangents. Finally one has to model the solution symbolically, moving in the opposite representational direction. Following the second tutorial the students were given the post-test, an attitude test and a questionnaire about the teaching programme, their learning and the calculator.

Results

Qualitative and quantitative analyses of the data were both used to try to determine:

- any differences in student performance on procedural and conceptual questions
- student attitudes to the work on the calculator
- any other apparent influences on learning.

Table 1
A comparison between the overall, section A and B pre– and post–test results

N=20	Pre-Test Mean	Post-Test Mean	t	p-value
Overall	5.8	13.05	6.16	<0.00001
Section A	3.15	7.15	10.28	<0.00001
Section B	2.7	5.90	3.26	<0.005

Firstly, as Table 1 shows, there was a significant overall improvement in the results of the students, in both the procedural and conceptual questions. This indicated that the work on the calculator module had influenced the learning of the NR method. Since these students had not studied the NR method previously other than for the 50 minute preliminary lesson, these results are not too surprising, but they do indicate that the programme using the calculator is a suitable method for learning both the skills involved and the concepts underlying the NR procedure. We are currently involved in comparing these results with another group of students who had previously learned the NR method without calculators and then used the same module afterwards.

Of course the benefit of this approach is not likely to be universal and so we compared the student performance on the individual questions to gauge where the improvement particularly occurred. Table 2 shows that the students did significantly better on questions A1–3, and B1, B3, B4, B6, and B7, while Questions A4 and B5 proved very resistant to improvement.

Table 2
The Korean pre– and post–test section A and B mean individual question scores

Question Number (Max score)	Pre–test mean	Post–test mean	t	p
A1 (3)	0.75	2.45	8.36	<0.000001
A2 (3)	1.6	2.9	6.72	<0.000001
A3* (2)	0.05	0.85	4	<0.0005
A4 (2)	0.75	0.95	0.78	n.s.
B1 (4)	1.3	2.4	2.60	<0.05
B2 (2)	1.2	1.35	0.65	n.s.
B3* (2)	0.15	0.65	2.03	<0.05
B4* (3)	0	0.4	1.90	<0.05
B5*(2)	0	0.05	1.45	n.s.
B6* (4)	0	0.45	2.02	<0.05
B7* (2)	0	0.6	2.85	<0.05.

* The questions on the Newton–Raphson Method

Question A4 is a standard calculus problem and the reason for the poor result here is not readily apparent. To answer question B5 (see Figure 1) students were required to carry out two procedures, doing so in the context of a geometrical representation of the NR method. It appears that the level of representational fluency required here to relate analytical and geometrical properties in a single context is difficult to acquire and requires further attention.

Individual students

A consideration of the work of some individual students will be informative in terms of their representational thinking about the method. Question B6 required students to be able to relate the relative position of the first and second approximations to the concavity of the graph of the function.

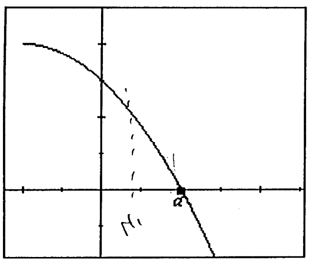
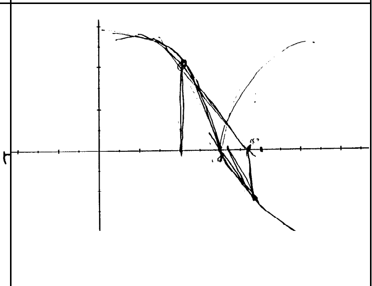
Lee 4 Post–Test	Lee 5 Post–Test
 답: x2보다 x3가 x3보다는 x4가 x4보다는 x5가 xi보다는 xi+1이 근에 가깝고 결국에 xi가 x=a가 된다. [x_3 less than x_2, x_4 less than x_3, x_5 less than x_4, and x_{i+1} less than x_i is getting closer to the root, finally x_i becomes $x = a$]	

Figure 3. Student Lee 5's pre-test and post-test working for question B6.

Neither Lee 4 nor Lee 5 made any attempt on this at the pre-test, however, at the post-test they both drew a graph concave down for $x < a$, showing conceptual understanding of the geometric significance of the first approximation (see Figure 3). Lee 5 actually shows the first approximation, on the left of a, and a tangent that crosses to the right of the root a, before subsequent approximations approach a. Lee 4 marks x_1 and then describes how each successive approximation after the first is less than the one before. Lee 4's section B showed a 74% improvement, with 2 (10%) in the pre-test but 16 (84.5%) out of 19 in the post-test. She made no attempt at question B7 on the pre-test, however, in the post-test her two solution methods (in Figure 4) clearly show that she understood how the method works geometrically as well as algebraically.

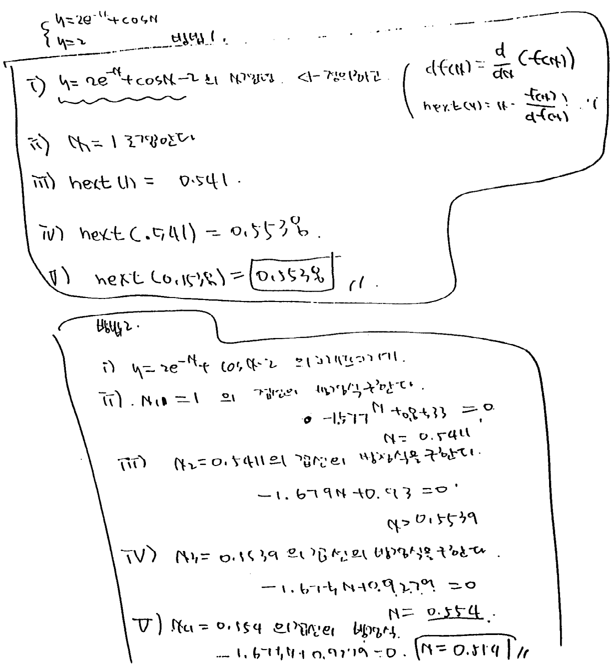
Post–Test Solution	
Translation from Korean: [Method 1: i) Define y=2e^x+cos$x$-2 of a point of intersection on x-axis. ii) Define x=1, (iii) next(1)=0.541, iv) next(.541)=0.5538, v) next(0.5538)=0.5538 Method 2: i) Draw the graph y=2e^x+cos$x$-2 ii) Find the tangent line on x_1=1, -1.577x+0.8533=0, x=0.5411 iii) Find the tangent line on x_2=0.5411, –1.679x+0.93=0, x=0.5539, iv) Find the tangent line on x_3=0.5539, -1.675x+0.9279=0, x=0.554, Find the tangent line on x_4=0.554, -1.675x+0.9279=0, x=0.554.]	

Figure 4. Student Lee 4's conceptual improvement on the Newton–Raphson method.

In both of these she uses a calculator method from the module. In the first she uses a calculator version of the 'standard' algorithm, including the English word 'next' which was introduced in the module, while the second involves finding the equation of the tangent and using its intersection with the x-axis to find the next approximation. While there is no diagram drawn, her use of the words 'tangent line' and understanding of the approach show her geometric understanding, and thus representational fluency is apparent in the solution. The working of Nam 2 and Kim 2 on question B7 (see Figure 5) also displays a conceptual knowledge of the geometrically based process of NR.

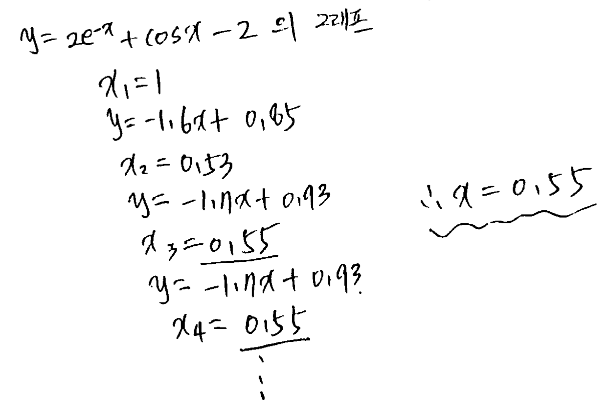
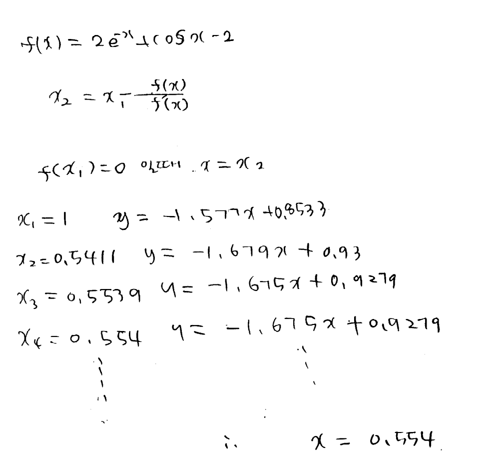
Nam 2 Post–Test	Kim 2 Post–Test
	

Figure 5. Post-test solutions to question B7 showing links to tangents.

Unable to tackle the question in the pre-test, here they have both correctly identified the function to use and Kim 2 has also written the normal algebraic formula. However, in the working, which has clearly been accompanied by the GC, like Lee 4 they too list the successive approximations accompanied by the equation of their tangents requiring an inter-representational perspective on the method. Question B4 (see Figure 1) presented a graph and required students to relate this to an algebraic expression for the range of values of x for which the first approximation x_1 would enable one to approach the root a. As Figure 6 shows, student Lee 4 didn't seem to understand the meaning of this question in the pre-test, he just defined the gradient function $f'(x)$ as $ax + b$, however, his solution in the post-test shows that he could relate the idea that the approximation needed to lie between a and b to an inequality. Of course x_1 may lie a little further to the left of a too, but he has the essence of the solution, probably referring to x_2.

Pre–Test	Post–Test
$f'(x) = ax+b$ 이때 $b \neq 0$ [When $f'(x) = ax + b$, $b \neq 0$]	$a \leqq x_1 < b$

Figure 6. Student Lee 4's pre-test and post-test working for question B4.

We do not need to speculate about the students' understanding of the geometric principles behind the NR method since a number of them described their conceptual understanding in response to the question "How does the Newton–Raphson method work?":

Lee 5: To find the root for the given function, find the gradient of the tangent line $f'(x)$, keep going the same way; finally we can find the closest root.

Lee 3 Take some point x_1, draw the tangent line at the point. Find the other intersection point x_2 on the graph. In the same way, finally find the intersection point x on the x-axis.

Chun Define some value x_1, draw the tangent line at x_1. In the same way, define again the other value x and draw the tangent line on the value.

Lee 2 Start with the initial point. Draw the tangent line. Take the approximation closer to the root.

All of these, and others, refer to tangents, and while the translation causes some problems, they clearly have the idea of using these to get closer to the root. They are developing representational fluency. When asked how the module had affected their understanding of the NR method they replied:

Lee 5 When I directly drew the graph, I could understand what the Newton-Raphson method is.

Um It was quite helpful. Sketching the graph and tangent line, the root could be found easily.

Lee 4 I could directly draw the tangent line and confirm what the method of Newton-Raphson is.

They also appreciated when things might go wrong:

Lee 4 If the differentiated value is 0, then we can not find the second approximation x_2 to the root.

Um If the tangent line for the value does not intersect the x-axis, we can not find the root.

Son If the first approximate root is not defined appropriately, Newton-Raphson method can not be used any more.

Student Attitudes

The Korean students had never used a calculator in their mathematics learning and live in a country where the prevailing opinion has been that mathematics for secondary school students should be conducted without technological aids, and hence it was of great interest to consider their view of the calculators. After their tutorials, when asked 'Were there any advantages in using the TI–92 graphic calculator?', 11 (55%) of the students replied that they made complicated calculations easy to solve, 6 (30%) of the students replied that the drawing of graphs was convenient, 3 (15%) of the students replied that the calculator was useful to get the concepts. However, when asked what difficulties they had encountered 7 (35%) of them replied that the commands on the calculator were difficult because they were in English, and 6 (30%) responded that using of calculator was confusing and difficult, primarily due to problems locating the correct keys for a given function. Generally their view of the technology was very positive and this was confirmed by the responses to the Likert attitude scale questions. Each question was scored with an integer from 1 to 5, and scores were reversed on negative questions so that in every case the higher the score the more positive the attitude to calculators. Overall the mean score was 3.96 (t=4.17, p<.0005), and the mean scores for some of the questions are given below.

I think the calculator is a very important tool for learning mathematics	4.0
I feel comfortable using a calculator for solving mathematical problems	4.0
Students should use calculators more often in mathematics	4.0
I want to improve my ability to use a calculator	4.5
All students should learn to use calculators	4.0
Mathematics is easier if a calculator is used to solve problem	4.7
When I use a calculator, my learning improves	4.0
Using calculators makes students better problem solvers	4.6

Of course this was a small scale study with a group of users new to the calculators, so it is difficult to separate out motivational aspects due to the novelty of the GC use from genuine progress in learning.

However, the study does suggest: that novice users can benefit in terms of both procedural and conceptual knowledge from the GCs; that the GC can be successfully applied to novel curriculum areas; that the students liked the calculator methods presented in the module of work and leaned to apply them; and that the technology is motivating for the students. Considering the short learning time these results are encouraging and suggest that it is worth pursuing this approach to representational fluency, helping students to make connections across representations. However, it is our opinion that much thought and care needs to be put into preparing well structured learning material and to assisting classroom teachers in its presentation. We hope that other studies may be undertaken to replicate the outcomes described here in other curriculum areas.

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