

SURVEY TEAM 4: KEY MATHEMATICAL CONCEPTS IN THE TRANSITION FROM SECONDARY TO UNIVERSITY

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In this report we present the work of ICME12 Survey Team 4, whose brief was to examine the transition from secondary school to university mathematics with a particular focus on mathematical concepts and aspects of mathematical thinking that are key to this transition. To this purpose we surveyed the recent literature in order to identify research that addresses issues of transition that are germane to the learning and teaching of: calculus and analysis; the algebra of generalised arithmetic and abstract algebra; linear algebra; reasoning, argumentation and proof; and modelling, applications and applied mathematics. The literature review revealed a multi-faceted web of cognitive, curricular and pedagogical issues, some spanning across the aforementioned mathematical topics (such as student cognitive preparedness for the requirements of university-level formal mathematical thinking) and some intrinsic to certain topics (such as little or no content coverage at school level). In addition to the literature review we surveyed the views on the transition of those engaged with teaching in university mathematics departments. Specifically, we aimed to elicit perspectives on: what topics are taught, and how, in the early parts of university-level mathematical studies; whether the transition should be smooth; student preparedness for university mathematics studies; and, what university departments do to assist those with limited preparedness. We present a summary of the survey results from 79 respondents from 21 countries.

Keywords: Transition, mathematics, secondary, university, survey.

BACKGROUND

Changing mathematics curricula and their emphases, along with distinct changes in an enlarged tertiary entrant profile (Hoyles, Newman, & Noss, 2001; Hockman, 2005), have provoked some international concern about the ability of students entering university with regard to their apparently decreasing levels of competence (PCAST, 2012; Smith, 2004) and the sometimes traumatic effect of the transition from school to university mathematics on many students (Engelbrecht, 2010). Changes in competency have been particularly apparent with regard to essential technical facility, analytical powers, and perceptions of the place of precision and proof in mathematics. Such mathematical under-preparedness of students entering university has been seen as an issue (Faulkner, Hannigan & Gill, 2011; Gill, O'Donoghue, Faulkner & Hannigan, 2010; Hourigan & O'Donoghue, 2007; Kajander & Lovric, 2005; Luk, 2005; Selden, 2005), and one that may impact on students' success in university mathematics (Anthony, 2000; D'Souza & Wood, 2003). However not all studies

agree on the extent of the problem (Engelbrecht & Harding, 2008) or that a high school focus on procedural tasks leads to undergraduate students who have difficulty with conceptual problems (Engelbrecht, Harding & Potgieter, 2005). Recently, for example, Faulkner, Hannigan and Gill (2011; also Gill, O'Donoghue, Faulkner & Hannigan, 2010) have noted the intensely shifting profile of students who take service mathematics courses along with a decline in the mathematical standard of these students entering university. Specifically, the study reports that between 1998 and 2010 the profile of students who take service mathematics courses in the University of Limerick (Ireland) changed dramatically: many more are diagnosed as at risk (increasing by around 25% to 58% for Technological Mathematics and 46% for Science Mathematics) and fewer have an advanced mathematics secondary qualification. On the other hand, research by James, Montelle and Williams (2008) analysed the relationship between the final secondary school qualifications in mathematics with calculus of incoming students and their results in the core first-year mathematics papers, and found that standards had been maintained. Some concern has been expressed about the levels of student enrolments in undergraduate mathematics programmes (The ICMI Pipeline Project, see Barton & Sheryn, 2009; and <http://www.mathunion.org/icmi/other-activities/pipeline-project/>) along with implications for the future of the subject. Furthermore the report of the President's Council of Advisors on Science and Technology (PCAST) (2012) states that in the USA alone there is a need to produce, over the next decade, around 1 million more college graduates in Science, Technology, Engineering, and Mathematics (STEM) fields than currently expected.

While recent research has specifically addressed these issues with regard to the transition from school to university (Brandell, Hemmi & Thunberg, 2008; Engelbrecht & Harding, 2008; James, Montelle & Williams, 2008; Jennings, 2009), overall the volume of research in tertiary mathematics education was, until about a decade ago, relatively modest (Selden & Selden, 2001). In response to this, the PCAST report (2012) in the USA recommends funding around 200 experiments at an average level of \$500,000 each to address mathematics preparation issues.

While we are aware that there are many aspects of the secondary school to university transition that are of interest to mathematics education practitioners and researchers, and relevant to the issues above, the brief of ICME-12 Survey Team 4 was restricted to a consideration of the role of mathematical thinking and concepts as they relate to transition. Hence, we sought to review the recent literature on transition to extract key issues that have been highlighted. We found relatively few papers that deal directly with mathematical thinking and concepts in transition. However, there were many other, relevant papers that discussed relevant mathematical issues involved and so our second aim was to review the recent literature analysing the learning of mathematics that occurs at either side (or both sides) of the transition boundary to present some of the mathematical transition issues identified. To achieve this we formed the, somewhat arbitrary, division of this mathematics into: calculus and analysis; the algebra of generalised arithmetic and abstract algebra; linear algebra; reasoning, argumentation and proof; and modelling, applications and applied mathematics, and report findings related to each of these fields.

Finally it was important to include the voices of those engaged in teaching in university mathematics departments and to record their perspectives on the mathematical thinking of students in transition. We wanted to know what topics are taught and how, if the professors think the transition should be smooth, or not, whether their students are well prepared mathematically, and what university departments do to assist those who are not. A summary of the survey results from 79 members of mathematics departments addressing these and other topics is included below.

THEORETICAL PERSPECTIVES IN THE TRANSITION LITERATURE

A number of different lenses have been used to analyse the mathematical transition from school to university. These have been summarised well elsewhere (see e.g., Winsløw, 2010) but we consider it useful to preface our findings with a very brief note on the major theoretical perspectives we find in the transition-related literature. One theory that is in common use is the Anthropological Theory of Didactics (ATD) based on the ideas of Chevallard (1985). This introduces the key concept of a *praxeology* defined to be a quadruple comprising (task, technique, technology, theory) and focuses on analysis of the organisation of praxeologies relative to institutions and the diachronic development of didactic systems. A second framework often employed is the Theory of Didactical Situations (TDS) of Brousseau (1997), which describes the use of *didactical situations* whereby the teacher orchestrates the elements of the didactical milieu under the constraints of a dynamic didactical contract. Some research uses the action-process-object-schema (APOS) framework for studying learning, as presented by Dubinsky and others (Dubinsky, 1991; 1997; Dubinsky & McDonald, 2001). This describes how a process may be constructed from actions by reflective abstraction, and subsequently an object is formed by encapsulation of the process. In turn the mental object can then become part of an appropriate mental schema. Other authors find the Three Worlds of Mathematics (TWM) framework of Tall (2004a, b, 2008) useful. This describes thinking and learning as taking place in three worlds: the embodied; the symbolic; and the formal. In the embodied world we build mental conceptions using visual and physical attributes of concepts, along with enactive sensual experiences. The symbolic world is where the symbolic representations of concepts are acted upon, or manipulated, and the formal world is where properties of objects are formalized as axioms, and learning comprises the building and proving of theorems by logical deduction from these axioms.

THE QUESTIONNAIRE

As a group we constructed an anonymous questionnaire on transition (see Appendix) using an Adobe Acrobat pdf form and each of us sent this out internationally by email to members of mathematics departments, with the responses collected electronically. There were 79 responses to this survey from 21 countries. Clearly the experience for beginning university students varies considerably depending on the country and the university that they attend. For example, while the majority teach pre-calculus (53, 67.1%), calculus (76, 96.2%) and linear algebra (49, 62%) in their first year, minorities taught complex analysis (1), topology (3), group theory (1), real analysis (5), number theory (9), graph theory (12), logic (15), set theory (17) and geometry (18), among other topics. Further, in response to 'Is the approach in **first** year mathematics at your university: Symbolic, Procedural; Axiomatic, Formal; Either,

depending on the course.’ 21 (26.6%) answered that their departments introduce symbolic and procedural approaches in first year mathematics courses, while 6 replied that their departments adapt axiomatic formal approaches. Most of the respondents (50, 63.3%) replied that their approach depended on the course.

When asked ‘Do you think students have any problems in moving from school to university mathematics?’ 72 (91.1%) responded “Yes” and 6 responded “No”. One third of those who answered “Yes” described these problems as coming from a lack of preparation in high school, supported by comments such as “They don't have a sufficiently good grasp of the expected school-mathematics skills that they need.” Further, two thirds of those who answered “Yes” described the problems as arising from the differences between high school classes and university (including more than 50% of the respondents from the USA, New Zealand, South Africa and Brazil, all of whom sent at least 5 responses), such as differences in class size and work load, with many specifically citing the conceptual nature of university mathematics as being different from the procedural nature of high school mathematics. Comments here included “university is much more theoretical” and “Move from procedural to formal and rigorous [sic], introduction to proof, importance of definitions and conditions of theorems/rules/statements/formulas.” Other responses cited: students’ weak algebra skills (12.5%); that university classes are harder (5%); personal difficulties in adjusting (10%); poor placement (3%); and, poor teaching at university (1%).

Looking at specific mathematical knowledge, we asked ‘How would you rate first year students’ mathematical understanding of each of the following on entry to university?’ With a maximum score of 5 for high, the mean scores of the responses were algebra or generalised arithmetic (3.0), functions (2.8), real numbers (2.7), differentiation (2.5), complex numbers (1.9), definitions (1.9), vectors (1.9), sequences and series (1.9), Riemann integration (1.8), matrix algebra (1.7), limits (1.7) and proof (1.6).

Since there has been some literature (e.g., Clark & Lovric, 2009) indicating that, rather than being ‘smooth’, the transition to university should require some measure of struggle by students, we asked ‘Do you think the transition from secondary to university education in mathematics should be smooth?’ Here, 54 (68.4%) responded “Yes” and 22 (27.8%) responded “No”. Of those who responded “No”, many of the comments were similar to the following, expressing the belief that this is a necessary transition: “Not necessarily smooth, because it is for most students a huge change to become more independent as learners.” Those who answered yes were then asked ‘what could be done to make the transition from secondary to university education in mathematics smoother?’ The majority of responses mentioned changes that could be made at the high school level, such as: encourage students to think independently and abstractly; change the secondary courses; have better trained secondary teachers; and, have less focus in secondary school on standardised tests and procedures. A few mentioned changes that could be made at the university, such as: better placement of students in classes; increasing the communication between secondary and tertiary teachers; and, addressing student expectations at each level. This lack of communication between the two sectors was also highlighted as a major area requiring

attention by the two-year study led by Thomas (Hong, Kerr, Klymchuk, McHardy, Murphy, Spencer, & Thomas, 2009).

Since one would expect that, seeing students with difficulties, universities might respond in some appropriate manner, we asked ‘Does your department periodically change the typical content of your first year programme?’. 33 (41.8%) responded “Yes” and 44 (55.7%) responded “No”. The responses to the question ‘How does your department decide on appropriate content for the first year mathematics programme for students?’ by those who answered yes to the previous question showed that departments change the content of the first year programme based on the decision of committees either on university level or on department level. Some respondents said that they change the course based on a decision by an individual member of faculty who diagnoses students’ need and background to change the course content for the first year students. 15 of the 35 responded that their universities try to integrate student, industry, and national needs into first year mathematics courses. The follow-up question ‘How has the content of your first year mathematics courses changed in the last 5 years?’ showed that 35 had changed their courses in the last 5 years, but 10 of these said that the change was not significant. 17 out of the 35 respondents reported that their departments changed the first year mathematics courses by removing complex topics, or by introducing practical mathematical topics. In some of the courses, students were encouraged to use tools for calculation and visualisation. However, there were also 6 departments that increased the complexity and the rigour of their first year mathematics courses.

LITERATURE REVIEW

Some authors (e.g. Eisenberg, Engelbrecht, & Mamona-Downs, 2010) consider transition in terms of the general differences in approach between the styles of mathematics taught at school against that at university. One aspect highlighted by Engelbrecht (2010) is that students are not familiar with logical deductive reasoning, required in advanced mathematics. Hence, it is necessary to assist students in this transition process, in moving from general to mathematical thinking, seeing the need for logical structure but recognising that mental processing, conceptualization and intuition also have a crucial, complementary role. Leviatan (2008) argues that school mathematics concentrates on problem solving skills, while tertiary mathematics is more abstract and emphasises the inquisitive as well as the rigorous nature of mathematics. Another perspective concerning transition considered by De Vleeschouwer (2010a, b) involves the movement from application of techniques to their justification and then significance within a mathematical theory.

Calculus and Analysis

A number of researchers have studied the problems of the learning of calculus and analysis in the transition between secondary school and university. Some of these studies focus on the specific topics of real numbers (Bergé, 2008, 2010; Bloch, Chiocca, Job & Scheider, 2006; Ghedamsi, 2008; Mamona-Downs, 2010), functions (Dias, Artigue, Jahn & Campos, 2008; Vandebrouk, 2010), limits (Bloch et al. 2006; Bloch & Ghedamsi, 2005), continuity (Artigue, 2008) and open and closed sets (Bridoux, 2010). Ghedamsi (2008a, b) and sequences and series (González-Martin, 2009; Gyöngyösi, Solovej & Winsløw, 2010). They were located in

several countries (Brazil, Canada, Denmark, France, Israel, Tunisia) and use different frameworks (such as ATD; textbook analysis, analysis of students' productions; use of CAS or innovative teaching and assessment methods). Some have shown that calculus conflicts that emerged from experiments with first year students could have their roots in a limited understanding of the concept of function, as well as suggesting the need for a more intensive exploration of the dynamical nature of the differential calculus (Junior, 2006). The transition from calculus to analysis has been extensively investigated within the Francophone community, with the research developed displaying a diversity of approaches and themes but a shared vision of the importance to be attached to epistemological and mathematical analyses. This has been demonstrated in studies by a number of researchers. For example, Robert (2010) shows that the Formalizing, Unifying, Generalizing (FUG) perspective can be useful for approaching the teaching and learning of some notions in analysis and Bridoux (2010), considering topological notions introduced in a first university course in Belgium, shows the FUG character of these notions. One important distinction in analysis, addressed by Artigue (2009), Rogalski (2008) and Vandebrouck (2010, 2011), is between local and global perspectives. They considered the evolution of functional thinking in transition from secondary to university, the latter using a TWM lens, and expressing the need to reconceptualise the concept of function in terms of its multiple registers and its process-object duality. He claims that university level work on functions requires that students can adopt a local point of view, whereas only pointwise (where functions are considered as a correspondence between two sets of numbers) and global points of view (where the representations are tables of variation) are constructed at secondary school. However, the emphasis on algebraic tasks at the end of the secondary school tends to erase the pointwise and global points of view and doesn't allow students to reach the local point of view. The claim is that the school approach also prevents consideration of functions as complex objects with pointwise as well as global properties and consequently students face difficulties developing the necessary local viewpoint when entering the formal axiomatic world at the start of university. To illustrate Vandebrouck presents a task which algebraic techniques are not sufficient to solve, giving rise to student difficulties. An ATD-based study of the transition from concrete to abstract perspectives in real analysis was that of Winsløw (2008), who considered real functions and the operations on these functions associated with the limit process. He claims that in secondary schools the focus is on the practical-theoretical blocks of concrete analysis, while at university level the focus is on more complex praxeologies of concrete analysis and on abstract analysis. He considers two kinds of transitions in the student's mathematical activity. The first is the transition from activity mainly centred on practical blocks to that of working with more comprehensive and structured mathematical organisations. The second is the transition to tasks with theoretical objects. Since the second kind of transition presupposes the first one an incomplete achievement of the first transition produces an obstacle for the second one by making the tasks to be worked on inaccessible.

One of the key concept changes in the transition from school calculus to university analysis is the need to work with limits, especially of infinite sequences or series. Two obstacles regarding the concept of infinite sum were identified by González-Martin (2009, see also González-Martín, Nardi, & Biza, 2011)), the intuitive and natural idea that the sum of infinity

of terms should also be infinite, and the conception that an infinite process must go through each step, one after the other and without stopping, which leads to the potential infinity concept. If the concept of series is reduced to its algorithmic aspects, as he claims it usually is, this leads to misconceptions of the integral concept. Employing epistemological, cognitive and didactic dimensions to textbook analysis he concludes that series represents 10% or more of the content, but that the textbooks do not foster the links between visual and algebraic representations. Further, there was no consensus on how way to introduce series, very few tasks showing real life applications and very few historical references. A useful approach to building thinking about limits, suggested by Mamona-Downs (2010), is the set-oriented characterization of convergence behaviour of sequences of that supports the mental image of ‘arbitrary closeness’ to a point. According to Oehrtman (2009), students’ reasoning about limit concepts appears to be influenced by metaphorical application of experiential conceptual domains. He identified strong metaphors for limit concepts by 120 students through analyses of their written assignments from an introductory calculus course and nine interviews. The metaphors were: collapse metaphors (for the definition of the derivative, the volume of solids of revolution, definite integrals and the fundamental theorem of calculus); approximation metaphors (for infinite series, the definition of the derivative); proximity metaphors (for the limit of function and continuity, infinite series, the definition of the derivative); infinity as number metaphors; and physical limitation metaphors (for a volume of revolution, the limit of a sequence of sets). He argues that the only metaphor cluster that was consistently detrimental to students’ understanding was that of the physical limitation metaphors. However, it could be a concern that students demonstrated an inability to apply abstract criteria for adopting, evaluating, or modifying particular metaphors, although this gives fertile opportunities for discussions.

The theoretical influence of TDS led to a long-term Francophone tradition of didactical engineering research, which in the last decade has been designed to support the transition from secondary calculus to university analysis. Ghedamsi (2008a, b) articulated knowledge and designed situations related to the nature and properties of real numbers and the notion of limit. Through the development and use of approximation methods, two situations allowed students to connect the intuitive, perceptual and formal dimensions of the limit concept productively. The aim was to enlarge the experimental field of students concerning the nature of real numbers and their appearance and hence develop conceptualisation from both natural and formal thinking. Two approximation methods were used as experimental situations: the construction of the better rational approximation of $\sqrt{2}$ and, if possible, its generalisation to other irrationals; and the cosine fixed point. The conclusion was that the irrational numbers situation gave a status to numbers that students have only considered as “notations”, while the cosine fixed point situation gives access to real numbers that we cannot make explicit, and consequently requires the implementation of formal procedures. Bridoux’s (2010) study designed and implemented a succession of situations for introducing the notions of interior and closure of a set and open and closed set, after identifying the FUG characteristics of these notions. This example of didactical engineering used meta-mathematical discourse and graphical representations to assist students to develop an intuitive insight into these notions that would then allow the teacher to characterise them in a formal language. The notion of

completeness was examined by Bergé (2008, 2010), who considered student thinking about the concept as they progress in their undergraduate career. Analysing whether they have an operational or conceptual view of completeness, or if it is something taken for granted, she concludes that many students have a weak understanding that does not include ideas such as: \mathbf{R} is the set that contains all the suprema of its bounded above subsets; Cauchy sequences come from the necessity of characterizing the kind of sequences that ‘must’ converge; and completeness is related to the issue whether a limit is guaranteed to lie in \mathbf{R} .

One aspect of transition highlighted by the ATD is that praxeologies exist in relation to institutions. Employing the affordances of ATD, and prior educational research, Praslon (2000) showed that by the end of high school in France a substantial institutional relationship with the concept of derivative is already established. Hence, for this concept he claims that the secondary-tertiary transition is not about intuitive and proceptual perspectives moving towards formal perspectives, as TWM might suggest, but is more complex, involving an accumulation of micro-breaches and changes in balance according several dimensions (tool/object dimensions, particular/general objects, autonomy given in the solving process, role of proofs, etc), something that university academics are not very sensitive to. Building on this work Bloch (2004) identified nine factors contributing to a discontinuity between high school and university in analysis. Further, using ATD Bosch, Fonseca and Gascón (2004) show the existence of strong discontinuities in the praxeological organization between high school and university, and build specific tools for qualifying and quantifying these. Another interesting anthropological contribution is that of Bergé (2008) who investigated the evolution of students’ relationships with real numbers and the idea of completeness, and linked these relationships with the characteristics of the different courses where students meet these notions and work with them. Also employing an institutional approach, Dias, Artigue, Jahn and Campos (2008) conducted a comparative ATD study of the secondary-tertiary transition in Brazil and France, using the concept of functions as a filter. They looked at the personal relationships developed by students with the concept of function and the continuities and discontinuities between teaching practices in secondary and tertiary institutions in the two countries. The analysis of institutional relationships considered evaluations used for the selection of students at university entrance or developed by specific universities. A typical task in Brazil involves the determination of terms of arithmetic and geometric sequences, with the associated praxeologies based on algebraic techniques and technology. A typical task in France is the study of the convergence of such sequences both qualitatively and quantitatively, with the associated praxeologies being the use of analytic techniques and technology, with a higher level of student guidance through hints and intermediate questions in France than in Brazil. The authors concluded that although contextual influences tend to remain invisible there is a need for those inside a given educational system to become aware of them in order to envisage productive collaborative work and evolution of the system. One crucial aspect of the institution is the teaching practice of the lecturers, and Smida and Ghedamsi (2006) studied the teaching practices of first year real analysis in mathematics/informatics courses in a Tunisian university. They distinguish two kinds of teaching projects leading to two different models of teaching practices: those where axiomatic, structures and formalism are the discourse that justify and generate the expected

knowledge and know-how (this model follows only mathematical logic); and projects where the variety of choices for proving, illustrating, applying or deepening the mathematical results highlights a declared intent – by teachers – to enrol in a constructivist setting (this model combines the logic of mathematics and cognitive demands). A questionnaire given to 57 lecturers from 4 universities highlighted 3 groups of lecturers: those with a logico-theoretical profile, who do not take into account cognitive demands (more or less 40%); those with a logico-constructivist profile, who have some cognitive concern (more or less 35%); and those who take into account cognitive demands (more or less 25%). However, 80% of the lecturers report hardly ever, or never, giving students tasks that lead them to formulate a conjecture. Further, more than 90% of the lecturers do not consider the proof in analysis as a means of convincing students of the validity of mathematical statements, and almost 60% do not consider proof as a priority, as a logico-theoretical tool for validation.

Some researchers consider possible ways to assist the transition. For example, Gyöngyösi, Solovej and Winsløw (2010, 2011) report an experiment using Maple CAS-based work to ease the transition from calculus to real analysis in Denmark. Using a combination of theoretical frameworks to study transition (an adaptation of ATD by Winsløw, Winsløw's semiotic representation, Artigue's notions of epistemic and pragmatic value, and Trouche's instrumental orchestration), they give examples of praxeologies to be developed by students and teachers and analyse them according to their *pragmatic value* (efficiency of solving tasks) and *epistemic value* (insight they provide into the mathematical objects and theories to be studied). They conclude that the use of instruments changes the kinds of mathematics students do, and those with an overall lower performance also commit more errors when using instrumented techniques. In a similar vein, Biehler, Fischer, Hochmuth and Wassong (2011) propose that blending traditional course attendance with systematic e-learning study can facilitate the bridging of school and university mathematics. Using a detailed calculus-based study, Farmaki and Paschos (2007) proposed that the transition from intuitive assumptions to mathematical argumentation of a first year student could suggest teaching interventions to develop students' intuitive strengths in a controlled manner, and hence promote formal mathematical thinking. Using graphing calculator technology in consideration of the Fundamental Theorem of Calculus, Scucuglia (2006) made it possible for the students to become gradually engaged in deductive mathematical discussions based on results obtained from experiments. Another approach suggested by Chorlay (2009) is to turn to the history of mathematics. His historical study of the different viewpoints on functions in elementary and non-elementary mathematics in the 19th century allowed him to formulate a series of hypotheses about the long-term development of functional thinking in the transition from secondary to university.

In our survey the mathematicians were asked whether students were well prepared for calculus study. Those whose students did not study calculus at school rated their students' preparation for calculus at 2.1 out of 5. Those whose students did, rated secondary school calculus as preparation to study calculus at university at 2.4, and as preparation to study analysis at university at 1.5. These results suggest that there is some room for improvement in school preparation for university study of calculus and analysis.

Generalised Forms of Arithmetic and Abstract Algebra

Understanding the constructs, principles, and eventually axioms, of the algebra of generalised arithmetic could be a way to assist students in the transition to study of more general algebraic structures. With a focus on students' work on solving a parametric system of simultaneous equations and the difficulties they experience with working with variables, parameters and unknowns, Stadler (2011) describes students' experience of the transition from school to university mathematics as an often perplexing re-visiting of content and ways of working that seems simultaneously both familiar and novel. Using a perspective that is discursive and enculturative, largely based on Sfard's commognition, the paper illustrates the multi-faceted nature of transition from school to university mathematical discourse through the extensive examination of a selected episode. While the analysis of the episode as a case illustrating several facets of the transition (individual, institutional, social) is not totally convincing (the students' difficulty with variables, parameters and unknowns is palpable and slightly overshadows the other aspects), the impression is that Sfard's perspective is a good match for studies of transition. The constructs of number, symbolic literals, operators, the '=' symbol itself, and the formal equivalence relation, as well as the principles of arithmetic, all contribute to building a deep understanding of equation. Godfrey and Thomas (2008), using the TWM framework, provide evidence that many students have a surface structure view of equation and hence fail to integrate the properties of the object with that surface structure. One example they provide concerns the way in which an embodied input-output, procedural, or operational, view of equation persists for approximately 25% of secondary school students, even when they reach university level. They also point out that equivalence is not well understood by school students, and that the reflexive, symmetric, and transitive properties forming an equivalence relation are rarely considered in schools, even though they are often assumed.

Students' encounter with abstract algebra at university marks a significant point in the transition to advanced mathematical formalism and abstraction. Topics such as group theory are characterised by deeper levels of insight and sophistication (Barbeau, 1995) and ask of students a commitment to what is often a fast-paced first encounter in lectures (Clark et al., 1997). Key to this encounter is the realisation of the need to "think selectively about its entities, paying attention to those aspects consistent with the context and ignoring those that are irrelevant." (Barbeau, 1995, p. 140). As Hazzan (1999) notes, students' difficulty with abstract algebra can be attributed to the novelty of dealing with concepts that are introduced abstractly, defined and presented by their properties along with an examination of what facts can be determined from these properties alone (Hazzan, 1999). Furthermore the way that students approach proof writing, and the type of practices and beliefs that they bring to the task often exacerbates some of this difficulty (Powers, 2010; Weber, 2001). In research spanning mathematical topics, but focusing on examples from group theory and linear algebra, Nardi (2011) makes use of Sfard's commognition perspective, and data from Nardi (2008), to analyse university mathematicians' comments on new Year 1 students' verbalisation skills. She notes: the role of verbal expression to drive noticing; the importance of good command of ordinary language; the role of verbalisation as a semantic mediator between symbolic and visual mathematical expression; and the precision proviso for the use of ordinary language in

mathematics. One observation that emerges from the analysis is that discourse on verbalisation in mathematics tends to be risk-averse and that more explicit, and less potentially contradicting, pedagogical action is necessary in order to facilitate students' move away from often wordless mathematical expression in school and appreciation of mathematical eloquence.

Below we summarise results from some studies that focus on the difficulties students experience in their first encounters with key concepts in abstract algebra – and a few that touch on pedagogical insights emerging from our understanding of these difficulties. As mentioned above, students' skills in proof production are central to the quality of their first encounter with Group Theory. According to Hart (1994):

- Students' conceptual schemas is the key element in the success of problem solving in Group Theory;
- Students' overreliance upon concrete examples of groups often causes operation confusion;
- The ability to translate concrete representations is critical in the students' proof production (as is the overreliance on concrete representations)
- Students need to learn how to apply domain-specific proving strategies

Dubinsky et al. (1994) made the first comprehensive attempt to explore student encounters with fundamental concepts of Group Theory (group, subgroup, coset, normality and quotient group). Written largely in the language of APOS, the study marked the importance of students' understanding of the process-object duality of mathematical concepts as a prerequisite for understanding in Group Theory. It highlighted the importance of the concept of function in building group-theoretical understanding, and identified specific issues of difficulty such as confusing normality with commutativity. Cosets and normality were also identified as major stumbling blocks in the early stages of students' learning.

As a particular, and important, form of the concept of function, the concept of group isomorphism has attracted attention in several studies. For example, Leron et al (1995) distinguished between students' *naive* and *formal* conceptualisations of *isomorphism* through an elaborate discussion of student attempts to distinguish isomorphic relations between two groups and isomorphism; to prove that a certain function is, or not, an isomorphism; to work with isomorphisms in the abstract or in concrete cases. The results highlighted students' difficulties to link isomorphic relations with group orders; to distinguish between properties of group elements and properties of groups', and to construct isomorphisms between certain groups.

Lagrange's theorem is another topic from the introductory parts of Group Theory that has attracted attention in several studies. For example, Hazzan and Leron (1996) noted that students may use theorems such as this (particularly those with recognisable names to them) as slogan-style references in their proofs (in their data students use Lagrange's theorem or some version of its converse where not appropriate or relevant to the problem and use the theorem and its converse, or 'naïve' versions of the converse, indistinguishably).

Other important introductory elements of Group Theory were treated in two papers published in the late 1990s in the *Journal of Mathematical Behavior*. Brown et al. (1997) focused on binary operations, groups and subgroups and Asiala et al. (1997) looked at cosets, normality and quotient groups. Soon after, an analogous report by Asiala et al. (1998) examined understanding of permutations of a finite set and symmetries of a regular polygon. Once again emphasis, in the context of groups of symmetries and dihedral groups, was on the need to facilitate students' transition to object understandings of key notions in Group Theory.

In addition to a consideration of fundamental Group Theory concepts, some studies have focused on issues such as the relationship between visual and analytic thinking (VA), and, largely, the need for both (Zazkis & Dubinsky, 1996). In these authors' VA model, whether external or internal, visual representations are in a constant interplay with analytical ones. Eventually it is of little concern whether the emerging complex construct is visual or analytic as the elements of both types of thinking have merged into it effectively. In resonance with the VA proposition by Zazkis and Dubinsky, Hazzan (1999) explored how students attempt to cope in Group Theory through reducing its high levels of abstraction. In a related paper Hazzan (2001) examines these attempts at reducing levels of abstraction in the context of a problem that asked students to construct the operation table for a group of order four.

Mirroring many of the difficulties outlined generally in the above in her analyses of student responses to introductory Group Theory problem sheets, Nardi (2000) identified students':

- difficulties with the static and operational duality within the concept of order of an element as well as the semantic abbreviation contained in $|g|$;
- often problematic use of 'times' and 'powers of' in association with the group operation;
- ambivalent use of geometric images as part of meaning bestowing processes with regard to the notion of coset;
- problematic conceptualization of multi-level abstractions embedded in the concept of isomorphism.

The duality underlying the concept of group, and the role of binary operation in this concept, were also discussed later by Iannone and Nardi (2002) who offered evidence of the students' tendency to: consider a group as a special kind of set, often ignoring the binary operation that is fundamental to its entity; consider the axioms in the definition of a group as properties of the group elements rather than the binary operation; and omit checking those axioms that they perceive as obvious (e.g. in some cases associativity). In addition, doctoral research by Ioannou, (see Ioannou & Nardi, 2009a, b; 2010; Ioannou & Iannone, 2011) considers students' first encounter with abstract algebra, focusing on the Subgroup Test, symmetries of a cube, equivalence relations, and employing the notions of kernel and image in the First Isomorphism Theorem. Provisional conclusions are that students' overall problematic experience of the transition to abstract algebra is characterised by the strong interplay between strictly conceptual matters, such as the ones addressed above, affective issues and those that are germane to the wider study skills and coping strategies that students arrive at university with.

Most of the above studies offer some pedagogical insight into how teaching can facilitate students' transition to this most abstract and formal topic in mathematics. Briefly, included in these recommendations is reversing the order of presentation, using examples and applications to stimulate the discovery of definitions and theorems through permutation and symmetry (Burn, 1996; 1998). Many of these were similarly re-stated by Larsen (2009) as presenting a series of tasks that explore the symmetries of an equilateral triangle and culminate in negotiating preliminary understandings of the order of a group and isomorphism, as well as getting students to construct multiplication tables for groups of small order as a stepping stone for an understanding of the general notion of group structure (Thrash & Walls, 1991). Other ideas are to start with an interactive approach involving computer-based experimentation with group structures, followed by a more formal introduction (Leron & Dubinsky, 1995), to use a set of group-work activities that take the place of the formal introduction in the lectures and stimulate students' to take over responsibility for learning (Cnop & Grandsard, 1998) or encourage independent study of proofs in Group Theory through carefully prepared workbooks (Alcock et al., 2008).

Linear Algebra

A sizeable amount of research in linear algebra has documented students' transition difficulties, particularly as these relate to students' intuitive or geometric ways of reasoning and the formal mathematics of linear algebra (Dogan-Dunlap, 2010; Gueudet-Chartier, 2004; Harel, 1990). Related to this work, Hillel (2000) constructed a theoretical framework for understanding student reasoning in linear algebra, and identifying three modes of description: geometric, algebraic, and abstract. Hillel found that the geometric and algebraic modes of relating to vectors and vector spaces could become obstacles for understanding the abstract modes because they limited the amount of generality that a student could draw from either geometric or algebraic examples. Corriveau (2009) suggests that one of the challenges of the transition from secondary algebra to university linear algebra is that the formalism obstacle appears when students work with expressions, losing sight of the mathematical objects that the symbols represent. Hence, when a new algebra (e.g., matrix algebra, etc.) is introduced as a tool for calculation for algorithmisation of procedures and reasoning through calculations and their rules, then students have to accept delegation of parts of the control of validity and meaning to this algebra, leading to a loss of control and meaning.

Wawro, Sweeney, and Rabin (2011) analyzed the ways that students used different modes of representation in making sense of the formal notion of subspace. Specifically, the authors studied the relationship between students' understanding of the definition of subspace and their concept images. In the study, students demonstrated a variety of ways of engaging with the formal definition and showed that they utilized geometric, algebraic and metaphoric ways of relating their concept image and the definition. The results of the study suggest that in generating explanations for the definition, students rely on their intuitive understandings of subspace. These intuitive understandings can be problematic, as in the case of seeing \mathbf{R}^2 as a subspace of \mathbf{R}^4 , but they can also be very powerful in developing a more comprehensive understanding of subspace.

In addition to the geometric mode of reasoning that Hillel references, problems with the symbolic notation of linear algebra have also been studied. Harel and Kaput (1991), for example, demonstrated that students have difficulties in generating relationships between many of the formal and algebraic symbols used in linear algebra and the conceptual entities that they are intended to represent. In examining students' decisions about whether a given set was in fact a vector space, the authors demonstrated that students who related to the vector space as a conceptual idea were better able to reason about whether a given set was a vector space than those who procedurally checked the axioms against the new set. Because symbols in advanced mathematics in general, and in linear algebra in particular, connect so many different ideas (e.g., formal notions, systems of equations, vector systems, etc.), developing an understanding of what a symbol represents conceptually is crucial to understanding linear algebra as a whole. Further evidencing students' difficulties with symbols in linear algebra, Britton and Henderson (2009) demonstrated that students had difficulties in dealing with the notion of closure. Specifically, the students had problems in moving between a formal understanding of subspace and the algebraic mode in which a problem was stated. These authors argued that student difficulties stemmed from an insufficient understanding of the various symbols used in the questions and in the formal definition of subspace.

Dorier, Robert, Robinet and Rogalski (2000a) expressed concern that in the French secondary school system the strong emphasis on algebraic concepts in linear algebra leaves little room for set theory and elementary logic. They contend that this absence leads to difficulty in working with the formal aspects of linear algebra. For example, students are often unable to reason with definitions and abstract concepts. Dorier, Robert, Robinet, and Rogalski (2000b) and Rogalski (2000) take an approach to dealing with these problems that involves teaching linear algebra as a long term strategy, having students revisit problems in a variety of different settings—geometric, algebraic, and formal. It also involves what the authors call the *meta-lever* in which students reflect on their activity in order to draw connections between the various settings and to build generalizations.

Another dimension of research deals with the relationships between linear algebra and geometry. These relationships were at the core of the doctoral thesis by Gueudet (Gueudet, 2004). Her habilitation dissertation (Gueudet, 2008a, b) synthesises ten years of research in that area and identifies specific views on students' difficulties, in the secondary-tertiary transition in linear algebra, resulting from different theoretical perspectives. The epistemological view leads to a focus on linear algebra as an axiomatic theory, which is very abstract for the students. Focusing on reasoning modes leads her to identify the need, in linear algebra, for various forms of flexibility, in particular flexibility between dimensions.

Other efforts to improve student learning include the work of Klapsinou and Gray (1999), who studied a course in which students were first given concrete instantiations of linear algebra concepts and then used those to generate understanding of the formal definitions of the concepts. The authors noted that students who were taught in this manner later had difficulty with understanding the definition and applying it to different situations. They argue that taking a computational approach and then developing the abstractions refines students' processes for doing computation in linear algebra, but not their understanding of certain

concepts as objects. Portnoy, Grundmeier and Graham (2006), in a study of pre-service teachers in a transformational geometry course, demonstrated that students who had been utilizing transformations as processes that transformed geometric objects into other geometric objects had difficulty writing proofs involving linear transformations. The authors argued that the process nature of students' understanding of transformation contributed to their understanding of the concept in general, but they may not have developed the necessary object understanding for writing correct proofs.

Other efforts to improve the learning and teaching of linear algebra have drawn on APOS theory, focusing on a variety of concepts including linear independence and dependence (Bogomolny, 2008). Recently, Stewart and Thomas (2009, also Thomas & Stewart, 2011) used a framework employing APOS theory in conjunction with TWM to analyze students' understanding of various concepts in linear algebra, including linear independence and dependence, eigenvectors, span and basis. In a series of studies, the authors found that students did not think of many of these concepts from an embodied standpoint, but instead tended to rely upon an action/process oriented, symbolic way of reasoning. Stewart and Thomas (2007) also conducted a study of two groups of linear algebra students. They employed a course in which the students were introduced to embodied, geometric representations in linear algebra along with the formal and the symbolic. The authors claim that the embodied view enriched students' understanding of the concepts and allowed them to bridge between concepts more effectively than employing just symbolic processes. In another study, Stewart and Thomas (2010) demonstrated that students viewed basis from an embodied perspective as a set of three non-coplanar vectors, symbolically, as the column vectors of a matrix with three pivot positions, and formally, as a set of three linearly independent column vectors. The students in this study, however, tended mostly toward a symbolic process-oriented matrix manipulation view for most concepts and hence did not attain the conceptually richer geometric aspects of linear algebra.

In order to address students' difficulties in bridging the many representational forms and the variety of concepts present in linear algebra, some researchers have turned to computers for aid in teaching (e.g., Berry, Lapp, & Nyman, 2008; Dogan-Dunlap & Hall, 2004; Hillel, 2001). Dreyfus, Hillel, and Sierpinska (1998) postulated that a geometric but coordinate-free approach to issues such as transformations and eigenvectors may be helpful in coming to understand these concepts. The authors found that the use of a computer environment and tasks enabled students to develop a dynamic understanding of transformation, but that it hindered their ability to understand transformation as relating a general vector to its image under the transformation. In another study the authors (Sierpinska, Dreyfus, & Hillel, 1999) investigated how students determined if a transformation was linear or not using Cabri. In this study, the researchers discovered that students made determinations about a transformation's linearity based upon a single example. Thus, using only one image of $k\mathbf{v}$ under the transformation they checked if for a vector, \mathbf{v} , a scalar k , and a transformation T , whether $T(k\mathbf{v})=kT(\mathbf{v})$, and did not vary \mathbf{v} using the program's capabilities. Recently, Meel and Hern (2005) created a series of interactive applets using Geometer's Sketchpad and *JavaSketchpad* to teach linear algebra. Their intention in developing and using these tools was "to help students experience the mathematics and then lead them to examine additional examples that

help them recognize the misinterpretation or mis-generalization” (*ibid*, p. 7). From anecdotal evidence, the authors noted that these activities have been largely successful in accomplishing this task.

More recently, different research teams have been spearheading innovations in the teaching and learning of linear algebra. Cooley, Martin, Vidakovic, and Loch (2007) developed a linear algebra course that combines the teaching of linear algebra with learning about APOS theory. By focusing on a theory for how mathematical knowledge is generated, students were made aware of their own thought processes and could then enrich their understanding of linear algebra accordingly. Other researchers have been working with Models and Modeling (Lesh & Doerr, 2003) and APOS to develop instruction that leverages students’ intuitive ways of thinking to teach linear algebra. For example, Possani, Trigueros, Preciado, and Lozano (2010) utilized a genetic composition of linear independence and dependence and systems of equations in order to aid in the creation of a task sequence. The task sequence, which asked students to model the coordination of the traffic flow in a particular area of town, was designed to present students with a problem that they could first mathematise and then use to understand linear independence and dependence.

In the United States, another group of researchers has been drawing on sociocultural theories (Cobb & Bauersfeld, 1995) and the instructional design theory of Realistic Mathematics Education (Freudenthal, 1973) to explore the prospects and possibilities for improving the teaching and learning of linear algebra. Using a design research approach (Kelly, Lesh, & Baek, 2008), these researchers are simultaneously creating instructional sequences and examining students’ reasoning about key concepts such as eigenvectors and eigenvalues, linear independence, linear dependence, span, and linear transformation (Henderson, Rasmussen, Zandieh, Wawro, & Sweeney, 2010; Larson, Zandieh, & Rasmussen, 2008; Sweeney, 2011). For example, these authors examined students’ various interpretations of the equation $A \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$, where $\begin{bmatrix} x \\ y \end{bmatrix}$ is a vector and A is a 2×2 matrix prior to any instruction on eigentheory. They identified three main categories of student interpretation and argue knowledge of student thinking prior to formal instruction is essential for developing thoughtful teaching that builds on and extends student thinking. This group has also begun to disseminate studies on the sequences of tasks for developing student reasoning of basis and constructing understanding of vectors, vector equations, linear dependence and independence and span. For example, Wawro, Zandieh, Sweeney, Larson, and Rasmussen (2011) report on student reasoning as they reinvented the concepts of span and linear independence. The reinvention of these concepts was guided by an innovative instructional sequence that began with vector equations (versus systems of equations, that most introductory texts employ) and successfully leveraged students’ intuitive imagery of vectors as movement to develop formal definitions. This more recent work challenges some of the earlier findings that students’ intuitive ways of reasoning are an obstacle to induction into formal mathematics.

Drawing on the TDS and ATD frameworks de Vleeschouwer and Gueudet (2011) put forward the view that some of the difficulties students experience may originate in the institutional experiences they have been offered (e.g. tasks). However, they observe that students can learn to appreciate the duality in linear forms (process-object or, to these authors,

micro-macro) if given an appropriate set of tasks that require them to engage with these concepts at both levels. Their perspective is that of the changing didactical contract between school and university mathematics, particularly with regard to ways of approaching mathematical content (and less of the more common research foci on more general aspects of the students' mathematical learning experiences such as teacher expectations, attitudes to proof etc.).

Discrete Mathematics

Discrete mathematics deals with finite or countable sets, bringing into play several overlapping domains, e.g. number theory, graph theory, and combinatorial geometry (Grenier, 2011). It occupies a rather variable place in mathematics education; in some countries, only a very small number of discrete mathematics concepts are taught, often those related to combinatorics and the basics of number theory. Discrete mathematics can be introduced, either as a mathematical theory, or as a set of tools to solve problems (a graph is a basic and intrinsic modelling tool). For example, mathematical games are often based on problems in discrete mathematics. We present here three contributions that illustrate how discrete mathematics can be used in mathematics in both high school and university for addressing important issues in the transition such as the nature and elaboration of mathematical definitions, and reasoning modes such as for instance reasoning by induction, necessary and sufficient conditions. Using contexts such as the Königsberg's bridges problem is suggested by Cartier and Moncel (2011) as a way to provide access to fundamental mathematical concepts like proofs, necessary and sufficient conditions, and modelling techniques. The elaboration of definitions was the topic of Ouvrier-Buffet (2011). She suggests that encouraging students to work on skills such as defining, proving and conceiving new concepts through various discrete mathematics concepts, such as trees, discrete straight lines and properties of displacements on a regular grid to generate knowledge and concepts. These types of activity give rise to specific reasoning modes and the potential for construction of new tools, such as coloring, proof by exhaustion of cases, proof by induction, and use of the Pigeonhole principle (Grenier, 2001, 2003). In this manner misconceptions that persist in the knowledge of many students as they are in transition to university, such as inaccurate knowledge about mathematical induction, may be addressed.

Logic and Proof

The difficulties met by transition students concerning logic are well recognized by teachers and mathematics educators around the world. In France, research on the role of logic in the learning and teaching of mathematics, and more specially proof and proving, has been developed since the eighties. Durand-Guerrier (2003), as well as Deloustal-Jorrand (2004) and Rogalski and Rogalski (2004) point out the importance of taking in account quantification matters in order to analyse difficulties related to implication, and more generally mathematical reasoning. In the same vein, in a Tunisian context, Chellougui (2004, 2009) investigates the use of quantification by new university students in Tunisia. Her didactic analysis of textbooks and course notes concerning upper limit, as well as an interview with pairs of students in a problem-solving situation, revealed, on the one hand, the didactic phenomena related to the alternation of the two types of quantifiers and, on the other

hand, difficulties in mobilizing the definition of the objects and the structures, which illustrate a major problem in the conceptualization process. These authors, as well as Durand-Guerrier and Arsac (2003, 2005) acknowledge that the importance of these questions seems to be largely underestimated by teachers at secondary school as well as at university level, as it appears in particular in textbooks. Durand-Guerrier and Arsac (2003, 2005) highlight the fact that a main challenge for novices is to develop together mathematical knowledge and logical skills, which are closely intertwined. Durand-Guerrier (2005) supports the relevance of a model theoretic point of view for analysing proof and proving in mathematics. These pieces of research concern mostly written mathematical discourse. In order to study deeply the oral interaction in argumentation and proof, Barrier (2009a, 2009b) introduces a semantic and dialogic perspective as developed by Hintikka. This permits one to highlight the importance of moving back and forth between syntax and semantics in the proving process in advanced mathematics (e.g. Blossier, Barrier & Durand-Guerrier, 2009). This research, together with research in other areas, calls for developing programmes allowing new university students to master the logical competencies required for the learning of advanced mathematics. The role of acquiring these competencies in a way that is similar to second language learning is developed by Durand-Guerrier and Njomgang Ngansop (2011), in a continuation of the work of Ben Kilani (2005) at secondary level.

A previous ICME survey report on proof (Mariotti et al., 2004) raised a number of questions that relate to transition issues. Among these were: “Is proof so crucial in the mathematics culture that it is worthwhile to include it in school curricula?”; “What are the meanings of proof and proving in school mathematics and how are these meanings introduced into curricula in different countries?”. Important aspects include: students’ conceptions of proof, students’ performance in proof tasks; teachers’ conceptions of proof; and, how research in mathematics education has approached the issue of proof. Of particular interest has been the question “is it possible to overcome the difficulties in introducing pupils to proof so often described by teachers?” (Mariotti et al., 2004, p. 184).

The key difference between school and university, which is expressed as a possible rupture, is that schools focus on argumentation while universities consider deductive proof (Mariotti et al., 2004, p. 193). Iannone and Inglis (2011) discuss a range of weaknesses in newly arriving Year 1 mathematics students’ production of deductive arguments (rather than in the oft-reported perception that a deductive argument was expected of them). Specifically, Year 1 mathematics students responded to four proof tasks and while they demonstrated a range of weaknesses in their production of deductive arguments, they were aware that when asked to generate a proof, they should provide a deductive argument. This is in some contrast to previous work in the field but this contrast may be accounted for by different student background and specialisms in the student sample.

In a translation of his own paper (Balacheff, 1999), Balacheff argues for the notion of *Cognitive Unity* (Boero, Garuti & Mariotti, 1996) as a potential bridge between them, saying “I would summarize in a formula the place that I find possible for argumentation in mathematics, according to the notion of Cognitive Unity as it was introduced by our Italian colleagues: argumentation relates to conjecture, like proof does to a theorem” (Mariotti et al.,

2004, p. 194). The survey report further proposed that “Research studies concerning the analysis of argumentation processes and their comparison with the production of mathematical proof appear to be very promising” (Mariotti et al., 2004, p. 201). The suggestion by Heinze and Reiss (2003) was that schools should move students away from inductive arguments toward formal argumentation. More recently, Antonini and Mariotti (2008) have applied the Cognitive Unity framework to the application of indirect proofs, such as contradiction and contraposition. They suggest that this provides a perspective, taking into account both epistemological and cognitive considerations, from which one may observe the relationship between argumentation and proof by focusing on analogies, without forgetting differences.

The 2004 survey report further recommended a cautious approach, suggesting that the inclusion of proof in the school or university curriculum is only a first step, and it is important to ensure that the goals for doing so should be clarified, along with processes for how they will be operationalized (Mariotti et al., 2004). In the years since that report there have been many studies considering the role of proof, both at school and university. However, there appear to have been few studies directly addressing proof as an issue of transition (we note that at the time of writing the book *Proof and Proving in Mathematics Education: The 19th ICMI study* (Hanna & de Villiers, 2012) was still in print). While this is the case, the research does point out some of the key differences between approaches to proof in school and in university and makes suggestions for pedagogical approaches that might assist in the transition. In this section we draw on some of these aspects of proof studies.

One theoretical perspective that may prove useful in considering the role of proof in transition is that of Harel (2008a, b), who proposes a framework called DNR-based instruction, which involves duality (D), necessity (N) and repeated reasoning (R). In this he distinguishes between ways of understanding, a generalisation of the idea of proof, and ways of thinking, which generalises the notion of proof scheme, but also includes problem solving approaches and beliefs about mathematics. In general, proof schemes are present at school, while learning and understanding in university is via proofs. One of the principal implications of defining mathematics as comprising both aspects is “that mathematics curricula at all grade levels, including curricula for teachers, should be thought of in terms of the constituent elements of mathematics—ways of understanding and ways of thinking—not only in terms of the former, as currently is largely the case.” (Harel, 2008, p. 490). However, such a definition of mathematics is consistent with mathematicians’ practice of mathematics, but not with their perception of it. There is a fundamental difference between the way mathematicians perceive mathematics and the way they practice it in their research. One reason for this may be, as Hanna and Janke (1993 – cited in Balacheff, 2008) hypothesise, that “Communication in scholarly mathematics serves mainly to cope with mathematical complexity, while communication at schools serves more to cope with epistemological complexity.” (Balacheff, 2008, p. 433).

According to Solomon (2006), enabling students to access academic proof processes in the transition from pre-university to undergraduate mathematics is a question of understanding and building on students’ own pre-existing epistemological resources in order to foster an

epistemic fluency that will allow them to recognize, and engage in, the process of creating and validating mathematical knowledge. Since transition involves maturation and its accompanying changes in thinking, Tall and Mejia-Ramos (2006) apply TWM to outline the changes in proof types that they suggest occur as students become more mathematically sophisticated. Firstly, in the embodied world, the individual begins with physical experiments to find how things fit together. Then in the symbolic world, arguments begin with specific numerical calculations and develop into the proof of algebraic identities by symbolic manipulation. However, it is only in the formal world where proof by formal deduction occurs. Thus as students develop cognitively, moving through the three worlds, their argument warrants (Toulmin, 1958) change, and the hope is that formal proof will become the only acceptable warrant. Tall (2004) refers to this as moving through the ‘three worlds’ of mathematics and characterizes development through the worlds, which impinges on production of proof schemes, as a move from perception and action, through operation and symbolism, to reason and formality (Tall, Yevdokimov, Koichu, Whiteley, Kondratieva & Cheng, 2012). Pinto and Tall (2002) also describe *natural thinking* as using thought experiments based on embodiment and symbolism to give meaning to definitions and to suggest possible theorems for formal proof.

Among the recommendations for pedagogical change that would have implications for transition is the point made by Balacheff (2008) and others (eg Hanna & de Villiers, 2008; Hemmi, 2008) that there is a need for more explicit teaching of proof, both in school and university. Some, (e.g., Stylianides & Stylianides, 2007; Hanna & Barbeau, 2008) argue for it to be made a central topic in both institutions. One reason given by Hanna and Barbeau (2008) is that, apart from their intrinsic value, proofs may display fresh methods, tools, strategies and concepts that are of wider applicability in mathematics and open up new mathematical directions for students. One example they cite, applicable to transition, is that an algebraic proof of the formula for solving a quadratic equation introduces the technique of adding a term and then subtracting it again. Hence they argue that “...proofs could be accorded a major role in the secondary-school classroom precisely because of their potential to convey to students important elements of mathematical elements such as strategies and methods.” (Hanna & Barbeau, 2008, p. 352). One way to make proof more central in the school mathematics classroom, proposed by Heinze et al. (2008) is the use of heuristic worked-out examples as an instrument for learning proof. While these kind of examples are based on traditional worked-out examples, they make explicit the heuristics of the problem solving process. The research showed some success with low- and average-achieving students, but there was no significant effect for high-achieving students (*ibid*). However, if proof is made more central, Balacheff, (2008) cautions that teaching of mathematical proof “must not lead to an emphasis on the form, but on the meaning of proof *within* the mathematical activity.” (p. 506). Further, he maintains that to understand what proving is about requires the systematic organization of validation (eg control), communication (eg representation) and the nature of knowing. Three requirements for successful engagement with proof are also listed by Stylianides and Stylianides (2007): to recognize the need for a proof; to understand the role of definitions in the development of a proof; and the ability to use deductive reasoning.

Two potential difficulties in any attempt to place proving more prominently in the transition years are the role of definitions, and the problem of student met-befores (Tall & Mejia-Ramos, 2006). A desire to use definitions as the basis of deductive reasoning in schools is likely to meet serious problems, since, according to Harel (2008), this form of reasoning is generally not available to school students. In fact he claims "...it does not become an integral part of the repertoire of students' ways of thinking until advanced grades (if at all)... Understanding the notion of mathematical definition and appreciating the role and value of mathematical definitions in proving is a developmental process, which is not achieved for most students until adulthood." (Harel, 2008, p. 495).

Evidence for this is that when asked to define an invertible matrix, many linear algebra students stated a series of equivalent properties (e.g., "a square matrix with a nonzero determinant", "a square matrix with full rank", etc.) rather than a definition. The conclusion is that the provision of more than one such property indicates that they were not thinking in terms of a mathematical definition (Harel, 2008). A study by Hemmi (2008) agrees that students have difficulties understanding the role of definitions in proofs and lack experience of proving in their secondary school mathematics. She advocates a style of teaching that uses the principle of *transparency*, making the difference between empirical evidence and deductive argument visible to students. In this manner proof techniques, key ideas, structures of proofs could be taught at the same time as proof is used by the teacher and the students to verify convince and explain mathematics. Her study showed that for students many aspects of proof remained invisible and they often wondered exactly what constituted a proof, since there were no discussions about proof or proof techniques for students new to it. Adding transparency would avoid students being left to find out by themselves and judge if their solutions are correct, and why. A study by Cartiglia et al. (2004) showed that the cognitive influence of student met-befores (Tall & Mejia-Ramos, 2006) was strong, with the most recent met-before for university students, namely a formal approach, having a strong influence on their reasoning. Having formed the habit of using formal mathematical knowledge as the only resource for doing mathematics inhibited their ability to look for meaning in algebraic formulas.

Another possible difficulty is the form of teaching in schools. It has been suggested that one of the major differences between argumentation and mathematical proof that could lead teachers to advance mostly argumentation skills with little or no deductive reasoning is the need to distinguish between the status and content of a proposition (Duval, 2002; Harel, 2008). A potential way forward, proposed by Inglis, Mejia-Ramos, and Simpson (2007), is the use of the full Toulmin argumentation scheme, including its modal qualifier and rebuttal. Their research indicates "non-deductive warrant-types play a crucial role in mathematical argumentation, *as long as they are paired with appropriate modal qualifiers*... they retain the use of the warrants that have been used in previous 'worlds' or 'proof schemes,' but *they qualify them appropriately* (where appropriateness is defined by expert practice)." (Inglis, Mejia-Ramos, & Simpson, 2007, p. 17) This has possible implications for transition, since it would not be necessary for teaching to go straight to the use of formal deductive warrants.

A positive pedagogical approach to the teaching of proving proposed by a number of researchers (eg Kondratieva, 2010; Pedemonte, 2007, 2008) is student construction and justification of conjectures. Pedemonte's (2007) conclusion was that teaching of proof based on presentation of proofs to students and getting them to reproduce them, rather than to construct them, appears to be unsuccessful. Instead she highlights the need for open problems that ask for a conjecture, which appears to be a very effective way to introduce the learning of proof. She also discusses (Pedemonte, 2007, 2008) the relationship between argumentation and proof in terms of *structural distance*, moving from abductive, or plausible, argumentation to a deductive proof, where in the former inferences are based on content rather than on a deductive scheme. She argues for an abductive step in the structural argumentation (coming after a conjecture, to justify it), since it "could be useful in maintaining the connection between the referential system in the constructive argumentation [contributing to construction of a conjecture] and the referential system in the proof, because it could help students to maintain the meaning of numerical examples used to construct the conjecture and algebraic letters used in the proof." (Pedemonte, 2008, p. 390). In this way it is hoped that the abductive step would decrease the gap between the arithmetic field in argumentation and the algebraic field in proof, and thus assist transition.

Another pedagogical approach, presented by Kondratieva (2010), uses the idea of an interconnecting problem to get students to construct and justify conjectures. The problem should allow simple formulation, solutions at various levels, be solvable using tools from different mathematical branches, and appropriate for different contexts. The value of conjecture production has also been espoused (Antonini & Mariotti, 2008) during production of indirect proofs, such as by contradiction and contraposition. The research, using a Cognitive Unity approach, showed that the production of indirect argumentation can hide some significant cognitive processes. Hence, they propose that task of producing a conjecture offers students the possibility both of activating these processes and of constructing a bridge to overcome the gaps. The conclusion is that "...without any conjecturing phase, some gaps could not be bridged or could require sacrifices and mental efforts that not all the students seem to be able to make." (Antonini & Mariotti, 2008, p. 411).

Two possible strategies to prepare upper secondary school students for transition to the rigour of tertiary proofs suggested by Yevdokimov (2003) include: the value of intuitive guesses, and experience in what distinguishes a reasonable guess from one that is less reasonable; and a consideration of restrictions on statements and proofs. This idea of considering restrictions, which links to ideas about the status of a proposition (Duval, 2002; Harel, 2008), has led some to propose the idea of pivotal and bridging examples, and suggest that a strategy using counterexamples can assist students with proof ideas (Zazkis & Chernoff, 2008). These authors claim that one benefit of a counterexample is to produce cognitive conflict in the student, and a pivotal example is designed to create a turning point in the learner's cognitive perception (*ibid*). In a similar vein Stylianides and Stylianides (2007) state that counterexamples also foster deductive reasoning, since we make deductions by building models and looking for counterexamples. For Zazkis and Chernoff (2008) a counterexample is a mathematical concept, while a pivotal example is a pedagogical concept, and it is important that pivotal examples are within, but pushing, the boundaries of the student's

potential example space (Watson & Mason, 2005 – the examples students have experienced). The importance of developing mathematical thinking through extension of example spaces by the addition of examples and counterexamples has been advocated by Mason and Klymchuk (2009). Another way to expand students' example spaces, researched by Iannone et al. (2011), was based on Dahlberg and Housman's (1997) idea that getting students to generate their own examples of mathematical concepts might improve their ability to produce proofs. However, the results did not support the hypothesis that generating examples is a more effective preparation for proof production tasks than reading worked examples. These authors conclude that this may be because of the examples employed, and believe that there is currently insufficient guidance available on how to generate suitable examples effectively (Iannone et al., 2011). The role of examples also arose in research by Weber and Mejia-Ramos (2011) on how to read proofs. They looked at proof reading by mathematicians and found that they were mainly concerned with understanding the key ideas, the structure and the techniques employed. Hence they suggest that "One implication for the design of learning environments is that students might be taught how to use examples to increase their conviction in, or understanding of, a proof in the same way that the mathematicians in this paper described the ways that they read proofs." (Weber and Mejia-Ramos, 2011. p. 14). One of these ways is that they might see the value or insight that understanding a proof may provide for them personally.

A pedagogical strategy proposed by Yevdokimov (2003) is that a way to arouse interest and free students from the monotony of 'standard' problems is to give them questions such as to find the mistakes in a given proof. However, when students check for errors in proofs they should be directed to consider three aspects of the methodological knowledge, proof scheme, proof structure and chain of conclusions (Heinze & Reiss, 2003).

Regardless of the route taken, there has been a discussion (Alcock & Inglis, 2008, 2009; Weber, 2009) on the relative roles of syntactic and semantic reasoning in proof construction. However, this seems to hinge on the definition of a syntactic proof, whether all, or just most, of the reasoning occurs within the representation system of proof. Alcock and Inglis (2008, 2009) argue that there are different strategies of proof construction among experts, and hence we need to identify these in order to know what skills we need to teach students and how they can be employed. They propose a need for large-scale studies to investigate undergraduate proof production, and an extension of this to include upper secondary school could be beneficial for transition.

One specific kind of problem that may be a good introduction to proof in schools, as suggested by Harel (2008), is one involving proof by mathematical induction. However, he claims that this method of proving is often considered too quickly and the DNR framework suggests that a slower approach is necessary for understanding (Harel, 2001). The research by Palla, Potari and Spyrou (2011) suggests that induction can be taught in a meaningful way at the upper secondary level if students are given tasks that encourage them to focus on the critical properties of mathematical induction. In addition, Man-Keung Siu (2008) recommends the use of history to help students engage with proof, thus humanising it, placing it in a cultural, socio-political and intellectual context. In a similar vein Nagafuchi (2009)

presents some elements that would make a historical-philosophical approach possible for mathematical proofs in undergraduate courses and Furinghetti (2000) provided students with a historical presentation of ‘definition’ in an attempt to encourage flexibility, open-mindedness and motivation towards mathematics.

Our survey considered proof in several questions. In response to ‘How important do you think definitions are in **first** year mathematics?’ 52 (65.8%) replied that definitions are important in first year mathematics, while 15 presented their responses as neutral. Only 8 respondents replied that definitions are not important in first year mathematics. Responses to the question ‘Do you have a course that explicitly teaches methods of proof construction?’ were evenly split with 49.4% answering each of “Yes” and “No”. Of those who responded “Yes”, 15 (38.4%) replied that they teach methods of proof construction during the first year, 23 (58.9%) during the second year and 5 (12.8%) in either third or fourth year. While some had separate courses (e.g. proof method and logic course) for teaching methods of proofs, many departments teach methods of proofs traditionally, by introducing examples of proof and exercises in mathematics class. Some respondents replied that they teach methods of proof construction in interactive contexts, citing having the course taught as a seminar, with students constructing proofs, presenting them to the class, and discussing/critiquing them in small size class. One respondent used the modified Moore method in interactive lecture. Looking at some specific methods of introducing students to proof construction was the question ‘How useful do you think that a course that includes assistance with the following would be for students?’ Four possibilities were listed, with mean levels of agreement out of 5 (high) being: Learning how to read a proof, 3.7; Working on counterexamples, 3.8; Building conjectures, 3.7; Constructing definitions, 3.6. These responses appear to show a good level of agreement with employing the suggested approaches as components of a course on proof construction.

Mathematical Modelling and Applications

Blum et al. (2002) wrote in the Discussion Document of the 14th ICMI Study: “It is not at all surprising that applications and modelling have been – and still are – a central theme in mathematics education. Nearly all questions and problems in mathematics education, that is questions and problems concerning the learning and teaching of mathematics, affect, and are affected by, relations between mathematics and the real world.” This might be the reason why research on mathematical modelling and applications has attracted an increasing interest in recent years. This trend can be noted from the fact that there is a growing research literature focusing on the teaching and learning of mathematical modelling and applications published in various mathematics education journals. In addition, there are also several international conferences/events dedicated to the teaching and learning of mathematical modelling and applications.

The literature reports many studies and practices on the teaching and learning of mathematical modelling and applications, for both the secondary and tertiary levels. The primary focus of much research is on practice activities, e.g. on constructing and trying out mathematical modelling examples for teaching and examinations, writing application-oriented textbooks, implementing applications and modelling into existing

curricula or developing innovative, modelling oriented curricula (Blum et al., 2002). There are also extensive studies on clarifying modelling concepts, characterising the features of modelling processes, classifying the modelling tasks, and investigating what are and how to evaluate and improve the students' modelling competencies and sub-competencies required for each modelling process. However, it appears that no literature exists explicitly discussing this topic with a focus on the 'transition' from the secondary to the university levels. One reason might be that until now there have been no roadmaps to sustained implementation of modelling education at all levels. As Blum et al. (2002) point out the role of applications and mathematical modelling in everyday teaching practice is still rather marginal for all levels of education. The big issue seems to be whether, and if so how, this trend can be reversed to ensure that applications and mathematical modelling is integrated and preserved at all levels of mathematics education.

There is recent literature partially relevant to the secondary-tertiary transition issue and this is briefly considered here. One crucial duality, mentioned by Niss et al. (2007), is the difference between 'applications and modelling for the learning of mathematics' and 'learning mathematics for applications and modelling'. They point out that in lower secondary levels this duality is seldom made explicit, and instead both orientations are simultaneously insisted on. However, at upper secondary or tertiary level the duality is often a significant one. Their analysis suggests that for students to develop applications and modelling competency as one outcome of their mathematical education, these have to be put explicitly on the agenda of the teaching and learning of mathematics.

The close relationship between modelling and problem solving is taken up by a number of authors and reports (see, for example, *Focus in High School Mathematics: Reasoning and Sense Making*, NCTM, 2009 and the *Common Core State Standards for Mathematics*, National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). For example, English and Sriraman (2010) suggest that mathematical modelling is a powerful option for advancing the development of problem-solving in the curriculum. However, according to Petocz et al. (2007), there are distinct advantages to using real world tasks in problem solving. They note that well-designed learning tasks that model the way mathematicians work can encourage students towards broader conceptions of mathematics, enabling explicit connections between students' courses and the world of professional work. One difficulty described by Ärlebäck and Frejd (2010) is that upper secondary students (in Sweden) do not have much experience working with real situations and modelling problems, making the incorporation of real problems from industry in the secondary mathematics classroom problematic. One possible solution they suggest is closer collaboration, with representatives from industry working directly with classroom teachers. A second potential difficulty arose in a survey of 62 secondary mathematics teachers by Gainsburg (2008): teachers do not tend to make many real-world connections in teaching. Reasons given for this were that it would take more time than teachers feel they can spend on most mathematics topics, it isn't stressed in the curriculum or assessment, and teachers feel a need for more resources, ideas, or training about what real world connections to make. One possible solution to this, suggested by the German experience, is to bring together combinations of students, teachers and mathematicians to work on modelling problems. An

example of this approach is reported by Kaiser and Schwarz (2006) who describe their experience of modelling projects where prospective teachers together with upper secondary level students carry out modelling examples either in ordinary lessons or special afternoon groups. Further, Kaland et al. (2010) present experiences with modelling activities known as the “modelling week”, in which small groups of students from upper secondary level work intensely for one week on selected modelling problems, while their work is supported by pre-service-teachers. These activities are unique because they create a setting where pre-service teachers and upper secondary students are afforded the opportunity to work on authentic problems that applied mathematicians tackle in industry. In other studies on modelling activities, Heilio (2010) reports tertiary level experiences with a “modelling week” project for undergraduate students across Europe and Göttlich (2010) reflects experiences in conducting “modelling week” projects and modelling courses with students (especially secondary level and undergraduates) at the University of Kaiserslautern, describing how practical implementations can be performed. Another way to assist teachers proposed by Maaß (2010) is a scheme for modelling tasks that provides an overview of the different features of modelling tasks, thus offering guidance in task design and selection processes for specific aims and predefined objectives and target groups. According to Bracke (2010) his twenty years experience of modelling with students suggest that mathematical modelling should be integrated into teacher training, including the learning by doing component, training of the supervisor role and learning how to find problems. To achieve this he proposes including student teachers in organisation and implementation of modelling events in schools, as implemented at Technische Universität Kaiserslautern, with promising results.

Some difference between problem solving at school and university are identified by Perrenet and Taconis (2009), who investigated changes in mathematical problem-solving beliefs and behaviour of mathematics students during the years after entering university. They report significant shifts for the group as a whole, such as the growth of attention to metacognitive aspects in problem-solving or the growth of the belief that problem-solving is not only routine but has many productive aspects. The students explain these shifts mainly by the change in the specific nature of the mathematics problems encountered at university compared to secondary school mathematics problems, with the latter not succeeding in presenting an authentic image of the culture of mathematics with regard to problem-solving.

There is some agreement that there is a need to target curriculum changes in the upper secondary school to include more modelling activities. For example, in a summary discussion of perspectives on mathematical modelling and applications in upper secondary and tertiary levels, Stillman (2007) points out that high-stakes assessment at the upper secondary-tertiary interface is often seen as an unresolved problem for the infusion of modelling into the secondary curriculum at this level. Explaining that other imperatives are uppermost in the minds of teachers and students due to the pressure from the external examination system, she advocates authentic evaluation of current upper secondary assessment practices so future planning and policy can be based on actualities. Other possible initiatives in this direction were suggested by Stillman and Ng (2010), who recognised two different models of curriculum embedding intended to bring authentic real world applications into secondary school curricula. The first has a system-wide focus emphasising an applications and

modelling approach to teaching and assessing all mathematics subjects in the last two years of pre-tertiary schooling. The second model involves interdisciplinary project work from upper primary through secondary school with mathematics as the anchor subject. Another initiative presented by Maaß and Mischo (2011) is the framework and methods of the project STRATUM (Strategies for Teaching Understanding in and through Modelling), whose aim is to design and evaluate teaching units for supporting the development of modelling competencies in low-achieving students at the German Hauptschule. Also, in the USA, Leavitt and Ahn (2010) have provided a teacher's guide to implementation strategies for Model Eliciting Activities (MEAs), which are becoming more popular in secondary schools. Another arena that might prove helpful to students making the secondary-tertiary transition in mathematical modelling and applications is entry to contests in mathematical modelling and applications, available to both high secondary and tertiary students. Examples include HiMCM (The High School version of the Mathematical Contest in Modeling) for high school students, MCM (Mathematical Contest in Modeling) and ICM (Interdisciplinary Contest in Modeling) for undergraduate students. Each of these operated annually by the Consortium for Mathematics and its Applications (COMAP, see <http://www.comap.com>). In addition, there is CUMCM (Contemporary Undergraduate Mathematical Contest in Modelling) for undergraduate students (<http://en.mcm.edu.cn>). This international contest is operated annually by the Chinese Society for Industrial and Applied Mathematics (CSIAM) and each year there are more than 1,000 institutions and about 50,000 students participate in it (Xie, 2010).

Our survey addressed the topic of mathematical modelling in universities. In response to the questions “Does your university have a mathematical course/activity dedicated to mathematical modeling and applications?” Or “Are mathematical modelling and applications contents/activities integrated into other mathematical courses?”, 44 replied that their departments offer dedicated courses for modelling, while 41 said they integrate teaching of modelling into mathematics courses such as calculus, differential equations, statistics, etc and 7 answered that their university does not offer mathematics courses for mathematical modelling and applications. Among the reasons given for choosing dedicated courses were that: the majority of all mathematics students will end up doing something other than mathematics so applications are far more important to them than are detailed theoretical developments; most of the mathematics teaching is service teaching for non-majoring students so it is appropriate to provide a course of modelling and applications that is relevant to the needs of the target audience; and if modelling is treated as an add-on then students do not learn the methods of mathematical modeling. Those who chose integrated courses did so because: for modeling, students need to be equipped with a wide array of mathematical techniques and solid knowledge base. Hence it is appropriate for earlier level mathematics courses to contain some theory, proofs, concepts and skills, as well as applications.

Considering what happens in upper secondary schools, 26 (33%) reported that secondary schools in their location have mathematical modelling and applications integrated into other mathematical courses, with only 4 having dedicated courses. 44 (56%) said that there were no such modelling courses in their area. When asked for their opinion on how modelling should be taught in schools, most of the answers stated that it should be integrated into other

mathematical courses. The main reasons presented for this were: the many facets of mathematics; topics too specialised to form dedicated courses; to allow cross flow of ideas, avoid compartmentalization; and students need to see the connection between theory and practice, build meaning, appropriate knowledge. The question ‘What do you see as the key differences between the teaching and learning of modelling and applications in secondary schools and university, if any?’ was answered by 33 (42%) of respondents. The key differences pointed out by those answering this question were: at school, modelling is poor, too basic and mechanical, often close implementation of simple statistics tests; students have less understanding of application areas; university students are more independent; they have bigger range of mathematical tools, more techniques; they are concerned with rigour and proof. Asked ‘What are the key difficulties for student transition from secondary school to university in the field of mathematical modelling and applications, if any?’ the 35 (44%) university respondents cited: lack of knowledge (mathematical theory, others subjects such as physics, chemistry, biology, ecology); difficulties in formulating precise mathematical problems/interpreting word problems/understanding processes, representations, use of parameters; poor mathematical skills, lack of logical thinking; no experience from secondary schools; and lack of support.

BRIDGING THE GAP AND WAYS FORWARD

In order to address how universities respond to assist students with transition problems our survey asked “Do you have any academic support structures to assist students in the transition from school to university? (e.g., workshops, bridging courses, mentoring, etc).”, and 56 (71%) replied ‘Yes’ and 22 ‘No’. Of those saying yes, 34% have a bridging course, 25% some form of tutoring arrangement, while 23% mentioned mentoring, with one describing it as a “Personal academic mentoring program throughout degree for all mathematics students” and another saying “We tried a mentoring system once, but there was almost no uptake by students.” Other support structures mentioned included ‘study skills courses’, ‘maths clinics’, ‘support workshops’, ‘pre-course’, ‘remedial mathematics unit’, and a ‘Mathematics Learning Service (centrally situated), consulting & assignment help room (School of Maths). The MLS has a drop-in help room, and runs a series of seminars on Maths skills. These are also available to students on the web.’ Others talked of small group peer study, assisted study sessions, individual consultations, daily help sessions, orientation programmes and remedial courses.

There is some evidence that bridging courses can assist in transition. A recent study by (Tempelaar, Rienties, Giesbers & Schim van der Loeff, 2012) showed that an online summer course with a broad coverage of basic mathematical topics and learning controlled by individual, adaptive testing, was very efficient in addressing skill deficiencies, with the treatment effect of successful summer course participation about 50% of the effect size of advanced prior math education. A description by Carmichael and Taylor (2005) of a study of students in a supportive bridging mathematics course indicates that student confidence contributes significantly to performance, even after accounting for prior knowledge, and for some this may be because they struggle with their learning of mathematics in English at undergraduate level much more than is sometimes appreciated (Barton, Chan, King,

Neville-Barton & Sneddon, 2005). In Australia, an increasing number of students elect not to undertake studies in mathematical methods in the final years of their secondary schooling, and hence some support structures are required. Some higher education providers offer pathways for these students to pursue mathematics studies up to a major specialization within the bachelor of science programme. The article by Varsavsky (2010) analyses the performance in, and engagement with, mathematics of students who elect to take up this option. Findings indicate that these are not very different when compared to students who enter university with an intermediate mathematics preparation. Leviatan (2008) presents details of a transition course aimed at bridging the gap for students of four-year secondary/high schoolteacher training programme. The objectives of this transition programme are: to identify and reinforce previous “core school mathematics”; to deepen and enrich the existing knowledge by adopting a more mature perspective to school mathematics; to introduce mathematical “culture” (language, rules of logic, etc.); to get acquaintance with typical mathematical activities (generalizations, deductions, definitions, proofs, etc.); to re-introduce central mathematical concepts and tools; and to provide a rigorous, yet only semiformal, exposure to selected new topics in advanced mathematics. Students’ evaluations of the programme report increasing self-confidence, as well as enjoyment of the sessions about misconceptions and playing the role of a reviewer. She concludes that a more systematic investigation is required and suggests possible follow-up. In other cases, a university first year programme of tutor training and collaborative tutorials, reported by Oates, Paterson, Reilly and Statham (2005), proved an effective way of addressing some of the mathematical issues in the transition.

The literature review presented here revealed a multi-faceted web of cognitive, curricular and pedagogical issues, some spanning across mathematical topics and some intrinsic to certain topics – and certainly exhibiting variation across the institutional contexts of the many countries our survey focused on. For example, most of the research we reviewed discusses the students’ limited cognitive preparedness for the requirements of university-level formal mathematical thinking (whether this concerns the abstraction, for example, within Abstract Algebra courses or the formalism of Analysis). Within other areas, such as discrete mathematics, much of the research we reviewed highlighted that students may arrive at university with little or no awareness of certain mathematical fields.

The literature review presented in this report is certainly not exhaustive. However we believe it is reasonable to claim that the bulk of research on transition is in a few areas (e.g. calculus, proof) and that there is little research in other areas (e.g. discrete mathematics). While this might simply reflect curricular emphases in the various countries that our survey focused on, it also indicates directions that future research may need to pursue. Furthermore across the preceding sections a pattern seems to emerge with regard to *how*, not merely *what*, students experience in their first encounters with advanced mathematical topics, whether at school or at university. Fundamental to addressing issues of transition seems also to be the coordination and dialogue across educational levels – here mostly secondary and tertiary – and our survey revealed that at the moment this appears largely absent.

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REFERENCES

- Alcock, L., & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics*, 69, 111-129.
- Alcock, L., & Inglis, M. (2009). Representation systems and undergraduate proof production: A comment on Weber. *Journal of Mathematical Behavior*, 28, 209-211.
- Alcock, L., Brown, G., & Dunning, C. (2009) Independent study workbooks for proofs in group theory. In *Proceedings of the 12th Special Interest Group of the Mathematical Association of America* Available at: http://sigmaa.maa.org/rume/crume2009/Alcock2_LONG.pdf
- Alsina, C. (2007). Teaching applications and modelling at tertiary level. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), (2007). *Modelling and applications in mathematics education. The 14th ICMI Study*. New York: Springer.
- Anthony, G. (2000). Factors influencing first year students' success in mathematics. *International Journal of Mathematical Education in Science and Technology*, 31(1), 3-14.
- Antonini, S., & Mariotti, M. A. (2008). Indirect proof: What is specific to this way of proving? *ZDM – The International Journal on Mathematics Education*, 40, 401-412.
- Ärlebäck J. B., & Frejd P. (2010). First results from a study investigating Swedish upper secondary students' mathematical modelling competencies. In A. Araújo, A. Fernandes, A. Azevedo, & J. F. Rodrigues (Eds.), *EIMI 2010 Conference (educational interfaces between mathematics and industry) Proceedings*. Comap Inc., Bedford, MA, USA.
- Artigue, M. (2008). Continu, Discontinu en mathématiques: Quelles perceptions en ont les élèves et les étudiants? In L. Viennot (Ed.), *Didactique, épistémologie et histoire des sciences. Penser l'enseignement* (pp. 151-173). Paris: Presses Universitaires de France.
- Artigue, M. (2009). L'enseignement des fonctions à la transition lycée – université. In B. Grugeon (Ed.), *Actes du XV^e Colloque CORFEM 2008*, pp. 25-44. Université de Cergy-Pontoise, IUFM de Versailles.
- Artigue, M., Batanero, C., & Kent, P. (2007). Learning mathematics at post-secondary level. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 1011-1049). Greenwich, Connecticut: Information Age Publishing, Inc.
- Asiala, M., Brown, A., Kleiman, J., & Mathews, D. (1998). The development of students' understanding of permutations and symmetries. *International Journal of Computers for Mathematical Learning*, 3, 13-43.
- Asiala, M., Dubinsky, E., Mathews, D., Morics, S. & Oktac, A. (1997). Development of students' understanding of cosets, normality and quotient groups. *Journal of Mathematical Behavior*, 16(3), 241-309.

- Balacheff N., & Gaudin N. (2002). *Students conceptions: An introduction to a formal characterization*. Cahier Leibniz, Numéro 65, Publication de Université Joseph Fourier. http://halshs.archives-ouvertes.fr/hal-00190425_v1/
- Balacheff, N. (2008). The role of the researcher's epistemology in mathematics education: An essay on the case of proof. *ZDM – The International Journal on Mathematics Education*, 40, 501–512.
- Barbeau, E. (1995). Algebra at the tertiary level. *Journal of Mathematical Behavior*, 14, 139-142.
- Barrier, T. (2009a), *Une perspective sémantique et dialogique sur l'activité de validation en mathématiques*. Thèse de Doctorat. Université Claude Bernard Lyon 1, Lyon. <http://tel.archives-ouvertes.fr/docs/00/04/58/16/PDF/tel-00003713.pdf>
- Barrier, T. (2009b). Quantification et Variation en Mathématiques: perspectives didactiques issues de la lecture d'un texte de Bolzano. In Kourkoulos M., Tzanakis C. (Ed.), *Proceedings of the 5th International Colloquium on the Didactics of Mathematics* (Vol. 2), University of Crete, Rethymnon, Greece.
- Barton, B., Chan, R., King, C., Neville-Barton, P., & Sneddon, J. (2005). EAL undergraduates learning mathematics. *International Journal of Mathematical Education in Science and Technology*, 36(7), 721-729.
- Barton, B., & Sheryn, L. (2009). The mathematical needs of secondary teachers: Data from three countries. *International Journal of Mathematical Education in Science and Technology*, 40(1), 101-8.
- Ben Kilani, I. (2005). Effets didactiques des différences de fonctionnement de la négation dans la langue arabe, la langue française et le langage mathématique. Thèse en co-tutelle, Université Lyon 1 et Université de Tunis.
- Bergé, A. (2008). The completeness property of the set of real numbers in the transition from calculus to analysis. *Educational Studies in Mathematics*, 67(3), 217-236.
- Bergé, A. (2010). Students' perceptions of the completeness property of the set of real numbers. *International Journal of Mathematical Education in Science and Technology*, 41(2), 217-227.
- Berry, J. S., Lapp, D. A., & Nyman, M. A. (2008). Using technology to facilitate reasoning: Lifting the fog from linear algebra. *Teaching Mathematics and its Applications*, 27, 102.
- Biehler, Fischer, Hochmuth & Wassong (2011). Designing and evaluating blended learning bridging courses in mathematics. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 1971-1980). Rzeszow, Poland.
- Bloch, I. (2004) The teaching of calculus at the transition between upper secondary school and the university: Factors of rupture. Communication to the Topic Study Group 12, Dans M. Niss (Eds.) *Actes de ICME10*. Copenhagen. Copenhagen: Roskilde University.
- Bloch, I., Chiocca, C. M., Job, P., & Scheider, M. (2006). Du numérique aux limites: Quelle forme prend la transition secondaire/supérieur dans le champ des nombres et de l'analyse?

- In R. Rouchier et al. (Eds.). *Actes de la XIIIème Ecole d'Été de Didactique des Mathématiques* (pp. 1-13). Cédérom. Grenoble: La Pensée Sauvage.
- Bloch, I., Ghedamsi I. (2010). From Numbers to Limits: Situations as a way to a process of abstraction. *Proceedings of CERME 6*, Lyon, France, <http://www.inrp.fr/publications/edition-electronique/cerme6/wg12-15-bloch.pdf>
- Blossier, T., Barrier, T., & Durand-Guerrier, V. (2009). Proof and quantification. In F.-L. Lin, F.-J. Hsieh, G. Hanna & M. de Villiers (Eds.), *ICMI Study 19 Conference Proceedings*, Taiwan: Taiwan University.
- Blum, W., Alsina, C., Biembengut, M. S., Bouleau, N., Confrey, J., Galbraith, P., Ikeda, T., Lingefjärd, T., Muller, E., Niss, M., Verschaffel, L., Wang, S., Hodgson, B. R. & Henn, H.-W. (2002). ICMI Study 14: Applications and modelling in mathematics education – Discussion Document. *Educational Studies in Mathematics*, 51, 149-171.
- Blum, W., Galbraith, P. L., Henn, H.-W., & Niss, M. (Eds.) (2007). *Modelling and applications in mathematics education. The 14th ICMI Study*. New York: Springer, New ICMI Studies series 10.
- Bogomolny, M. (2007). Raising students' understanding: Linear algebra. In Woo, J. H., Lew, H. C., Park, K. S. & Seo, D. Y. (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 65-72. Seoul: PME.
- Bosch, M., Fonseca, C., Gascón, J. (2004). Incompletitud de las organizaciones matemáticas locales en las instituciones escolares. *Recherches en Didactique des Mathématiques*, 24/2.3, 205-250.
- Bracke, M. (2010). Modelling reality: Motivate your students! In A.D. Fitt et al. (Eds.), *Progress in Industrial Mathematics at ECMI 2008, Mathematics in Industry 15*, Springer-Verlag Berlin Heidelberg. DOI: 10.1007/978-3-642-12110-4104.
- Brandell, G., Hemmi, K., & Thunberg, H. (2008). The widening gap—A Swedish perspective. *Mathematics Education Research Journal*, 20(2), 38-56.
- Bridoux, S. (2010). Une séquence d'introduction des notions de topologie dans l'espace \mathbb{R}^n : de la conception à l'expérimentation. In A. Kuzniak & M. Sokhna (Eds.) *Actes du Colloque International Espace Mathématique Francophone 2009, Enseignement des mathématiques et développement, enjeux de société et de formation*, *Revue Internationale Francophone*, N° Spécial 2010, <http://fastef.ucad.sn/EMF2009/Groupes%20de%20travail/GT7/Bridoux.pdf>
- Britton, S., & Henderson, J. (2009). Linear algebra revisited: An attempt to understand students' conceptual difficulties. *International Journal of Mathematical Education in Science and Technology*, 40, 963-974.
- Brousseau, G. (1997). *Theory of didactical situations in mathematics* (N. Balacheff, M. Cooper, R. Sutherland & V. Warfield: Eds. and Trans.). Dordrecht: Kluwer Academic Publishers.
- Brown, A., DeVries, D. J., Dubinsky, E. & Thomas, K. (1997). Learning binary operations, groups, and subgroups. *Journal of Mathematical Behaviour*, 16 (3), 187–239.

Thomas, de Freitas Druck, Huillet, Ju, Nardi, Rasmussen, Xie

- Burn, R. P. (1996). What are the fundamental concepts of Group Theory? *Educational Studies in Mathematics*, 31(4), 371-377.
- Burn, R. P. (1998). Participating in the learning of Group Theory. *PRIMUS VIII*(4), 305-316.
- Carmichael, C., & Taylor, J. A. (2005). Analysis of student beliefs in a tertiary preparatory mathematics course. *International Journal of Mathematical Education in Science and Technology*, 36(7), 713-719.
- Cartier, L., & Moncel, J. (2008). Learners' conceptions in different class situations on Königsberg's bridges problem. *Paper presented at TSG-15, 11th International Congress on Mathematical Education*, Monterrey.
- Cartiglia, M., Furinghetti, F., & Paola, D. (2004). Patterns of reasoning in classroom. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 287-294.
- Chellougui . (2004) L'utilisation des quantificateurs universel et existentiel en première année d'université. Entre l'implicite et l'explicite. Thèse de l'université Lyon 1.
- Chellougui F. (2009), L'utilisation des quantificateurs universel et existentiel en première année d'université, entre l'explicite et l'implicite, *Recherches en Didactique des Mathématiques*, 29(2), 123-154.
- Chevallard, Y. (1985). *La transposition didactique*. Grenoble: La Pensée Sauvage.
- Chorlay, R. (2009). From historical analysis to classroom work: Function variation and long-term development of functional thinking. *Proceedings of CERME 6*, Lyon, France, <http://www.inrp.fr/publications/edition-electronique/cerme6/wg12-16-chorlay.pdf>
- Clark, J. M., DeVries, D. J., Hemenway, C., St. John, D., Tolia, G., & Vakil, R. (1997). An investigation of students' understanding of abstract algebra (binary operations, groups and subgroups) and the use of abstract structures to build other structures (through cosets, normality and quotient groups). *Journal of Mathematical Behavior*, 16(3), 181-185.
- Clark, M. & Lovric, M. (2009). Understanding secondary-tertiary transition in mathematics. *International Journal of Mathematical Education in Science and Technology*, 40(6), 755-776. doi:10.1080/00207390902912878
- Cnop, I., & Grandsard, F. (1998). Teaching abstract algebra concepts using small group instruction. *International Journal of Mathematical Education in Science and Technology*, 29(6), 843-850.
- Cobb, P., & Bauersfeld, H. (1995). The coordination of psychological and sociological perspectives in mathematics education. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Cooley, L., Martin, W., Vidakovic, D., & Loch, S. (2007). Coordinating learning theories with linear algebra. *International Journal for Mathematics Teaching and Learning* [online journal], University of Plymouth: U.K. <http://www.cimt.plymouth.ac.uk/journal/default.htm>
- Corriveau, C. (2009). Formalisme et démonstration en algèbre linéaire. Paper presented at *Groupe de travail 7, EMF 2009*, Dakar.

- D'Souza, S., & Wood, L. (2003). Rationale for collaborative learning in first year engineering mathematics. *New Zealand Journal of Mathematics*, 32(Supplementary Issue), 47-55.
- De Vleeschouwer, M. (2010a) *Enseignement à l'Université, perspective institutionnelle et contrat didactique: le cas de la dualité en algèbre linéaire*. Thèse de doctorat de l'Université de Namur, Presses universitaires de Namur, Belgique. <http://hdl.handle.net/2078.2/33480>
- De Vleeschouwer, M. (2010b). An institutional point of view of the secondary–university transition: The case of duality. *International Journal of Mathematical Education in Science and Technology*, 41(2), 155-171. doi:10.1080/00207390903372445
- De Vleeschouwer, M. & Gueudet, G. (2011). Secondary-tertiary transition and evolutions of didactic contract: The example of duality in linear algebra. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 2113-2122). Rzeszow, Poland.
- Deloustal-Jorrand (2004). L'implication mathématique: étude épistémologique et didactique. Étude sous trois points de vue: raisonnement déductif, logique formelle et théorie des ensembles. Construction d'une situation didactique qui problématise l'implication. Thèse de l'Université Joseph Fourier, Grenoble 1.
- Dias, M., Artigue, M., Jahn A., & Campos, T. (2008). A comparative study of the secondary-tertiary transition. In M. F. Pinto & T. F. Kawasaki (Eds.). *Proceedings of the 34th Conference of the PME* (Vol. 2, pp. 129-136). Belo Horizonte, Brazil: IGPME.
- Dogan-Dunlap, H. (2010). Linear algebra students' modes of reasoning: Geometric representations. *Linear Algebra and its Applications*, 432, 2141-2159.
- Dogan-Dunlap, H., & Hall, B. (2004). Computers and linear algebra. *WSEAS Transactions on Mathematics*, 3, 537-542.
- Dorier, J.-L., (Ed.) (2000). *Research into the teaching and learning of linear algebra*, Kluwer Academic Publishers.
- Dorier, J.-L., Robert, A., Robinet, J., & Rogalski, M. (2000a). The obstacle of formalism in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (Vol. 23, pp. 85-124): Springer Netherlands.
- Dorier, J. L., Robert, A., Robinet, J., & Rogalski, M. (2000b). On a research programme concerning the teaching and learning of linear algebra in the first-year of a French science university. *International Journal of Mathematical Education in Science and Technology*, 31, 27-35.
- Dreyfus, T., Hillel, J. & Sierpinska, A. (1998). Cabri–based linear algebra: Transformations. Article presented at *CERME–1 (First Conference on European Research in Mathematics Education)* Osnabrück. Obtained from <http://www.fmd.uni-osnabrueck.de/ebooks/erme/cerme1-proceedings/papers/g2-dreyfus-et-al.pdf>
- Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Dordrecht: Kluwer Academic Publishers.

- Dubinsky, E. (1997). Some thoughts on a first course in linear algebra at the college level. In D. Carlson, C. Johnson, D. Lay, A. D. Porter, A. Watkins, & W. Watkins (Eds.), *Resources for teaching linear algebra* (pp. 85-106). MAA notes volume 42, Washington, DC: The Mathematical Association of America.
- Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning. In D. Holton (Ed.) *The teaching and learning of mathematics at university level: An ICMI study* (pp. 275–282). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1997). A reaction to Burn's "What are the fundamental concepts of Group Theory?" *Educational Studies in Mathematics*, 34(3), 249-253.
- Durand-Guerrier, V. (2003) Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in Mathematics* 53, 5-34.
- Durand-Guerrier, V. (2005). *Recherches sur l'articulation entre la logique et le raisonnement mathématique dans une perspective didactique. Apports de la théorie élémentaire des modèles pour une analyse didactique du raisonnement mathématique*. Habilitation à Diriger des Recherche, Université Claude Bernard Lyon 1, Lyon.
- Durand-Guerrier, V., & Arsac, G. (2003), Méthode de raisonnement et leurs modélisations logiques. Le cas de l'analyse. Quelles implications didactiques? *Recherches en didactiques des mathématiques*, 23(3), 295-342.
- Durand-Guerrier, V., & Arsac, G. (2005). An epistemological and didactic study of a specific calculus reasoning rule. *Educational Studies in Mathematics*, 60(2), 149-172
- Durand-Guerrier, V., & Njomgang Ngansop, J. (2010). Questions de logique et de langage à la transition secondaire – supérieur. L'exemple de la négation. In A. Kuzniak & M. Sokhna (Eds.), Actes du Colloque International Espace Mathématique Francophone 2009, Enseignement des mathématiques et développement, enjeux de société et de formation, *Revue Internationale Francophone*, N° Spécial. http://fastef.ucad.sn/EMF2009/Groupes%20de%20travail/GT7/Durand_Ngansop.pdf
- Duval, R. (2002). Proof understanding in mathematics. In *Proceedings of the 2002 International Conference on Mathematics: Understanding Proving and Proving to Understand* (pp. 23–44). Department of Mathematics, National Taiwan Normal University.
- Eisenberg, T., Engelbrecht, J., & Mamona-Downs, J. (2010). Advanced mathematical topics: Transitions, evolutions and changes of foci. *International Journal of Mathematical Education in Science and Technology*, 41(2), 139-141. doi:10.1080/00207391003607870.
- Engelbrecht, J. (2010). Adding structure to the transition process to advanced mathematical activity. *International Journal of Mathematical Education in Science and Technology*, 41(2), 143-154. doi:10.1080/00207390903391890
- Engelbrecht, J., & Harding, A. (2008). The impact of the transition to outcomes-based teaching on university preparedness in mathematics in South Africa, *Mathematics Education Research Journal*, 20(2), 57-70.

- Engelbrecht, J., Harding, A., & Potgieter, M. (2005). Undergraduate students' performance and confidence in procedural and conceptual mathematics. *International Journal of Mathematical Education in Science and Technology*, 36(7), 701-712.
- English, L. & Sriraman, B. (2010). Problem solving for the 21st Century. In B. Sriraman, & L. English (Eds.), *Theories of mathematics education, advances in mathematics education* (pp. 263-290). Berlin: Springer-Verlag. DOI 10.1007/978-3-642-00742-2_27
- Farmaki, V., & Paschos, T. (2007). The interaction between intuitive and formal mathematical thinking: A case study. *International Journal of Mathematical Education in Science and Technology*, 38(3), 353-365. doi:10.1080/00207390601035302
- Faulkner, F., Hannigan, A., & Gill, O. (2011). The changing profile of third level service mathematics in Ireland and its implications for the provision of mathematics education (1998-2010). In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 1992-2001). Rzeszow, Poland.
- Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.
- Gainsburg, J. (2008). Real-world connections in secondary mathematics teaching. *Journal of Mathematics Teacher Education*, 11, 199-219. DOI 10.1007/s10857-007-9070-8
- Ghedamsi, I. (2008a). Apports de situations de contrôles mixtes pragmatique/formel: Etude de CAS. Université de Tunis.
- Ghedamsi, I. (2008b). Enseignement du début de l'analyse réelle à l'entrée à l'université: Articuler contrôles pragmatique et formel dans des situations à dimension a-didactique. Unpublished doctoral dissertation, University of Tunis.
- Gill, O., O'Donoghue, J., Faulkner, F., & Hannigan, A. (2010). Trends in performance of science and technology students (1997-2008) in Ireland. *International Journal of Mathematical Education in Science and Technology*, 41(3), 323-339. doi:10.1080/00207390903477426
- Godfrey, D., & Thomas, M. O. J. (2008). Student perspectives on equation: The transition from school to university. *Mathematics Education Research Journal*, 20(2), 71-92. doi:10.1007/BF03217478
- González-Martín, A. (2009). L'introduction du concept de somme infinie : une première approche à travers l'analyse des manuels. *Actes du colloque EMF 2009*. Groupe de travail 7, 1048-1061.
- González-Martín, A. S., Nardi, E., & Biza, I. (2011). Conceptually-driven and visually-rich tasks in texts and teaching practice: The case of infinite series. *International Journal of Mathematical Education in Science and Technology*, 42(5), 565-589.
- Göttlich, S. (2010). Modelling with students – A practical approach. In A. Araújo, A. Fernandes, A. Azevedo, J. F. Rodrigues (Eds.), *EIMI 2010 Conference (educational interfaces between mathematics and industry) Proceedings*. Comap Inc., Bedford, MA, USA.

- Thomas, de Freitas Druck, Huillet, Ju, Nardi, Rasmussen, Xie
- Grenier, D. (2001). Learning proof and modeling. Inventory of fixtures and new problems. *Actes du 9ème International Congress for Mathematics Education (ICME9)*.
- Grenier, D. (2003). The concept of induction, *Mediterranean Journal For Research in Mathematics Education*, 2(1), 55-64.
- Grenier, D. (2011). Some specific concepts and tools of discrete mathematics. *Paper presented at TSG-15, 11th International Congress on Mathematical Education*, Monterrey.
- Gueudet, G. (2004). Rôle du géométrique dans l'enseignement de l'algèbre linéaire *Recherches en Didactique des Mathématiques* 24/1, 81-114, France.
- Gueudet, G. (2008a). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics*, 67(3), 237-254.
- Gueudet, G. (2008b). *Note d'habilitation à diriger les recherches. Université Paris 7.* http://tel.archivesouvertes.fr/index.php?halsid=t223ort8k4mca13u95ece8qgt0&view_this_doc=tel-00349254&version=1
- Gueudet-Chartier, G. (2004). Should we teach linear algebra through geometry? *Linear Algebra and its Applications*, 379, 491-501.
- Gyöngyösi, E., Solovej, J., & Winsløw, C. (2010). Using CAS based work to ease the transition from calculus to real analysis. In *Proceedings of CERME7, Working Group 14, University mathematics education.* <http://www.cerme7.univ.rzeszow.pl/index.php?id=wg14>
- Gyöngyösi, E., Solovej, J. P., & Winsløw, C. (2011). Using CAS based work to ease the transition from calculus to real analysis. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 2002-2011). Rzeszow, Poland.
- Hanna, G., & Barbeau, E. (2008). Proofs as bearers of mathematical knowledge. Barbeau – *The International Journal on Mathematics Education*, 40, 345-353.
- Hanna, G., & de Villiers, M. (2008). ICMI Study 19: Proof and proving in mathematics education. *ZDM – The International Journal on Mathematics Education*, 40, 329–336. DOI 10.1007/s11858-008-0073-4
- Hanna, G., & Janke, N. (1993). Proof and application. *Educational Studies in Mathematics*, 24(4), 421–428.
- Harel, G. (1990). Using geometric models and vector arithmetic to teach high-school students basic notions in linear algebra. *International Journal of Mathematical Education in Science and Technology*, 21, 387-392.
- Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell & R. Zaskis (Eds.), *Learning and Teaching Number Theory* (pp. 185–212). Ablex Publishing Corporation.
- Harel, G. (2008). DNR perspective on mathematics curriculum and instruction, Part I: focus on proving. *ZDM – The International Journal on Mathematics Education*, 40(3), 487–500.

- Harel, G., & Kaput, J. (1991). The role of conceptual entities in building advanced mathematical concepts and their symbols. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 82-94). Dordrecht: Reidel.
- Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics III* (pp.234–282). Providence, Rhode Island: American Mathematical Society.
- Hart, E. W. (1996). *Research Issues in Undergraduate Mathematics Learning: Preliminary Analysis and Results*, MAA Notes 33.
- Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics*, 40, 71-90.
- Hazzan, O. (2001). Reducing abstraction: The case of constructing an operation table for a group. *Journal of Mathematical Behavior*, 20, 163-172.
- Hazzan, O., & Leron, U. (1996). Students' use and misuse of mathematical theorems: The case of Lagrange's Theorem. *For the learning of mathematics*, 16(1), 23-26.
- Heilio, M. (2010). Mathematics in industry and teachers' training. In A. Araújo, A. Fernandez, A. Azevedo, & J. F. Rodrigues (Eds.), *EIMI 2010 Conference (educational interfaces between mathematics and industry) Proceedings*. Comap Inc., Bedford, MA, USA.
- Heinze, A., & Reiss, K. (2003). Reasoning and proof: Methodological knowledge as a component of proof competence. *Proceedings of the 3rd conference of the European Society for Research in Mathematics Education*, Available from http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG4/TG4_Heinze_cerme3.pdf.
- Heinze, A., Cheng, Y-H., Ufer, S., Lin, F-L., & Reiss, K. M. (2008). Strategies to foster students' competencies in constructing multi-steps geometric proofs: Teaching experiments in Taiwan and Germany. *ZDM – The International Journal on Mathematics Education*, 40, 443-453.
- Hemmi, K. (2008). Students' encounter with proof: The condition of transparency. *ZDM – The International Journal on Mathematics Education*, 40, 413-426.
- Henderson, F., Rasmussen, C., Zandieh, M., Wawro, M., & Sweeney, G. (2010). Symbol sense in linear algebra: A start toward eigen theory. *Proceedings of the 14th Annual Conference for Research in Undergraduate Mathematics Education*. Raleigh, N.C.
- Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra* (Vol. 23, pp. 191-207). Springer: Netherlands.
- Hillel, J. (2001). Computer algebra systems in the learning and teaching of linear algebra: Some examples. In D. Holton (Ed.) *The teaching and learning of mathematics at university level* (pp. 371-380). Dordrecht: Kluwer Academic Publishers.

Thomas, de Freitas Druck, Huillet, Ju, Nardi, Rasmussen, Xie

- Hockman, M. (2005). Curriculum design and tertiary education. *International Journal of Mathematical Education in Science and Technology*, 36(2-3), 175-191. doi:10.1080/0020739042000196331
- Holton, D. (Ed.) (2001). *The teaching and learning of mathematics at university level. An ICMI Study*, Dordrecht: Kluwer Academic Publishers.
- Hong, Y., Kerr, S., Klymchuk, S., McHardy, J., Murphy, P., Spencer, S., & Thomas, M. O. J. (2009). A comparison of teacher and lecturer perspectives on the transition from secondary to tertiary mathematics education. *International Journal of Mathematical Education in Science and Technology*, 40(7), 877-889. doi:10.1080/00207390903223754
- Hourigan, M., & O'Donoghue, J. (2007). Mathematical under-preparedness: The influence of the pre-tertiary mathematics experience on students' ability to make a successful transition to tertiary level mathematics courses in Ireland. *International Journal of Mathematical Education in Science and Technology*, 38(4), 461-476.
- Hoyles, C., Newman, K. & Noss, R. (2001). Changing patterns of transition from school to university mathematics. *International Journal of Mathematical Education in Science and Technology*, 32(6), 829-845. doi:10.1080/00207390110067635
- Iannone, P. & Inglis, M. (2011). Undergraduate students' use of deductive arguments to solve "prove that..." tasks. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 2012-2021). Rzeszow, Poland.
- Iannone, P., & Nardi, E. (2002). A group as a 'special set'? Implications of ignoring the role of the binary operation in the definition of a group. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 121-128). Norwich, UK.
- Iannone, P., Inglis, M., Mejia-Ramos, J. P., Simpson, A., & Weber, K. (2011). Does generating examples aid proof production? *Educational Studies in Mathematics*, 77, 1-14 [online version].
- Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, 66, 3-21.
- Ioannou, M. (in preparation). [working title: *Cognitive, affective and social aspects of Year 1 mathematics undergraduates' learning experience in Abstract Algebra*]. Unpublished doctoral thesis. University of East Anglia: Norwich, UK.
- Ioannou, M. & Iannone, P. (2011). Students' affective responses to the inability to visualise cosets. *Research in Mathematics Education* 13(1), 81-82.
- Ioannou, M., & Nardi, E. (2009). Engagement, abstraction and visualisation: Cognitive and emotional aspects of Year 2 mathematics undergraduates' learning experience in Abstract Algebra. *Proceedings of the British Society for Research into Learning Mathematics*, 29(2), 35-40.
- Ioannou, M., & Nardi, E. (2009). Visualisation as a meaning-bestowing process in the learning and teaching of Abstract Algebra. In M. Tzekaki, M. Kaldrimidou & C. Sakonidis

(Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education (PME)* (Vol. 1, pp. 297). Thessaloniki, Greece: PME.

Ioannou, M., & Nardi, E. (2010). Mathematics undergraduates' experience of visualisation in Abstract Algebra: The metacognitive need for an explicit demonstration of its significance. In *Proceedings of the 13th Special Interest Group of the Mathematical Association of America (SIGMAA) Conference on Research in Undergraduate Mathematics Education (RUME)*. Available at: <http://sigmaa.maa.org/rume/crume2010/Archive/Ioannou%20&%20Nardi.pdf>

James, A., Montelle, C., & Williams, P. (2008). From lessons to lectures: NCEA mathematics results and first-year mathematics performance. *International Journal of Mathematical Education in Science and Technology*, 39(8), 1037-1050. doi:10.1080/00207390802136552

Jennings, M. (2009). Issues in bridging between senior secondary and first year university mathematics. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides* (Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp. 273-280). Palmerston North, NZ: MERGA.

Junior, O. (2006). *Compreensões de conceitos de cálculo diferencial no primeiro ano de matemática uma abordagem integrando oralidade, escrita e informática*. Unpublished PhD Thesis, Universidade Estadual Paulista, Rio Claro, Brazil.

Kaiser, G., & Schwarz, B. (2006). Mathematical modelling as bridge between school and university. *ZDM*, 38(2), 196-208.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM*, 38(3), 302-310.

Kajander, A., & Lovric, M. (2005). Transition from secondary to tertiary mathematics: McMaster University Experience. *International Journal of Mathematical Education in Science and Technology*, 36(2-3), 149-160

Kaland, K., Kaiser, K., Ortlieb, C. P., & Struckmeier, J. (2010). Authentic modelling problems in mathematics education. In A. Araújo, A. Fernandes, A. Azevedo, & J. F. Rodrigues (Eds.). *EIMI 2010 Conference (educational interfaces between mathematics and industry) Proceedings*. Comap Inc., Bedford, MA, USA.

Kelly, A. E., Lesh, R., & Baek, J. (Eds.) (2008). *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching*. New York, NY: Routledge.

Klapsinou, A., & Gray, E. (1999). The intricate balance between abstract and concrete in linear algebra. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 153-160). Haifa, Israel: PME.

Larsen, S. (2009). Reinventing the concepts of group and isomorphism: The case of Jessica and Sandra. *Journal of Mathematical Behavior*, 28, 119-137.

Larsen, S. & Zandieh, M. (2007). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*, 67, 205-216.

- Larson, C., Zandieh, M., & Rasmussen, C. (2008). A trip through eigen-land: Where most roads lead to the direction associated with the largest eigenvalue. *Proceedings of the 11th Annual Conference for Research in Undergraduate Mathematics Education*. San Diego, CA.
- Leavitt, D. R., & Ahn, C. M. (2010). A middle grade teacher's guide to model eliciting activities. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies: ICTMA13*, New York: Springer.
- Lebeau C., & Schneider, M. (2010). Equations incomplètes de plans et obstacles à la nécessité épistémique, *Recherches en Didactique des Mathématiques* 30(1), 11-46.
- Leron, U., & Dubinsky, E. (1995). An abstract algebra story. *The American Mathematical Monthly*, 102(3), 227-242.
- Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics*, 29, 153-174.
- Lesh, R., & Doerr, H. M. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. Lesh & H. Doerr (Eds.), *Beyond Constructivism: Models and modeling perspectives on mathematics problem solving, teaching, and learning* (pp. 3-33). Mahwah, NJ: Lawrence Erlbaum Associates.
- Leviatan, T. (2008). Bridging a cultural gap. *Mathematics Education Research Journal*, 20(2), 105-116.
- Luk, H. S. (2005). The gap between secondary school and university mathematics. *International Journal of Mathematical Education in Science and Technology*, 36(2-3), 161-174.
- Maaß, K. (2010). Classification scheme for modelling tasks. *J. Math Didakt* 31, 285–311. DOI 10.1007/s13138-010-0010-2
- Maaß, K., & Mischo, C. (2011). Implementing modelling into day-to-day teaching practice – The project STRATUM and its framework. *J Math Didakt* 32, 103–131. DOI 10.1007/s13138-010-0015-x
- Mamona-Downs, J. (2010). On introducing a set perspective in the learning of limits of real sequences. *International Journal of Mathematical Education in Science and Technology*, 41(2), 277-291.
- Man-Keung, S. (2008). Proof as a practice of mathematical pursuit in a cultural, socio-political and intellectual context. *ZDM – The International Journal on Mathematics Education*, 40, 355–361.
- Martin, W., Loch, S., Cooley, L., Dexter, S., & Vidakovic, D. (2010). Integrating learning theories and application-based modules in teaching linear algebra. *Linear Algebra and its Applications*, 432, 2089-2099.
- Mason, J., & Klymchuk, S. (2009). *Counterexamples in calculus*. London: Imperial College Press.
- Meel, D. E. & Hern, T. A. (2005). Tool building: Web-based linear algebra modules. *Journal of Online Mathematics and its Applications*. Obtained from

<http://mathdl.maa.org/mathDL/4/?pa=content&sa=viewDocument&nodeId=560&pf=1>
on April 30, 2011.

- Nagafuchi, T. (2009). *A historical-philosophical study about the role of mathematical proof on bachelor of mathematics undergraduate courses*. Unpublished masters thesis, Universidade Estadual de Londrina, Paraná, Brasil. Available from http://www2.uel.br/cce/pos/mecem/pdf/Dissertacoes/thiago_nagafuchi_texto.pdf
- Najar, R. (2010). Effet des choix institutionnels d'enseignement sur les possibilités d'apprentissage des étudiants. Cas des notions ensemblistes fonctionnelles dans la transition Secondaire/Supérieur. Doctoral Thesis. Université Paris Diderot – Paris 7. http://tel.archives-ouvertes.fr/index.php?halsid=laq1i5j6ihpj73e32tn4kst545&view_this_doc=tel-00564191&version=1
- Nardi, E. (1996). *The novice mathematician's encounter with mathematical abstraction: Tensions in concept-image construction and formalisation* Unpublished doctoral thesis, University of Oxford, UK (available at <http://www.uea.ac.uk/~m011>).
- Nardi, E. (2000). Mathematics undergraduates' responses to semantic abbreviations, 'geometric' images and multi-level abstractions in Group Theory. *Educational Studies in Mathematics*, 43, 169-189.
- Nardi, E. (2008). *Amongst mathematicians: Teaching and learning mathematics at university level*. New York: Springer.
- Nardi, E. (2011). 'Driving noticing' yet 'risking precision': University mathematicians' pedagogical perspectives on verbalisation in mathematics. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 2053-2062). Rzeszow, Poland.
- National Council of Teachers of Mathematics. (2009). *Focus in high school mathematics: Reasoning and sense making*. Reston VA: Author.
- National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Author. Retrieved from Common Core State Standards website: http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf
- Niss, M., Blum, W., & Galbraith, P. L. (2007). Part 1: Introduction. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education. The 14th ICMI Study*. New York/etc.: Springer, New ICMI Studies series 10.
- Oates, G., Paterson, J., Reilly I., & Statham, M. (2005). Effective tutorial programmes in tertiary mathematics. *International Journal of Mathematical Education in Science and Technology*, 36(2), 731-739. doi:10.1080/0020739042000196331
- Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, 40(4), 396-426.
- Ouvrier-Buffet, C. (2011). A mathematical experience involving defining processes: in-action definitions and zero-definitions. *Educational Studies in Mathematics*, 76(2), 165-182.

- Palla, M., Potari, D., & Spyrou, P. (2011). Secondary school students' understanding of mathematical induction: structural characteristics and the process of proof construction. *International Journal of Science and Mathematics Education*, Published Online 4 August 2011.
- Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66, 23–41.
- Pedemonte, B. (2008). Argumentation and algebraic proof. *ZDM – The International Journal on Mathematics Education*, 40(3), 385–400.
- Perrenet, J., & Taconis, R. (2009). Mathematical enculturation from the students' perspective: Shifts in problem-solving beliefs and behaviour during the bachelor programme. *Educational Studies in Mathematics*, 71, 181–198. DOI 10.1007/s10649-008-9166-9
- Petocz, P., Reid, A., Wood, L. N., Smith, G. H., Mather, G., Harding, A., Engelbrecht, J., Houston, K., Hillel, J., & Perrett, G. (2007). Undergraduate students' conceptions of mathematics: An international study. *International Journal of Science and Mathematics Education*, 5, 439-459.
- Pinto, M., & Tall, D. O. (2002). Building formal mathematics on visual imagery: A case study and a theory. *For the Learning of Mathematics*, 22(1), 2–10.
- Portnoy, N., Grundmeier, T. A., & Graham, K. J. (2006). Students' understanding of mathematical objects in the context of transformational geometry: Implications for constructing and understanding proofs. *The Journal of Mathematical Behavior*, 25, 196-207.
- Possani, E., Trigueros, M., Preciado, J., & Lozano, M. (2010). Use of models in the teaching of linear algebra. *Linear Algebra and its Applications*, 432, 2125-2140.
- Powers, R. A., Craviotto, C. & Grassl, R. M. (2010). Impact of proof validation on proof writing in abstract algebra. *International Journal of Mathematical Education in Science and Technology*, 41(4), 501-514.
- Praslon, F. (1999). Discontinuities regarding the secondary/university transition: The notion of derivative as a specific case. In O. Zaslavsky (Ed.), *Proceedings of the 23rd International Conference on the Psychology of Mathematics Education* (Vol. 4, pp. 73-80). Haifa, Israel: Technion – Israel Institute of Technology.
- Praslon, F. (2000). Continuités et ruptures dans la transition Terminale S/DEUG Sciences en analyse. Le cas de la notion de dérivée et son environnement. In, T. Assude & B. Grugeon (Eds.), *Actes du Séminaire National de Didactique des Mathématiques* (pp. 185-220). IREM Paris 7.
- President's Council of Advisors on Science and Technology (PCAST) (2012). *Engage to excel: Producing one million additional college graduates with Degrees in Science, Technology, Engineering, and Mathematics*. Washington, DC: The White House.
- Reiss, K. M., Heinze, A., Renkl, A., & Groß, C. (2008). Reasoning and proof in geometry: Effects of a learning environment based on heuristic worked-out examples. *ZDM – The International Journal on Mathematics Education*, 40, 455–467.

- Robert, A. (2011). Recherches de type ingénierie. In C. Margolinas, M. Abboud-Blanchard, L. Bueno-Ravel, N. Douek, A. Fluckiger, P. Gibel, et al. (Eds.), *En amont et en aval des ingénieries didactiques*. Grenoble: La Pensée Sauvage.
- Robert, A. (to appear) Savoirs mathématiques à enseigner : différents types de notions, d'une généralisation à une autre. In C. De Hosson (Ed.) *Actes de la 11^{ème} journée de l'école doctorale « Savoirs scientifiques : Epistémologie, histoire des sciences et didactique des disciplines »*, Université Paris Diderot- Paris 7.
- Rogalski, M. (2008). Les rapports entre local et global: mathématiques, rôle en physique élémentaire, questions didactiques. In L. Viennot (Ed.) *Didactique, épistémologie et histoire des sciences* (pp. 61-87). Paris: Presses Universitaires de France.
- Rogalski, M. (2008) Trois entrées dans l'intelligibilité de contenus à enseigner en mathématique, In, C. De Hosson & A. Robert (Eds.), *Actes de la 8^{ème} journée de l'école doctorale « Savoirs scientifiques : Epistémologie, histoire des sciences et didactique des disciplines »*. Université Paris Diderot – Paris 7.
- Rogalski, J. & Rogalski, M. (2004). Contribution à l'étude des modes de traitement de la validité de l'implication par de futurs enseignants de mathématiques, in *Annales de Didactique et de Sciences Cognitives Vol.9*, Actes du colloque Argentoratum de juillet 2002, 175-203.
- Rogalski, M. (2000). The teaching experiment in Lille. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 133-149). Dordrecht, The Netherlands: Kluwer.
- Sackur, C., Assude, T., Maurel, M., Drouhard, J. P., & Paquelier, Y. (2005). L'expérience de la nécessité épistémique. *Recherches en Didactique des Mathématiques*, 25(1), 57-90.
- Scucuglia, R. (2006). *A investigação do teorema fundamental do cálculo com calculadoras gráficas*. Unpublished PhD Thesis, Universidade Estadual Paulista, Rio Claro, Brazil.
- Selden, A. (2005). New developments and trends in tertiary mathematics education: Or more of the same? *International Journal of Mathematical Education in Science and Technology*, 36(2-3), 131-147.
- Selden, A., & Selden, J. (2001). Tertiary mathematics education research and its future. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at the University Level: An ICMI Study* (pp. 207-220). The Netherlands: Kluwer Academic Publishers.
- Sierpinska, A. Dreyfus, T., & Hillel, J. (1999). Evaluation of a teaching design in linear algebra: The case of linear transformations. *Recherches en Didactique des Mathématiques*, 19, 7–40.
- Smida, H., & Ghedamsi, I. (2006). Pratiques enseignantes dans la transition lycée/université en analyse. Dans N. Bednarz (Dir.). *Actes du 3e colloque international Espace Mathématique Francophone: Thème 6* (pp. CD-ROM). Sherbrooke: Université de Sherbrooke.
- Solomon, Y. (2006). Deficit or difference? The role of students' epistemologies of mathematics in their interactions with proof. *Educational Studies in Mathematics*, 61, 373–393. DOI: 10.1007/s10649-006-6927-1

- Souto Rubio, B., & Gómez-Chacón, I. (2011). Challenges with visualization at university level: The concept of integral. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 2073-2082). Rzeszow, Poland.
- Stadler, E. (2011). The same but different – novice university students solve a textbook exercise. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 2083-2092). Rzeszow, Poland.
- Stewart, S., & Thomas, M. O. J. (2007). Embodied, symbolic and formal thinking in linear algebra. *International Journal of Mathematical Education in Science and Technology*, 38, 927-937.
- Stewart, S., & Thomas, M. O. J. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology*, 40, 951 - 961.
- Stewart, S., & Thomas, M. O. J. (2010). Student learning of basis, span and linear independence in linear algebra. *International Journal of Mathematical Education in Science and Technology*, 41, 173-188.
- Stillman, G. (2007). Upper secondary perspectives on applications and modelling. In W. Blum, P. L. Galbraith, H.-W. Henn, M. & Niss (Eds.). *Modelling and applications in mathematics education. The 14th ICMI Study*. New York/etc.: Springer, New ICMI Studies series 10.
- Stillman, G., & Ng, D. (2010). The other side of the coin-attempts to embed authentic real world tasks in the secondary curriculum. In A. Araújo, A. Fernandes, A. Azevedo, & J. F. Rodrigues (Eds.), *EIMI 2010 Conference (educational interfaces between mathematics and industry) Proceedings*. Comap Inc., Bedford, MA, USA.
- Stylianides, A. J., & Stylianides, G. J. (2007). The mental models theory of deductive reasoning: Implications for proof instruction. *Proceedings of CERME5, the 5th Conference of European Research in Mathematics Education*, 665-674.
- Sweeney, G. (2011). Classroom activity with vectors and vector equations: Integrating informal and formal ways of symbolizing \mathbb{R}^n . *Paper presented at the 14th Conference on Research in Undergraduate Mathematics Education*, Portland, OR.
- Tall, D. O. (2004). Building theories: The three worlds of mathematics: A comment on Inglis. *For the Learning of Mathematics*, 23(3), 29–32.
- Tall, D. O. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, 20(2), 5–24.
- Tall, D. O., & Mejia-Ramos, J. P. (2006). The long-term cognitive development of different types of reasoning and proof. *Conference on Explanation and Proof in Mathematics: Philosophical and Educational Perspectives*, Universität Duisburg-Essen, Campus Essen, 1-11.
- Tall, D. O., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y-H. (2012). The cognitive development of proof. In G. Hanna & M. de Villiers (Eds.) *Proof and Proving in Mathematics Education: The 19th ICMI study* (In Press). York: Springer.

- Tempelaar, D. T., Rienties, B., Giesbers, B., & Schim van der Loeff, S. (2012). Effectiveness of a voluntary postsecondary remediation program in mathematics. In P. Van den Bossche et al. (Eds.), *Learning at the Crossroads of Theory and Practice*, Advances in Business Education and Training 4 (pp. 199-222), Dordrecht: Springer. DOI 10.1007/978-94-007-2846-2_13
- Thomas, M. O. J., & Stewart, S. (2011). Eigenvalues and eigenvectors: Embodied, symbolic and formal thinking. *Mathematics Education Research Journal*, 23(3), 275-296. DOI 10.1007/s13394-011-0016-1
- Thrash, K. R., & Walls, G. L. (1991). A classroom note on understanding the concept of group isomorphism. *Mathematics and Computer Education*, 25(1), 53-55.
- Toulmin, S. E. (1958). *The uses of argument*. Cambridge University Press.
- Turner, P., & Fowler, K. (2010). A holistic approach to applied mathematics education for middle and high schools. In A. Araújo, A. Fernandes, A. Azevedo, & J. F. Rodrigues (Eds.). *EIMI 2010 Conference (educational interfaces between mathematics and industry) Proceedings*. Bedford, MA: Comap Inc.
- Vandebrouck, F. (2010). Students' conceptions of functions at the transition between secondary school and university. CERME. <http://www.cerme7.univ.rzeszow.pl/index.php?id=wg14>
- Vandebrouck, F. (2011, à paraître). Points de vue et domaines de travail en analyse. *Annales de didactique de Strasbourg*.
- Vandebrouck, F. (2011). Students' conceptions of functions at the transition between secondary school and university. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education* (pp. 2093-2102). Rzeszow, Poland.
- Varsavsky, C. (2010). Chances of success in and engagement with mathematics for students who enter university with a weak mathematics background. *International Journal of Mathematical Education in Science and Technology*, 41(8), 1037-1049. doi:10.1080/0020739X.2010.493238
- Vos, P. (2010). The Dutch maths curriculum: 25 Years of modelling. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.). *Modeling students' mathematical modeling competencies: ICTMA13*, New York: Springer.
- Wawro, M., Sweeney, G., & Rabin, J. (2011). Subspace in linear algebra: investigating students' concept images and interactions with the formal definition. *Educational Studies in Mathematics*. doi: 10.1007/s10649-011-9307-4
- Wawro, M., Zandieh, M., Sweeney, G., Larson, C., & Rasmussen, C. (2011). Using the emergent model heuristic to describe the evolution of student reasoning regarding span and linear independence. Paper presented at the 14th Conference on Research in Undergraduate Mathematics Education, Portland, OR.
- Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48(1), 101-119.

Thomas, de Freitas Druck, Huillet, Ju, Nardi, Rasmussen, Xie

Weber, K. (2009). How syntactic reasoners can develop understanding, evaluate conjectures, and generate counterexamples in advanced mathematics. *Journal of Mathematical Behavior*, 28, 200–208.

Weber, K., & L. Alcock (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, 56(2/3), 209-234.

Weber, K., & Mejia-Ramos, J. P. (2011). Why and how mathematicians read proofs: An exploratory study. *Educational Studies in Mathematics*, DOI 10.1007/s10649-010-9292-z.

Winsløw, C. (2007). Les problèmes de transition dans l'enseignement de l'analyse et la complémentarité des approches diverses de la didactique. *Annales de Sciences Cognitives et Didactique*, 12, 189-204.

Winsløw, C. (2008). Transformer la théorie en tâches: La transition du concret à l'abstrait en analyse réelle (Turning theory into tasks: Transition from concrete to abstract in calculus). In A. Rouchier, et al. (Eds.), *Actes de la XIII^{ième} école d'été de didactique des mathématiques*. Grenoble: La Pensée Sauvage.

Xie, J. (2010). Mathematical modeling courses and related activities in China universities. In A. Araújo, A. Fernandes, A. Azevedo, & J. F. Rodrigues (Eds.). *EIMI 2010 Conference (educational interfaces between mathematics and industry) Proceedings*. Comap Inc., Bedford, MA, USA.

Yevdokimov, O. (2003). The place and significance of the problems for proof in learning mathematics. *Proceedings of the 3rd conference of the European Society for Research in Mathematics Education*, Available from http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG4/TG4_Yevdokimov_cerme3.pdf.

Zazkis, R., & Chernoff, E. J. (2008). What makes a counterexample exemplary? *Educational Studies in Mathematics*, 68, 195–208.

Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating visual and analytic strategies: A study of students' understanding of the group D_4 . *Journal for Research in Mathematics Education*, 27(4), 435-457.