AN INTRODUCTION TO SOBOLEV SPACES

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Preface: These notes were written to supplement the graduate level PDE course at Montana State University. Sobolev Spaces have become an indispensable tool in the theory of partial differential equations and all graduate-level courses on PDE's ought to devote some time to the study of the more important properties of these spaces. The object of these notes is to give a self-contained and brief treatment of the important properties of Sobolev spaces. The main aim is to give clear proofs of all of the main results without writing an entire book on the subject! Why did I write these notes? Much of the existing literature on the subject seems to fall into two categories, either long treatises on the subject with the most general assumptions possible (and thus unsuitable for part of a PDE course), or very sketchy discussions confined to a chapter of a PDE text.

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In these notes, Ω is a domain (i.e. an open, connected set) in R^n.

1. The Spaces $W^{j,p}(\Omega)$ and $W_0^{j,p}(\Omega)$

Definitions: Suppose $1 \le p < \infty$. Then

(i) $L^p_{loc}(\Omega) = \{u: u \in L^p(K)$ for every compact subset K of $\Omega\}$

(ii) u is *locally integrable* in Ω if $u \in L^1_{loc}(\Omega)$.

(iii) Let u and v be locally integrable functions defined in Ω. We say that v is the α *th weak derivative* of u if for every $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega uD^\alpha \phi\ dx = (-1)^{|\alpha|}\int_\Omega v\phi\ dx,$$

and we say that $D^\alpha u = v$ *in the weak sense.*

(iv) Let u and v be in $L^p_{loc}(\Omega)$. We say that v is the α *th strong derivative* of u if for each compact subset K of Ω there exists a sequence $\{\phi_j\}$ in $C^{|\alpha|}(K)$ such that $\phi_j \to u$ in $L^p(K)$ and $D^\alpha \phi_j \to v$ in $L^p(K)$.

THEOREM 1 *If* $D^\alpha u = v$ *and* $D^\beta v = w$ *in the weak sense then* $D^{\alpha+\beta}u = w$ *in the weak sense.*

PROOF Let $\psi \in C_0^\infty(\Omega)$ and $\phi = D^\beta \psi$. Then

$$\int_\Omega uD^{\alpha+\beta}\psi\ dx = (-1)^{|\alpha|}\int_\Omega \phi v\, dx = (-1)^{|\alpha|}\int_\Omega vD^\beta\psi\ dx = (-1)^{|\alpha|+|\beta|}\int_\Omega \psi w\, dx.$$

Definition (mollifiers): Let $\rho \in C_0^\infty(R^n)$ be such that

(i) Supp $\rho \subset B_1(0)$, (recall that "supp" denotes the support of a function, and $B_r(p)$ denotes an open ball of radius r and center p).

(ii) $\int \rho(x)\, dx = 1$,

(iii) $\rho(x) \geq 0$.

If $\varepsilon > 0$ then we set (provided that the integral exists)

$$J_\varepsilon u(x) = \frac{1}{\varepsilon^n}\int_\Omega \rho(\frac{x-y}{\varepsilon})u(y)\, dy.$$

$J_\varepsilon u$ is called a mollifier of u. Note that if u is locally integrable in Ω and if K is a compact subset of Ω then $J_\varepsilon u \in C^\infty(K)$ provided that $\varepsilon < dist(K, \partial\Omega)$. Suppose now that $u \in L^p_{loc}(\Omega)$. Clearly

$$J_\varepsilon u(x) = \int_{B_1(0)} \rho(y)u(x-\varepsilon y)\, dy,$$

so for $p > 1$ we have (if $1/p + 1/q = 1$)

$$|J_\varepsilon u(x)| \leq \int_{B_1(0)} \{\rho(y)\}^{1/q}\{\rho(y)\}^{1/p}|u(x-\varepsilon y)|\, dy$$

$$\le \left(\int_{B_1(0)} (\{\rho(y)\}^{1/q})^q \, dx\right)^{1/q} \left(\int_{B_1(0)} (\{\rho(y)\}^{1/p} |u(x-\varepsilon y)|)^p \, dy\right)^{1/p} .$$

Hence $|J_\varepsilon u(x)|^p \le \int_{B_1(0)} \rho(y) |u(x-\varepsilon y)|^p \, dy$, and this trivially holds if $p=1$ too. Integrating this, we see that

$$\begin{aligned}\int_K |J_\varepsilon u(x)|^p \, dx &\le \int_{B_1(0)} \rho(y) \int_K |u(x-\varepsilon y)|^p \, dx \, dy \\ &\le \int_{B_1(0)} \rho(y) \int_{K_0} |u(x)|^p \, dx \, dy \\ &= \int_{K_0} |u(x)|^p \, dx ,\end{aligned}$$

where K_0 is a compact subset of Ω, $K \subset Interior(K_0)$ and $\varepsilon < dist(K, \partial K_0)$. i.e. we have

$$\| J_\varepsilon u \|_{L^p(K)} \le \| u \|_{L^p(K_0)} . \tag{1}$$

Lemma 2 *If $u \in L^p_{loc}(\Omega)$ and K is a compact subset of Ω then $\| J_\varepsilon u - u \|_{L^p(K)} \to 0$ as $\varepsilon \to 0$.*

Proof Let K_0 be a compact subset of Ω where $K \subset Interior(K_0)$ and let $\varepsilon < dist(K, \partial K_0)$. Let $\delta > 0$ and let $w \in C^\infty(K_0)$ be such that $\| u - w \|_{L^p(K_0)} < \delta$. Then applying (1) to $u - w$, we obtain

$$\| J_\varepsilon u - J_\varepsilon w \|_{L^p(K)} < \delta . \tag{2}$$

But $J_\varepsilon w(x) - w(x) = \int_{B_1(0)} \rho(y) \{ w(x-\varepsilon y) - w(x) \} \, dy$, and this goes to zero uniformly on K as $\varepsilon \to 0$. Hence, if ε is sufficiently small, we have

$$\| J_\varepsilon w - w \|_{L^p(K)} < \delta . \tag{3}$$

Hence, by (2) and (3)

$$\| J_\varepsilon u - u \|_{L^p(K)} \le \| w - u \|_{L^p(K)} + \| J_\varepsilon u - J_\varepsilon w \|_{L^p(K)} + \| J_\varepsilon w - w \|_{L^p(K)} < 3\delta . \tag{4}$$

Since δ is arbitrary, $\| J_\varepsilon u - u \|_{L^p(K)} \to 0$ as $\varepsilon \to 0$.

The proof of the following theorem contains some other important approximating properties of mollifiers.

THEOREM 3 *Suppose that u and v are in* $L^p_{loc}(\Omega)$. *Then* $D^\alpha u = v$ *in the weak sense if and only if* $D^\alpha u = v$ *in the strong* L^p *sense.*

PROOF Suppose that $D^\alpha u = v$ in the strong L^p sense. Let $\phi \in C_0^\infty(\Omega)$ and let K = supp ϕ. Let $\varepsilon > 0$ and take $\psi \in C^{|\alpha|}(K)$ so that $\|\psi - u\|_{L^p(K)} < \varepsilon$ and $\|D^\alpha \psi - v\|_{L^p(K)} < \varepsilon$. Then

$$\begin{aligned}
\left|\int_K uD^\alpha\phi\ dx - (-1)^{|\alpha|}\int_K v\phi\ dx\right| &\le \left|\int_K \psi D^\alpha\phi\ dx - (-1)^{|\alpha|}\int_K \phi D^\alpha\psi\ dx\right| \\
&\quad + \left|\int_K (u-\psi)D^\alpha\phi\ dx\right| + \left|\int_K (v - D^\alpha\psi)\phi\ dx\right| \\
&\le \|u-\psi\|_{L^p(K)} \|D^\alpha\phi\|_{L^q(K)} + \|v - D^\alpha\psi\|_{L^p(K)} \|\phi\|_{L^q(K)} \\
&\le \varepsilon(\|D^\alpha\phi\|_{L^q(K)} + \|\phi\|_{L^q(K)}),
\end{aligned}$$

where q is the conjugate exponent of p (if $p=1$ then $q=\infty$ and if $p>1$ then $1/p+1/q=1$). But ε is arbitrary, so the LHS must be zero. So $D^\alpha u = v$ in the weak sense.

Conversely, suppose that $D^\alpha u = v$ in the weak sense and let K be a compact subset of Ω. Then $J_\varepsilon u \in C^\infty(K)$ if $\varepsilon < dist(K, \partial\Omega)$ and we have for all x in K

$$\begin{aligned}
D^\alpha J_\varepsilon u(x) &= \varepsilon^{-n}\int_\Omega D_x^\alpha \rho\left(\frac{x-y}{\varepsilon}\right)u(y)\ dy \\
&= \varepsilon^{-n}(-1)^{|\alpha|}\int_\Omega D_y^\alpha \rho\left(\frac{x-y}{\varepsilon}\right)u(y)\ dy \\
&= \varepsilon^{-n}\int_\Omega \rho\left(\frac{x-y}{\varepsilon}\right)v(y)\ dy \\
&= J_\varepsilon v(x).
\end{aligned}$$

But by Lemma 2, $\|J_\varepsilon u - u\|_{L^p(K)} \to 0$ and $\|D^\alpha J_\varepsilon u - v\|_{L^p(K)} = \|J_\varepsilon v - v\|_{L^p(K)} \to 0$ as $\varepsilon \to 0$. Thus $D^\alpha u = v$ in the strong sense.

Definitions (i) $|u|_{j,p}^\Omega = \left(\sum_{|\alpha|\le j}\int_\Omega |D^\alpha u(x)|^p\ dx\right)^{1/p}$.

(ii) $\hat{C}^{j,p}(\Omega) = \{u \in C^j(\Omega) : |u|_{j,p}^\Omega < \infty\}$.

(iii) $H^{j,p}(\Omega)$ = completion of $\hat{C}^{j,p}(\Omega)$ with respect to the norm $|\ |_{j,p}^\Omega$.

$H^{j,p}(\Omega)$ is called a Sobolev space. We will encounter other such spaces as well. Recall that the completion of a normed linear space is a larger space in which all Cauchy sequences converge (i.e. it is a Banach space). It is constructed by first defining a space of equivalence classes of Cauchy sequences. Two Cauchy sequences $\{x_m\}$, $\{y_m\}$ are said to be in the same equivalence class if $\lim_{m\to\infty} ||x_m - y_m|| = 0$. A member x of the old space is identified with the equivalence class of the sequence $\{x, x, x, \dots\}$ of the new space and in this sense the new space contains the old space. Further, the old space is dense in its completion. Moreover, if a normed linear space X is dense in a Banach space Y, then Y is the completion of X.

Recall that for $1 \le p < \infty$, $L^p(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the usual "p norm". This knowledge allows us to see what members of $H^{j,p}(\Omega)$ "look like". Members of $L^p(\Omega)$ are equivalence classes of measurable functions with finite p norms, two functions being in the same equivalence class if they differ only on a set of measure zero.

Suppose that $\{u_m\}$ is a Cauchy sequence in $\hat{C}^{j,p}(\Omega)$. Then for $|\alpha| \le j$, $\{D^\alpha u_m\}$ is a Cauchy sequence in $L^p(\Omega)$. Hence, there are members u^α of $L^p(\Omega)$ such that $D^\alpha u_m \to u^\alpha$ in $L^p(\Omega)$. Hence, according to our definition of strong derivatives, u^0 is in $L^p(\Omega)$ and u^α is the α strong derivative of u^0. Hence we see that

$$H^{j,p}(\Omega) = \{u \in L^p(\Omega) : u \text{ has strong } L^p \text{ derivatives of order} \le j \text{ in } L^p(\Omega) \text{ and there exists a sequence } \{u_m\} \text{ in } \hat{C}^{j,p}(\Omega) \text{ such that } D^\alpha u_m \to D^\alpha u \text{ in } L^p(\Omega)\}.$$

Definition $W^{j,p}(\Omega) = \{u \in L^p(\Omega)$: the weak derivatives of order $\le j$ of u are in $L^p(\Omega)\}$

Note that by Theorem 3, an equivalent definition of $W^{j,p}(\Omega)$ is obtained by writing "strong derivatives" instead of "weak derivatives". Because of this, we see easily that $H^{j,p}(\Omega) \subset W^{j,p}(\Omega)$. In fact, $H^{j,p}(\Omega) = W^{j,p}(\Omega)$. This is not obvious because for members of $H^{j,p}(\Omega)$ we can find sequences of nice functions such that $D^\alpha u_m \to D^\alpha u$ in the topology of $L^p(\Omega)$, while according to our definition of strong derivatives, such limits exist only in the topology of $L^p_{loc}(\Omega)$ for members of $W^{j,p}(\Omega)$. Before proving that $H^{j,p}(\Omega) = W^{j,p}(\Omega)$, we need the concept of a partition of unity.

LEMMA 4 *Let $E \subset R^n$ and let* G *be a collection of open sets U such that $E \subset \{\cup U: U \in \mathsf{G}\}$. Then there exists a family* F *of non-negative functions $f \in C_0^\infty(R^n)$ such that $0 \le f(x) \le 1$ and*

(i) *for each $f \in \mathsf{F}$, there exists $U \in \mathsf{G}$ such that* $\operatorname{supp} f \subset U$,

(ii) *if $K \subset E$ is compact then* $\operatorname{supp} f \cap K$ *is non-empty for only finitely many $f \in \mathsf{F}$,*

(iii) $\sum_{f \in \mathsf{F}} f(x) = 1$ *for each $x \in E$ (because of* (ii), *this sum is finite),*

(iv) *if $\mathsf{G} = \{\Omega_1, \Omega_2, \ldots\}$ where each Ω_i is bounded and $\overline{\Omega}_i \subset E$ then the family* F *of such functions can be constructed so that $\mathsf{F} = \{f_1, f_2, ..\}$ and* $\operatorname{supp} f_j \subset \Omega_j$.

The family of functions F is called a *partition of unity subordinate to the cover* G.

PROOF Suppose first that E is compact, so there exists a positive integer N such that $E \subset \cup_{i=1}^N U_i$, where each $U_i \in \mathsf{G}$. Pick compact sets $E_i \subset U_i$ such that $E \subset \cup_{i=1}^N E_i$. Let $g_i = J_{\varepsilon_i} \chi_{E_i}$, where ε_i is chosen to be so small that $\operatorname{supp} g_i \subset U_i$. Then $g_i \in C_0^\infty(U_i)$ and $g_i > 0$ on a neighborhood of E_i. Let $g = \sum_{i=1}^N g_i$, and let $S = \operatorname{supp} g \subset \cup_{i=1}^N U_i$. If $\varepsilon < dist(E, \partial S)$ then $k = J_\varepsilon \chi_S$ is zero on E and $h = g + k \in C^\infty(R^n)$. Further, $h > 0$ on R^n and $h = g$ on E. Thus $\mathsf{F} = \{f_i: f_i = g_i / h\}$ does the job.

If E is open, let

$$E_i = E \cap \overline{B}_i(0) \cap \{x: dist(x, \partial E) \ge \frac{1}{i}\}.$$

Thus E_i is compact and $E = \cup_{i=1}^N E_i$. Let G_i be the collection of all open sets of the form $U \cap [Interior(E_{i+1}) - E_{i-2}]$, where $U \in \mathsf{G}$ and $E_0 = E_{-1} = \varnothing$. The members of G_i provide an open cover for the compact set $E_i - Interior(E_{i-1})$, so they possess a partition of unity F_i with finitely many elements. We let

$$s(x) = \sum_{i=1}^\infty \sum_{g \in \mathsf{F}_i} g(x)$$

and observe that only finitely many terms are represented and that $s > 0$ on E. Now we let F be the collection of all functions of the form

$$f(x)=\begin{cases}\dfrac{g(x)}{s(x)}, & x\in E\\ 0, & x\notin E\end{cases}$$

This F does the job.

If E is not open, note that any partition of unity for $\cup U$ is a partition of unity for E.

For the proof of (iv), let H be the partition of unity obtained above and let f_i = sum of functions h in H such that $\operatorname{supp} h\subset\Omega_i$, but $\operatorname{supp} h\not\subset\Omega_j$, $j<i$. Note that each h is represented in one and only one of these sums and that the sums are finite since each $\overline{\Omega}_i$ is a compact subset of E. Thus the functions f_i provide the required partition of unity.

THEOREM 5 (Meyers and Serrin, 1964) $H^{j,p}(\Omega)=W^{j,p}(\Omega)$.

PROOF We already know that $H^{j,p}(\Omega)\subset W^{j,p}(\Omega)$. The opposite inclusion follows if we can show that for every $u\in W^{j,p}$ and for every $\varepsilon>0$ we can find $w\in\hat{C}^{j,p}$ such that for $|\alpha|\le j$, $\|D^\alpha w-D^\alpha u\|_{L^p(\Omega)}<\varepsilon$.

For $m\ge 1$ let

$$\Omega_m=\{x\in\Omega:\|x\|<m,\ dist(x,\partial\Omega)>\frac{1}{m}\}$$

and let $\Omega_0=\Omega_{-1}=\varnothing$. Let $\{\psi_m\}$ be the partition of unity of part (iv), Theorem 4, subordinate to the cover $\{\Omega_{m+2}-\overline{\Omega}_m\}$. Each $u\psi_m$ is j times weakly differentiable and has support in $\Omega_{m+2}-\overline{\Omega}_m$. As in the "conversely" part of the proof of Theorem 3, we can pick $\varepsilon_m>0$ so small that $w_m=J_{\varepsilon_m}(u\psi_m)$ has support in $\Omega_{m+3}-\overline{\Omega}_{m-1}$ and $|w_m-u\psi_m|_{j,p}<\dfrac{\varepsilon}{2^m}$. Let $w=\Sigma_{m=1}^{\infty}w_m$. This is a C^∞ function because on each set $\Omega_{m+2}-\overline{\Omega}_m$ we have $w=w_{m-2}+w_{m-1}+w_m+w_{m+1}+w_{m+2}$. Further,

$$\begin{aligned}\|D^\alpha w-D^\alpha u\|_{L^p(\Omega)}&=\|\Sigma_{m=1}^{\infty}D^\alpha(w_m-u\psi_m)\|_{L^p(\Omega)}\\&\le\Sigma_{m=1}^{\infty}\|D^\alpha(w_m-u\psi_m)\|_{L^p(\Omega)}\\&\le\Sigma_{m=1}^{\infty}\varepsilon/2^m=\varepsilon\,.\end{aligned}$$

Remarks

(i) The proof shows that in fact $C^{\infty}(\Omega)\cap\hat{C}^{j,p}(\Omega)$ is dense in $W^{j,p}(\Omega)$.

(ii) Clearly members of $C^{\infty}(\Omega)\cap\hat{C}^{j,p}(\Omega)$ are not necessarily continuous on $\partial\Omega$ or even bounded near $\partial\Omega$. It would be very useful to have the knowledge that $C^{\infty}(\overline{\Omega})\cap\hat{C}^{j,p}(\Omega)$ or $C^{j}(\overline{\Omega})\cap\hat{C}^{j,p}(\Omega)$ is also dense in $W^{j,p}(\Omega)$. But the following example shows that this cannot always be expected.

Problem 1 Let $\Omega=\{(x,y):1<x^2+y^2<2, y\neq 0 \text{ if } x>0\}$, i.e. an annulus minus the positive x-axis. Let $w(x,y)=\theta$, the angular polar coordinate of (x,y). Clearly w is in $W^{1,1}(\Omega)$ because it is a bounded continuously differentiable function. Show that we cannot find a $\phi\in C^1(\overline{\Omega})$ such that $|u-\varphi|_{1,1}<2\pi$. (Note that $\overline{\Omega}$ is the whole annulus).

The reason for the failure of the domain in Problem 1 is that the domain is on each side of part of its boundary. The following definition expresses the idea of a domain lying on only one side of its boundary.

Definition A domain Ω has the *segment property* if for each $x\in\partial\Omega$ there exists an open ball U centered at x and a vector y such that if $z\in\overline{\Omega}\cap U$ then $z+ty\in\Omega$ for $0<t<1$.

We will not need the following theorem, so we don't prove it. For a proof, see Adam's book. However, see Lemma 9 for the simpler version of the result that we will need.

THEOREM 6 *If Ω has the segment property then the set of restrictions to Ω of functions in $C_0^{\infty}(R^n)$ is dense in $W^{m,p}(\Omega)$.*

THEOREM 7 Change of Variables and the Chain Rule. *Let V, Ω be domains in R^n and let $T\colon V\to\Omega$ be invertible. Suppose that T and T^{-1} have continuous, bounded derivatives of order $\leq j$. Then if $u\in W^{j,p}(\Omega)$ we have $v=u\circ T\in W^{j,p}(V)$ and the derivatives of v are given by the chain rule.*

PROOF Let y denote coordinates in Ω and let x denote coordinates in V ($y=T(x)$). If $f\in L^p(\Omega)$ then $f\circ T\in L^p(V)$ because

$$\int_V |f\circ T|^p\,dx=\int_\Omega |f|^p\,J\,dy\leq const.\int_\Omega |f|^p\,dy \tag{5}$$

(Here J is the Jacobian of T^{-1}).

If $u \in W^{j,p}(\Omega)$, let $\{u_m\}$ be a sequence in $\hat{C}^{j,p}(\Omega)$ converging to u in $W^{j,p}(\Omega)$ and set $v_m = u_m \circ T$. By the chain rule, if $|\alpha| \le j$

$$D_x^\alpha v_m = \sum_{\beta \le \alpha} (D_y^\beta u_m) \circ T \ R_{\alpha,\beta}$$

where the $R_{\alpha,\beta}$ are bounded terms involving T and its derivatives. But for $|\beta| \le j$ $D_y^\beta u \in L^p(\Omega) \Rightarrow (D_y^\beta u) \circ T \in L^p(V) \Rightarrow (D_y^\beta u) \circ T R_{\alpha,\beta} \in L^p(V)$ since the $R_{\alpha,\beta}$ are bounded. Further,

$$\begin{aligned} \| D_x^\alpha v_m - \Sigma_{\beta \le \alpha} (D_y^\beta u) \circ T R_{\alpha,\beta} \|_{L^p(V)} &= \| \Sigma_{\beta \le \alpha} (D_y^\beta u_m - D_y^\beta u) \circ T R_{\alpha,\beta} \|_{L^p(V)} \\ &\le \Sigma_{\beta \le \alpha} \| (D_y^\beta u_m - D_y^\beta u) \circ T R_{\alpha,\beta} \|_{L^p(V)} \\ &\le const. \Sigma_{\beta \le \alpha} \| (D_y^\beta u_m - D_y^\beta u) \circ T \|_{L^p(V)} \\ &\le const. \Sigma_{\beta \le \alpha} \| D_y^\beta u_m - D_y^\beta u \|_{L^p(\Omega)} \end{aligned}$$

by (5). So ($\alpha = 0$ case), $v_m \to v = u \circ T$ in $L^p(V)$ and $D_x^\alpha v_m \to \Sigma_{\beta \le \alpha} (D_y^\beta u) \circ T \ R_{\alpha,\beta}$ in $L^p(V)$. This shows that $v \in W^{j,p}(V)$ and $D_x^\alpha v = \Sigma_{\beta \le \alpha} (D_y^\beta u) \circ T \ R_{\alpha,\beta}$.

Definition $W_0^{j,p}(\Omega)$ =completion of $C_0^\infty(\Omega)$ with respect to the norm $| \ |_{j,p}^\Omega$.

Remarks (i) Clearly $W_0^{j,p}(\Omega) \subset W^{j,p}(\Omega)$ because $C_0^\infty(\Omega) \subset \hat{C}^{j,p}(\Omega)$.

(ii) Saying that $f \in W_0^{j,p}(\Omega)$ is a generalized way of saying that f and its derivatives of order $\le j-1$ vanish on $\partial\Omega$. e.g. $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ is a useful space for studying solutions of the Dirichlet problem for second order elliptic PDE's.

(iii) $C_0^j(\Omega) \subset W_0^{j,p}(\Omega)$ because if $f \in C_0^j(\Omega)$, we know that if ε is sufficiently small then $J_\varepsilon f \in C_0^\infty(\Omega)$ and $J_\varepsilon f \to f$ in $| \ |_{j,p}^\Omega$ norm.

Problem 2 Show that $W^{j,p}(R^n) = W_0^{j,p}(R^n)$. Hint: Why is it enough to show that $\hat{C}^{j,p}(R^n) \subset W_0^{j,p}(R^n)$?

Problem 3 Show that if Ω is a domain in R^n, $f \in W_0^{j,p}(\Omega)$ and if f is extended to be zero outside Ω then the new function is in $W^{j,p}(R^n)$.

Problem 4 Show that if $y \in C^1[0,1]$ and $y(0) = y(1) = 0$ then $y \in W_0^{1,p}(0,1)$. Use this fact to show that for any $f \in L^p(0,1)$ there is a unique $y \in W_0^{1,p}(0,1) \cap W^{2,p}(0,1)$ such that $y'' - y = f$. Hint: Solve the problem first with $f \in C_0^\infty(0,1)$ and then take limits.

2. Extension Theorems

Most of the important Sobolev inequalities and imbedding theorems that we will derive in the next section are most easily derived for the space $W_0^{j,p}(\Omega)$ which (see Problem 3) can be viewed as being a subspace of $W^{j,p}(R^n)$. Direct derivations of these results for the spaces $W^{j,p}(\Omega)$ are tedious and difficult because of the boundary behavior of the functions (Adams uses the direct derivation approach in his book). In this section we investigate the existence of extension operators that allow us to extend functions in $W^{j,p}(\Omega)$ to be functions in $W^{j,p}(R^n)$. This will allow us to easily deduce the Sobolev imbedding theorems for the spaces $W^{j,p}(\Omega)$ from the corresponding results for $W^{j,p}(R^n)$.

LEMMA 8 *Let $u \in R^n$ and $f \in L^p(R^n)$. Set $f_\delta(x) = f(x+\delta u)$. Then $\lim_{\delta \to 0} f_\delta = f$ in $L^p(R^n)$.*

PROOF Given $\varepsilon > 0$, let $\phi \in C_0^\infty(R^n)$ be such that $\| f - \phi \|_{L^p} < \varepsilon$. Since $\phi_\delta \to \phi$ uniformly on a sufficiently large ball containing the supports of all ϕ_δ (say, for $\delta \le 1$), we can pick δ so small that $\| \phi - \phi_\delta \|_{L^p} < \varepsilon$. Then

$$\| f - f_\delta \|_{L^p} \le \| f - \phi \|_{L^p} + \| \phi - \phi_\delta \|_{L^p} + \| \phi_\delta - f_\delta \|_{L^p} < 3\varepsilon .$$

LEMMA 9 *Let $R_+^n = \{x \in R^n : x_n > 0\}$. $C^\infty(\overline{R}_+^n) \cap \hat{C}^{j,p}(R_+^n)$ is dense in $W^{j,p}(R_+^n)$.*

PROOF Suppose *f* is in $W^{j,p}(R_+^n)$ let $\varepsilon > 0$ and pick $\phi \in C^\infty(R_+^n) \cap \hat{C}^{j,p}(R_+^n)$ so that $\| D^\alpha \phi - D^\alpha f \|_{L^p(R_+^n)} < \varepsilon$ for all $|\alpha| \le j$. We take the vector of Lemma 8 to be $u = (0,0,0,\ldots,1)$ and define functions $\psi^\alpha \in L^p(R^n)$ as

$$\psi^\alpha(x) = \begin{cases} D^\alpha \phi(x) & , x_n > 0 \\ 0 & , x_n \le 0 \end{cases}$$

Observe that for each $\delta > 0$, $\phi_\delta \in C^\infty(\overline{R}_+^n) \cap \hat{C}^{j,p}(R_+^n)$. By Lemma 8, we can pick $\delta > 0$ so that, for all $|\alpha| \le j$, $\| \psi_\delta^\alpha - \psi^\alpha \|_{L^p(R^n)} < \varepsilon$. But this implies that $\| D^\alpha \phi_\delta - D^\alpha \phi \|_{L^p(R_+^n)} < \varepsilon$. Hence

$$\| D^\alpha \phi_\delta - D^\alpha f \|_{L^p(R_+^n)} \le \| D^\alpha \phi_\delta - D^\alpha \phi \|_{L^p(R_+^n)} + \| D^\alpha \phi - D^\alpha f \|_{L^p(R_+^n)} < 2\varepsilon .$$

LEMMA 10 *There exists a linear mapping* $E_0\colon W^{j,p}(R_+^n) \to W^{j,p}(R^n)$ *such that* $E_0 f = f$ *in* R_+^n *and* $|E_0 f|_{j,p}^{R^n} \le C|f|_{j,p}^{R_+^n}$*, where C depends on only n and p.*

PROOF If $f \in C^\infty(\overline{R}_+^n)$, define

$$E_0 f(x) = \begin{cases} f(x) & , x_n \ge 0 \\ \sum_{k=1}^{j+1} c_k f(x_1, x_2, \ldots, x_{n-1}, -kx_n) & , x_n < 0 \end{cases}$$

where the constants c_k are chosen so that $E_0 f(x) \in C^j(R^n)$, i.e.

$$\sum_{k=1}^{j+1} (-k)^m c_k = 1, \quad m = 0,1,2,\ldots,j\,.$$

It is easy to check that there is a constant C depending on only n and p such that

$$\|D^\alpha E_0 f\|_{L^p(R^n)} \le C\|D^\alpha f\|_{L^p(R_+^n)}. \tag{6}$$

If now $f \in W^{j,p}(R_+^n)$, take a sequence $f_m \in C^\infty(\overline{R}_+^n) \cap \hat{C}^{j,p}(R_+^n)$ converging to f in $W^{j,p}(R_+^n)$ (we can do this by Lemma 9). Then f_m is a Cauchy sequence and (6) implies that $E_0 f_m$ is a Cauchy sequence in $W^{j,p}(R^n)$. We denote the limit by $E_0 f$. Since $\|D^\alpha E_0 f_m\|_{L^p(R^n)} \le C\|D^\alpha f_m\|_{L^p(R_+^n)}$, taking limits shows that f satisfies (6).

Definition A domain Ω is of class C^m if $\partial\Omega$ can be covered by bounded open sets Ω_j such that there are mappings $\psi_j\colon \overline{\Omega}_j \to \overline{B}$, where B is the unit ball centered at the origin and

(i) $\psi_j(\Omega_j \cap \Omega) = B \cap R_+^n$

(ii) $\psi_j(\Omega_j \cap \partial\Omega) = B \cap \partial R_+^n$

(iii) $\psi_j \in C^m(\overline{\Omega}_j)$ and $\psi_j^{-1} \in C^m(\overline{B})$.

(Because of (iii), all derivatives of order $\le m$ of ψ_j and its inverse are bounded).

THEOREM 11 *If* Ω *is a bounded domain of class* C^m *then there exists a bounded linear extension operator* $E\colon W^{m,p}(\Omega) \to W^{m,p}(R^n)$.

PROOF Since $\partial\Omega$ is compact (boundaries are always closed), we might as well assume that the number of sets Ω_j covering $\partial\Omega$ is a finite number N. Let $U = \cup_{j=1}^{N}\Omega_j$ and let $d = dist(\partial\Omega, \partial U)$. Setting $\Omega_0 = \{x \in \Omega : dist(x, \partial\Omega) > d/2\}$, we see that $\Omega_0, \Omega_1, \Omega_2, .., \Omega_N$ cover Ω. These sets also cover $\overline{\Omega}$, which is compact, so by the first part of the proof of Lemma 4, there exists a finite partition of unity $\theta_0, \theta_1, \theta_2, .., \theta_N$ for Ω and supp $\theta_j \subset \Omega_j$. Recall that the support of a function is the closure of the set on which that function is non-zero. Hence, supp θ_j is even bounded away from $\partial\Omega_j$.

Let $f \in W^{m,p}(\Omega)$. Then $f\theta_j \in W^{m,p}(\Omega \cap \Omega_j)$, so by our chain rule theorem (Theorem 7) $w_j = (f\theta_j) \circ \psi_j^{-1} \in W^{m,p}(R_+^n \cap B)$. Clearly supp w_j is bounded away from ∂B, so we can extend w_j to be a member of $W^{m,p}(R_+^n)$ by letting it be zero in $R_+^n - B$. We can further extend w_j to all of R^n by use of the extension operator E_0 of Lemma 10. Let $\tilde{w}_j = E_0 w$. If $\rho < 1$ is chosen so that supp$\theta_j \circ \psi_j^{-1} \subset \overline{B}_\rho(0)$, then we observe from the way that E_0 was constructed that supp $\tilde{w}_j \subset \overline{B}_\rho(0)$. Consequently, supp $\tilde{w}_j \circ \psi_j \subset \psi_j(\overline{B}_\rho(0))$ is bounded away from $\partial\Omega_j$. Further, again by Theorem 7, this function is in $W^{m,p}(\Omega_j)$. We extend it to be in $W^{m,p}(R^n)$ by defining it to be zero outside Ω_j. If we call the extended function g_j, it is clear from our construction that $g_j = f\theta_j$ on $\Omega \cap \Omega_j$ and that $|g_j|_{m,p}^{R^n} \le C|f|_{m,p}^{\Omega}$, where C is independent of f. Finally, we let g_0 denote the function obtained by extending $f\theta_0$ to be zero outside Ω and define $Ef = \Sigma_{j=0}^{N} g_j$.

Remarks The theorem can be improved in a number of ways:

(i) We can allow Ω to be unbounded if $\partial\Omega$ is bounded (e.g. Ω is the exterior of a bounded domain).

(ii) We can allow Ω to be of class $C^{m-1,1}$ instead of C^m (i.e. the derivatives of order $m-1$ of the functions ψ_j are Lipschitz continuous. The proof of this requires a better version of Theorem 7 which we don't have time to prove here. Note that for the case $m = 1$, the boundary could have corners.

(iii) Calderón has proved an extension theorem for domains satisfying the cone property (see the definition below) and a few other minor assumptions. The proof is much too time-consuming for us and it relies on the Calderón-Zygmund inequality, which also has a very lengthy proof. (See [Ad] for this).

Definition A domain Ω is said to satisfy the *cone property* if there exist positive constants α, h such that for each $x \in \Omega$ there exists a right spherical cone $V_x \subset \Omega$ with height h and opening α.

3. Sobolev Inequalities and Imbedding Theorems

THEOREM 12 *If $\Omega \subset R^n$ satisfies the cone condition (with height h and opening α) and if $p > 1$, $mp > n$ then $W^{m,p}(\Omega) \subset C_B(\Omega)$ and there is a constant C depending on only α, h, n and p such that for all $u \in W^{m,p}(\Omega)$, $\sup|u| \le C|u|_{m,p}$.*

Note: Ω does not have to be bounded as Friedman suggests in his Theorem 9.1!

PROOF Initially, suppose that u is in $\hat{C}^{m,p}(\Omega)$. Let $g \in C^\infty(R)$ be such that $g(t) = 1$ if $t \le 1/2$ and $g(t) = 0$ if $t \ge 1$. Let $x \in \Omega$ and let (r, θ) denote polar coordinates centered at x. Here, $\theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$ denotes the angular coordinates and we can describe the cone with vertex x in polar coordinates as $V_x = \{(r,\theta) : 0 \le r \le h, \theta \in A\}$. Clearly, we have

$$\begin{aligned} u(x) &= -\int_0^h \frac{\partial}{\partial r}\{g(r/h)u(r,\theta)\}\,dr \\ &= \frac{(-1)^m}{(m-1)!}\int_0^h r^{m-1}\frac{\partial^m}{\partial r^m}\{g(r/h)u(r,\theta)\}\,dr, \end{aligned}$$

after m-1 integrations by parts. Next, we integrate with respect to the angular measure dS_θ, noting that the left-hand-side becomes a constant times $u(x)$.

$$\begin{aligned} u(x) &= c\int_A\int_0^h r^{m-1}\frac{\partial^m}{\partial r^m}\{g(r/h)u(r,\theta)\}\,drdS_\theta \\ &= c\int_A\int_0^h r^{m-n}\frac{\partial^m}{\partial r^m}\{g(r/h)u(r,\theta)\}\,r^{n-1}drdS_\theta \\ &= c\int_{V_x} r^{m-n}\frac{\partial^m}{\partial r^m}\{g(r/h)u(r,\theta)\}\,dV. \end{aligned}$$

Applying Hölder's inequality to this, we obtain

$$\begin{aligned} |u(x)| &\le const.\|r^{m-n}\|_{L^q(V_x)}\|\tfrac{\partial^m}{\partial r^m}\{g(r/h)u(r,\theta)\}\|_{L^p(V_x)} \\ &\le const.\|r^{m-n}\|_{L^q(V_x)}|u|^{\Omega}_{m,p}. \end{aligned}$$

But r^{m-n} is in $L^q(V_x)$ if $n-1+(m-n)q > -1$, which is the case because $q = \frac{p}{p-1}$ and $mp > n$. Thus, we obtain $\sup|u| \le C|u|_{m,p}$. To extend this result to arbitrary $u \in W^{m,p}(\Omega)$, take a sequence $\{u_k\}$ of functions in $\hat{C}^{m,p}(\Omega)$ converging to u in the $|\ |^{\Omega}_{m,p}$ norm. Then $\sup|u_j - u_k| \le C|u_j - u_k|_{m,p}$, showing that the sequence is a Cauchy sequence in $C_B(\Omega)$.

Thus u is in $C_B(\Omega)$ and taking the limit of $\sup|u_j| \le C|u_j|_{m,p}$ shows that u satisfies the same inequality.

Problem 5. Modify the proof to show that the theorem also applies to the case $p = 1$, $m = n$.

Problem 6. Show that a similar theorem holds for $W_0^{m,p}(\Omega)$ and the cone condition is not required. Note that here we can even conclude that $W_0^{m,p}(\Omega) \subset \{u \in C_B(\overline{\Omega}) : u = 0$ on $\partial\Omega\}$.

COROLLARY 13 *If $\Omega \subset R^n$ satisfies the cone condition (with height h and opening α) and if $p > 1$, $(m-k)p > n$ then $W^{m,p}(\Omega) \subset C_B^k(\Omega)$ and there is a constant C depending on only α, h, n, k and p such that for all $u \in W^{m,p}(\Omega)$ $\sup_{|\alpha| \le k} |D^\alpha u| \le C|u|_{m,p}$.*

PROOF Apply the previous theorem to the derivatives $D^\alpha u$ for $|\alpha| \le k$.

Problem 7 What can you conclude if $p = 1$ and $m - k = n$? See Problem 5.

Problem 8 What is the corresponding theorem for $W_0^{m,p}(\Omega)$? See Problem 6.

THEOREM 14 *If $\Omega \subset R^n$ is any domain and $p > n$ then $W_0^{1,p}(\Omega) \subset C^{0,\alpha}(\overline{\Omega})$, where $\alpha = 1 - \frac{n}{p}$ and there exists a constant C depending on only p and n such that for all $u \in W_0^{1,p}(\Omega)$*

$$\frac{|u(x) - u(y)|}{||x - y||^\alpha} \le C \sum_{i=1}^{n} \| D_i u \|_{L^p(\Omega)} .$$

PROOF Let $u \in C_0^\infty(\Omega)$. We might as well assume that $u \in C_0^\infty(R^n)$. Let $d = \|x - y\|$, $S_x = B_d(x)$, $S_y = B_d(y)$ and $S = S_x \cap S_y$. Then

$$\begin{aligned} |u(x) - u(y)|\, vol(S) &= \int_S |u(x) - u(y)|\, dz \\ &\le \int_S |u(x) - u(z)| + |u(z) - u(y)|\, dz \\ &\le \int_{S_x} |u(x) - u(z)|\, dz + \int_{S_y} |u(z) - u(y)|\, dz \end{aligned}$$

But if (r, θ) are the polar coordinates of z in a coordinate system centered at x, we get $|u(x) - u(z)| \le \int_0^r |\frac{\partial u}{\partial \rho}|\, d\rho$, which implies

$$
\begin{aligned}
\int_{S_x} |u(x)-u(z)|\,dz &\le \int\int_0^d\int_0^r \left|\frac{\partial u}{\partial\rho}\right| d\rho\, r^{n-1} dr dS_\theta \\
&\le \int\int_0^d\int_0^d \left|\frac{\partial u}{\partial\rho}\right| d\rho\, r^{n-1} dr dS_\theta \\
&= \frac{d^n}{n}\int\int_0^d \left|\frac{\partial u}{\partial\rho}\right| d\rho dS_\theta \\
&= \frac{d^n}{n}\int\int_0^d \rho^{1-n}\left|\frac{\partial u}{\partial\rho}\right| \rho^{n-1} d\rho dS_\theta \\
&= \frac{d^n}{n}\int_{S_x} \rho^{1-n}\left|\frac{\partial u}{\partial\rho}\right| dz \\
&\le \frac{d^n}{n}\|\rho^{1-n}\|_{L^q(S_x)}\left\|\frac{\partial u}{\partial\rho}\right\|_{L^p(S_x)}
\end{aligned}
$$

where $q=\dfrac{p}{p-1}$. A simple calculation shows that

$$\|\rho^{1-n}\|_{L^q(S_x)} = const.\; d^{1-\frac{n}{p}}$$

and it is easy to see that

$$\left\|\frac{\partial u}{\partial\rho}\right\|_{L^p(S_x)} \le const. \sum_{i=1}^{n} \|D_i u\|_{L^p(\Omega)}\,.$$

Also, $vol(S) = const.\; d^n$ and the integral over S_y can be estimated in a similar fashion. Putting this together yields

$$|u(x)-u(y)| \le C d^{1-\frac{n}{p}} \sum_{i=1}^{n} \|D_i u\|_{L^p(\Omega)}$$

which is precisely the inequality that we wanted. Further, we know from Theorem 12 applied to R^n that $\sup|u| \le C|u|_{1,p}$. Combining this with the previous inequality shows that for $u \in C_0^\infty(\Omega)$ we have $\|u\|_{C^{0,\alpha}(\overline{\Omega})} \le C|u|_{1,p}$. Thus, if we now let $u \in W_0^{1,p}(\Omega)$ and take a sequence $\{u_m\}$ of functions in $C_0^\infty(\Omega)$ converging to u in $|\ |_{1,p}$ norm, it follows that $\{u_m\}$ converges in $C^{0,\alpha}(\overline{\Omega})$. Thus $u \in C^{0,\alpha}(\overline{\Omega})$, and taking limits shows that u satisfies the inequality in the statement of the theorem.

THEOREM 15 *If* $\Omega \subset R^n$ *is any domain and* $p < n$ *then* $W_0^{1,p}(\Omega) \subset L^r(\Omega)$, *where* $r = \dfrac{np}{n-p}$ *and there exists a constant* C *depending on only* p *and* n *such that for all* $u \in W_0^{1,p}(\Omega)$

$$\|u\|_{L^r(\Omega)} \le C \sum_{i=1}^{n} \|D_i u\|_{L^p(\Omega)} .$$

Remark The proof relies on a simple generalization of Hölder's inequality which can be proved by induction by using Hölder's inequality. The inequality states

$$\int_\Omega |u_1 u_2 u_3 \ldots u_m|\, dx \le \|u_1\|_{L^{p_1}} \|u_2\|_{L^{p_2}} \ldots \|u_m\|_{L^{p_m}} \qquad (7)$$

where $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \dfrac{1}{p_3} + \ldots + \dfrac{1}{p_m} = 1.$

PROOF of Theorem 15. It suffices to prove the result for $u \in C_0^1(R^n)$. First we prove the result for the case $p = 1$. For each i we have

$$|u(x)| \le \int_{-\infty}^{x_i} |D_i u|\, dx_i \le \int_{-\infty}^{\infty} |D_i u|\, dx_i .$$

Multiplying these n inequalities together and taking the $n-1$ th root gives

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |D_i u|\, dx_i\right)^{\frac{1}{n-1}} \qquad (8)$$

Observe that $\int_{-\infty}^{\infty} |D_i u|\, dx_i$ does not depend on x_i, but it does depend on all $n-1$ of the remaining variables. We integrate each side of (8) with respect to x_1 and use the generalized Hölder inequality with $p_i = m = n-1$ to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\, dx_1 &\le \left(\int_{-\infty}^{\infty} |D_1 u|\, dx_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |D_i u|\, dx_i\right)^{\frac{1}{n-1}} dx_1 \\ &\le \left(\int_{-\infty}^{\infty} |D_1 u|\, dx_1\right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_i u|\, dx_i\, dx_1\right)^{\frac{1}{n-1}} .\end{aligned}$$

The RHS is still a product of $n-1$ functions of x_2, so we integrate each side with respect to x_2, again applying (7) with $p_i = m = n-1$. Continuing in this manner, we finally obtain

$$\int_{R^n} |u(x)|^{\frac{n}{n-1}}\, dx \le \Big(\prod_{i=1}^{n} \int_{R^n} |D_i u|\, dx\Big)^{\frac{1}{n-1}}$$

i.e.

$$\|u\|_{L^{\frac{n}{n-1}}} \le \Big(\prod_{i=1}^{n} \int_{R^n} |D_i u|\, dx\Big)^{\frac{1}{n}} \qquad \text{(geometric mean)}$$

$$\le \frac{1}{n}\sum_{i=1}^{n} \int_{R^n} |D_i u|\, dx \qquad \text{(arithmetic mean)}$$

Here we have used the fact that an arithmetic mean is no less than a geometric mean of the same numbers. This proves the result for the case $p=1$.

For $p>1$, let $\gamma = \dfrac{(n-1)p}{n-p} = 1 + \dfrac{n(p-1)}{n-p}$. Since $\gamma > 1$ and $u \in C_0^1(R^n)$, it follows that $|u|^\gamma \in C_0^1(R^n)$. Clearly

$$D_i |u|^\gamma = \frac{(n-1)p}{n-p} |u|^{\frac{n(p-1)}{n-p}} (\pm D_i u).$$

We apply the $p=1$ case to $|u|^\gamma$ and obtain

$$\begin{aligned}
\Big(\int_{R^n} |u|^{\frac{np}{n-p}}\, dx\Big)^{\frac{n-1}{n}} &\le \sum_{i=1}^{n} \frac{1}{n} \int_{R^n} \frac{(n-1)p}{n-p} |u|^{\frac{n(p-1)}{n-p}} |D_i u|\, dx \\
&\le \frac{(n-1)p}{n(n-p)} \sum_{i=1}^{n} \Big(\int_{R^n} \big(|u|^{\frac{n(p-1)}{n-p}}\big)^{\frac{p}{p-1}}\, dx\Big)^{\frac{p-1}{p}} \|D_i u\|_{L^p} \\
&= \frac{(n-1)p}{n(n-p)} \sum_{i=1}^{n} \Big(\int_{R^n} |u|^{\frac{np}{n-p}}\, dx\Big)^{\frac{p-1}{p}} \|D_i u\|_{L^p}
\end{aligned}$$

Hence

$$\Big(\int_{R^n} |u|^{\frac{np}{n-p}}\, dx\Big)^{\frac{n-p}{np}} \le \frac{(n-1)p}{n(n-p)} \sum_{i=1}^{n} \|D_i u\|_{L^p},$$

which is the desired result. As usual, to obtain the same result for a function $u \in W_0^{1,p}(\Omega)$, we just take a sequence of functions in C_0^1 converging to *u*.

Remark According to the theorem, $W_0^{1,p}(\Omega) \subset L^r(\Omega)$, where r is given above. But obviously $W_0^{1,p}(\Omega) \subset L^p(\Omega)$, so by the following interpolation lemma, $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for all q satisfying $p \le q \le r$. If Ω is bounded then clearly this holds for all q satisfying $1 \le q \le r$.

Lemma 16 *If $s \le q \le r$ and $\phi \in L^s(\Omega) \cap L^r(\Omega)$, then $\phi \in L^q(\Omega)$ and*

$$\|\phi\|_{L^q} \le \|\phi\|_{L^s}^{\lambda} \|\phi\|_{L^r}^{1-\lambda},$$

where $\lambda = \dfrac{s(r-q)}{q(r-s)}$.

Proof Apply Hölder's inequality to the integral of $|\phi|^q$, using the facts that $|\phi|^{(1-\lambda)q} \in L^{\frac{r}{(1-\lambda)q}}$ and $|\phi|^{\lambda q} \in L^{\frac{s}{\lambda q}}$.

Problem 9 Modify the proof of Theorem 15 to show that if $p = n > 1$ then $W_0^{1,p}(\Omega) \subset L^r(\Omega)$ for every $r \ge p$. Hint: First prove this for the case $r > \dfrac{n^2}{n-1}$ by setting $\gamma = r\dfrac{n-1}{n}$ and applying the $p = 1$ result to $|u|^{\gamma}$. The $r - \dfrac{n}{n-1}$ norm that shows up on the RHS after applying Hölder's inequality (as we did in our proof above) can be estimated in terms of the $n = p$ norm and the r norm by use of Lemma 16. Finally, obtain the result for all $r \ge p$ by applying Lemma 16 to the result that you have just proved.

Corollary 17 *For every domain Ω in R^n there exists a constant C depending on only n and p such that*

a) *if $kp < n$ then $W_0^{k,p}(\Omega) \subset L^{\frac{np}{n-kp}}(\Omega)$ and for each $u \in W_0^{k,p}(\Omega)$*

$$\|u\|_{L^{\frac{np}{n-kp}}} \le C|u|_{k,p}^{\Omega}$$

b) *if $kp > n$ then $W_0^{k,p}(\Omega) \subset C^{m,\alpha}(\overline{\Omega})$, where m is the integer satisfying $0 < k - m - \dfrac{n}{p} < 1$ and $\alpha = k - m - \dfrac{n}{p}$. Further, if $u \in W_0^{k,p}(\Omega)$ then*

$$\|u\|_{C^{m,\alpha}(\overline{\Omega})} \le C|u|_{k,p}^{\Omega}.$$

PROOF a) If $|\beta| \le k-1$ and $u \in W_0^{k,p}$ then $D^\beta u \in W_0^{1,p}$, which is contained in $L^{\frac{np}{n-p}}$ by Theorem 15. Thus $W_0^{k,p} \subset W_0^{k-1,\frac{np}{n-p}}$. Iterating this process once more, we find $W_0^{k,p} \subset W_0^{k-2,\frac{np}{n-2p}}$. Continuing the iterations culminates in the desired result.

b) Since $(k-m-1)p < n$, case (a) implies that $W_0^{k-m-1,p} \subset L^{\frac{np}{n-(k-m-1)p}} = L^{\frac{n}{(1-\alpha)}}$. Thus if $u \in W_0^{k,p}$ and $|\beta| \le m+1$, then $D^\beta u \in W_0^{k-m-1,p}$. Hence $W_0^{k,p} \subset W_0^{m+1,\frac{n}{(1-\alpha)}}$. But this shows that if $u \in W_0^{k,p}$ and $|\beta| \le m$ then $D^\beta u \in W_0^{1,\frac{n}{(1-\alpha)}}$, which is contained in $C^{0,\alpha}(\overline{\Omega})$ by Theorem 14. Thus $W_0^{k,p}(\Omega) \subset C^{m,\alpha}(\overline{\Omega})$.

Remarks A few "particular cases" have been left out because they require separate proofs (see Problem 10 below). They are:

i) If $kp = n$ and $p > 1$ then $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ for all q satisfying $p \le q < \infty$.

ii) If $kp = n$ and $p = 1$ (so that $k = n$) then $W_0^{k,p}(\Omega) \subset C_B(\overline{\Omega})$.

iii) If $kp > n$, $p > 1$ and $\dfrac{n}{p}$ is an integer then $W_0^{k,p}(\Omega) \subset W_0^{k-\frac{n}{p},q}(\Omega)$ for all q satisfying $p \le q < \infty$.

iv) If $kp > n$ and $p = 1$ (so $\dfrac{n}{p}$ is obviously an integer) then $W_0^{k,p}(\Omega) \subset C_B^{k-n}(\overline{\Omega})$.

All of the particular cases listed above have the appropriate norm inequalities associated with them.

Problem 10 Use the results of Problems 5, 6, 7, 8 and 9 to prove the particular cases listed above.

COROLLARY 18 *If Ω is a bounded C^1 domain in R^n (or any other domain such that there exists a bounded extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(R^n)$) then the statements concerning the spaces $W_0^{k,p}(\Omega)$ in Corollary 17 and in the remark following the corollary also apply to the spaces $W^{k,p}(\Omega)$. However, the constant C may also depend on Ω.*

Proof The cases for $k=1$ dealt with in Theorems 14 and 15 are easily seen to have their counterparts here because of the extension operator. Inspection of the proof of Corollary 17 shows how the results for $k>1$ may be derived from the results for $k=1$ without any additional assumptions on the domain.

Remark One can show that extension operators exist for Lipschitz domains and even domains satisfying certain cone conditions (see the remarks following the proof of Theorem 11). This Sobolev imbedding theorem is thus valid for such domains.

Definition Let A and B be Banach spaces. If $A \subset B$, we say that A is *continuously imbedded* in B (in symbols, this is written A\o(\s\do3($\to$),$\subset$)B)if there is a constant C such that $\|x\|_B \le C\|x\|_A$ for all $x \in A$.

The theorems in this section provide examples of imbeddings and are called Sobolev Imbedding Theorems. e.g. $W_0^{1,p}(\Omega)$ \o(\s\do3($\to$),$\subset$) $L^{\frac{np}{n-p}}$ for $p<n$.

It is easy to see that A\o(\s\do3($\to$),$\subset$)B is equivalent to the identity mapping from A into B being continuous (i.e. bounded).

4. Compactness Theorems

Definition Suppose that A\o(\s\do3($\to$),$\subset$)B. We say that A is *compactly imbedded* in B if every sequence bounded in A has a subsequence that converges in B.

e.g. If K is compact then any bounded sequence in $C^1(K)$ is a set of equicontinuous functions so, by the Arzela-Ascoli Theorem, it has a subsequence that converges in $C(K)$. i.e. $C^1(K)$ is compactly imbedded in $C(K)$.

Recall that if A and B are Banach spaces and if $M: A \to B$ is a bounded linear mapping then M is said to be compact if for every bounded sequence $\{x_m\}$ in A the sequence $\{Mx_m\}$ has a subsequence that converges. Thus, saying that A is compactly imbedded in B is equivalent to saying that the identity mapping from A into B is compact. It is easy to see that if $M: A \to B$ and $P: B \to C$ are bounded linear mappings and A, B and C are Banach spaces then PM is compact if one of the mappings A or B is compact. Consequently, we obtain the very useful result that if A\o(\s\do3($\to$),$\subset$)B and B\o(\s\do3($\to$),$\subset$)C then the imbedding A\o(\s\do3($\to$),$\subset$)C is compact if one of the other two imbeddings is compact.

LEMMA 19 *Suppose that* Ω *is a bounded domain. If*

a) $0 < \lambda \le 1$ *then* $C^{m,\lambda}(\overline{\Omega})$ *is compactly imbedded in* $C^{m}(\overline{\Omega})$.

b) $0 < \nu < \lambda \le 1$ *then* $C^{m,\lambda}(\overline{\Omega})$ *is compactly imbedded in* $C^{m,\nu}(\overline{\Omega})$.

PROOF It suffices to prove the results for $m = 0$ because, once this is done, we can apply this case to the derivatives of the functions and deduce the result for general m.. Let $\{f_j\}$ be a sequence in $C^{0,\lambda}(\overline{\Omega})$ such that $\| f_j \|_{C^{0,\lambda}} \le M$. But this implies $| f_j(x) - f_j(y) | \le M \| x - y \|^{\lambda}$, showing that the sequence is a bounded, equicontinuous set of functions. By the Arzela-Ascoli Theorem, there exists a subsequence $\{f_{j_k}\}$ that converges in $C(\overline{\Omega})$. Thus $C^{0,\lambda}(\overline{\Omega})$ is compactly imbedded in $C(\overline{\Omega})$.

We show below that the same subsequence also converges in $C^{0,\nu}(\overline{\Omega})$. Suppose that $\psi \in C^{0,\lambda}(\overline{\Omega})$. Then

$$
\begin{aligned}
[\psi]_{0,\nu} &= \sup \frac{|\psi(x)-\psi(y)|}{\|x-y\|^{\nu}} = \sup \left(\frac{|\psi(x)-\psi(y)|}{\|x-y\|^{\lambda}} \right)^{\frac{\nu}{\lambda}} |\psi(x)-\psi(y)|^{1-\frac{\nu}{\lambda}} \\
&\le 2^{1-\frac{\nu}{\lambda}} ([\psi]_{0,\lambda})^{\frac{\nu}{\lambda}} (\max|\psi|)^{1-\frac{\nu}{\lambda}}
\end{aligned}
$$

We apply this to $f_{j_k} - f_{j_r}$, noting that $[f_{j_k} - f_{j_r}]_{0,\lambda} \le [f_{j_k}]_{0,\lambda} + [f_{j_r}]_{0,\lambda} \le 2M$, and obtain

$$
[f_{j_k} - f_{j_r}]_{0,\nu} \le 2M^{\frac{\nu}{\lambda}} (\max|f_{j_k} - f_{j_r}|)^{1-\frac{\nu}{\lambda}},
$$

showing that the subsequence is a Cauchy sequence in $C^{0,\nu}(\overline{\Omega})$ (because it converges in $C(\overline{\Omega})$). Thus the subsequence converges in $C^{0,\nu}(\overline{\Omega})$.

COROLLARY 20 *If* Ω *is bounded,* $kp > n$ *and* $0 < k - m - \frac{n}{p} < 1$ *then* $W_0^{k,p}(\Omega)$ *is compactly imbedded in* $C^{m,\beta}(\overline{\Omega})$ *if* $\beta < k - m - \frac{n}{p}$.

PROOF Let $\alpha = k - m - \frac{n}{p}$. Then $W_0^{k,p}(\Omega)$\o(\s\do3($\rightarrow$),$\subset$)$C^{m,\alpha}(\overline{\Omega})$\o(\s\do3($\rightarrow$),$\subset$)$C^{m,\beta}(\overline{\Omega})$, and the second imbedding is compact.

COROLLARY 21 *If Ω is a bounded C^1 domain (or any other domain for which there is a bounded extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(R^n)$), $kp > n$ and $0 < k - m - \dfrac{n}{p} < 1$ then $W^{k,p}(\Omega)$ is compactly imbedded in $C^{m,\beta}(\overline{\Omega})$ if $\beta < k - m - \dfrac{n}{p}$.*

PROOF Let $\phi \in C_0^\infty(R^n)$ be such that supp ϕ is contained in some ball B containing Ω and $\phi = 1$ on Ω. Then we can define $\tilde{E} : W^{1,p}(\Omega) \to W_0^{1,p}(B)$ by $\tilde{E}(f) = \phi E(f)$. By Corollary 20, $W_0^{1,p}(B)$ is compactly imbedded in $C^{0,\beta}(\overline{B})$. Hence $W^{1,p}(\Omega)$ is compactly imbedded in $C^{0,\beta}(\overline{\Omega})$. The result for general k can be deduced from the $k = 1$ case by considering derivatives of the functions (as in the proof of Corollary 17 (b), deduce that if $u \in W^{k,p}(\Omega)$ and $|\beta| \le m$ then $D^\beta u \in W^{1,\frac{n}{(1-\alpha)}}(\Omega)$, which is contained in $C^{0,\alpha}(\overline{\Omega})$**)**.

Remark The statements of Corollaries 20 and 21 continue to hold if we replace the condition $0 < k - m - \dfrac{n}{p} < 1$ by the condition $0 < k - m - \dfrac{n}{p} \le 1$, i.e. we can include the case of $\dfrac{n}{p}$ being an integer. The proof of this is left to the next problem.

Problem 11 Show that if Ω is bounded then the statements in the Remark above hold for the spaces $W_0^{k,p}(\Omega)$. Hint: You need to use the particular cases (iii) and (iv) following the proof of Corollary 17. For the case $p = 1$, you need to use the fact that $C^{m+1}(\overline{\Omega})$ is compactly imbedded in $C^{m,\beta}(\overline{\Omega})$, which follows easily from Lemma 19 and the fact that C^{m+1} is continuously imbedded in $C^{m,1}(\overline{\Omega})$. Can you see how to modify your proof so that it deals with $W^{k,p}(\Omega)$ if Ω is a bounded C^1 domain ?

Definition A subset E of a metric space is said to be *totally bounded* if for each $\varepsilon > 0$, E can be covered by finitely many balls of radius ε.

The following theorem is a standard result that can be found in most books on topology and in many books on functional analysis (e.g. Rudin's "Functional Analysis", Appendix A4). We omit the proof.

THEOREM 22 *Let E be a subset of a complete metric space X. Then the following statements are equivalent.*

(i) $\overline{E}$ *is compact.*

(ii) *Every sequence in E has a convergent subsequence.*

(iii) *E is totally bounded.*

The Theorem gives us two other very useful characterizations of compact mappings and imbeddings.

THEOREM 23 *If Ω is bounded and $p < n$, then $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ for all $q < \dfrac{np}{n-p}$.*

PROOF Consider first the case $q = 1$. Let A be a bounded set in $W_0^{1,p}(\Omega)$. We may consider the members of A as members of $W^{1,p}(R^n)$ with supports contained in $\overline{\Omega}$. Let $A_h = \{J_h u : u \in A\}$. Note that we have

$$|J_h u(x)| \le h^{-n} \int_\Omega \rho(\frac{x-z}{h}) |u(z)|\, dz \le h^{-n} (\max \rho) \|u\|_{L^1(\Omega)}$$

and
$$|D_i J_h u(x)| \le h^{-n-1} \int_\Omega |D_i \rho(\frac{x-z}{h})| |u(z)|\, dz \le h^{-n-1} (\max |D_i \rho|) \|u\|_{L^1(\Omega)}.$$

Since Ω is bounded, $\|u\|_{L^1} \le const. \|u\|_{L^p}$. The inequalities above show that A_h is a bounded equicontinuous set of functions in $C(\overline{\Omega})$. By the Arzela-Ascoli Theorem, every sequence in A_h has a subsequence that converges in $C(\overline{\Omega})$. Obviously, such subsequences also converge in $L^1(\Omega)$, so we see that A_h is totally bounded in $L^1(\Omega)$.

If $u \in A$ then

$$\begin{aligned} u(x) - J_h u(x) &= \int_{|z| \le 1} \rho(z)(u(x) - u(x - hz))\, dz \\ &= \int_{|z| \le 1} \rho(z) \int_0^{h\|z\|} -\frac{\partial}{\partial r} u(x - r\frac{z}{\|z\|})\, dr\, dz . \end{aligned}$$

Thus
$$|u(x) - J_h u(x)| \le \int_{|z| \le 1} \rho(z) \int_0^{h\|z\|} \sum_{i=1}^n |D_i u(x - r\frac{z}{\|z\|})|\, dr\, dz .$$

Integrating this with respect to x, we find

$$\int_\Omega |u(x)-J_h u(x)|\,dx \le \int_{|z|\le 1} \rho(z)\int_0^{h\|z\|}\sum_{i=1}^n \int_{R^n} |D_i u(x - r\frac{z}{||z||})|\,dx\,dr\,dz$$
$$= \int_{|z|\le 1} \rho(z)\int_0^{h\|z\|}\sum_{i=1}^n \int_\Omega |D_i u(x)|\,dx\,dr\,dz$$
$$= \int_{|z|\le 1} \rho(z)h||z||\sum_{i=1}^n \int_\Omega |D_i u(x)|\,dx\,dz$$
$$\le h\sum_{i=1}^n \int_\Omega |D_i u(x)|\,dx$$
$$\le hB, \tag{9}$$

where B is a constant depending on our bound of members of A in $W_0^{1,p}(\Omega)$ (again we use the fact that the L^1 norm is weaker than the L^p norm on a bounded domain).

Let $\varepsilon > 0$. Since A_h is totally bounded in $L^1(\Omega)$, we can cover A_h by a finite number of balls B_i of radius $\varepsilon/2$. Let $h = \frac{\varepsilon}{2B}$. By (9), if $J_h u \in B_i$, then u is contained in a ball of radius ε centered at the center of B_i. Thus, A is covered by a finite number of balls of radius ε. i.e. A is totally bounded in $L^1(\Omega)$. Thus $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^1(\Omega)$.

Suppose $\phi \in W_0^{1,p}(\Omega)$. Then $\phi \in L^{\frac{np}{n-p}}$ by Theorem 15 and we get from Lemma 16 (with $s=1$ and $r = \frac{np}{n-p}$) that

$$\|\phi||_{L^q} \le \|\phi\|_{L^1}^{\lambda}\,\|\phi\|_{L^{\frac{np}{n-p}}}^{1-\lambda} \le C\|\phi\|_{L^1}^{\lambda}\,(\|\sum_{i=1}^n ||D_i\phi||_{L^p})^{1-\lambda}$$

Now let $\{u_m\}$ be a bounded sequence in $W_0^{1,p}(\Omega)$ and assume $|u_m|_{1,p} \le M$. Since $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^1(\Omega)$, we can extract a subsequence $\{u_{m_j}\}$ that converges in $L^1(\Omega)$. Applying the inequality above to $u_{m_j} - u_{m_k}$, noting that $|u_{m_j} - u_{m_k}|_{1,p} \le 2M$, we obtain

$$\|u_{m_j} - u_{m_k}||_{L^q} \le const.||u_{m_j} - u_{m_k}||_{L^1}^{\lambda},$$

showing that the subsequence is a Cauchy sequence in $L^q(\Omega)$. Hence the subsequence converges in $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$.

COROLLARY 24 *If $kp < n$ and Ω is bounded then $W_0^{k,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ for all $q < \dfrac{np}{n-kp}$.*

PROOF $W_0^{k,p}(\Omega)$ is continuously imbedded in $W_0^{1,\frac{np}{n-(k-1)p}}(\Omega)$, which is compactly imbedded in $L^q(\Omega)$ if $q < \dfrac{np}{n-kp}$, by Theorem 23.

COROLLARY 25 *The same compactness results hold for $W^{k,p}(\Omega)$ if Ω is a bounded, C^1 domain (or any other type of bounded domain for which there is an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(R^n)$.*

PROOF See the proof of Corollary 21.

Remark The case $kp = n$ is missing from the previous results. But since $W^{k,p}$ is continuously imbedded in $W^{k,r}$ for all $r < p$ (provided that the domain is bounded), it follows from Corollary 25 that $W^{k,p}$ is compactly imbedded in $L^q(\Omega)$ for all $q < \infty$. The same applies to $W_0^{k,p}(\Omega)$.

5. Interpolation Results

The following results are very useful in PDE theory. We make use of Theorem 26 in our proof of Gårding's Inequality in our study of elliptic problems.

THEOREM 26 *Let $u \in W_0^{k,p}(\Omega)$. Then for any $\varepsilon > 0$ and any $0 < |\beta| < k$*

$$\| D^\beta u \|_{L^p} \leq \varepsilon |u|_{k,p} + C\varepsilon^{\frac{-|\beta|}{k-|\beta|}} \|u\|_{L^p}$$

where C is a constant depending only on k.

PROOF We prove the result for $|\beta| = 1$, $k = 2$. The general result is easily obtained from this case by induction. In fact, we show that for each i

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p} \leq \varepsilon \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|_{L^p} + \frac{72}{\varepsilon} \|u\|_{L^p} \tag{10}$$

First suppose that $u \in C_0^2(R)$ and consider an interval (a,b) of length $b-a=\varepsilon$. If $y \in (a, a+\varepsilon/3)$ and $z \in (b-\varepsilon/3, b)$, then by the Mean Value Theorem there is a $p \in (a,b)$ such that

$$|u'(p)|=|\frac{u(z)-u(y)}{z-y}|\leq \frac{3}{\varepsilon}(|u(z)|+|u(y)|)$$

Consequently, for every $x \in (a,b)$, we obtain

$$|u'(x)|=|u'(p)+\int_p^x u''(t)\,dt|\leq \frac{3}{\varepsilon}(|u(z)|+|u(y)|)+\int_a^b |u''(t)|\,dt.$$

Integrating with respect to y and z over the intervals $(a, a+\varepsilon/3)$ and $(b-\varepsilon/3, b)$ respectively, we obtain

$$|u'(x)|\leq \int_a^b |u''(t)|\,dt+\frac{18}{\varepsilon^2}\int_a^b |u(t)|\,dt,$$

so by Hölder's inequality and the inequality $(A+B)^p \leq 2^{p-1}(A^p+B^p)$,

$$\begin{aligned}|u'(x)|^p &\leq 2^{p-1}(\{\int_a^b |u''(t)|\,dt\}^p+\frac{(18)^p}{\varepsilon^{2p}}\{\int_a^b |u(t)|\,dt\}^p)\\ &\leq 2^{p-1}(\{\int_a^b |u''(t)|^p\,dt\}\{\int_a^b 1\,dt\}^{p-1}+\frac{(18)^p}{\varepsilon^{2p}}\{\int_a^b |u(t)|^p\,dt\}\{\int_a^b 1\,dt\}^{p-1})\\ &= 2^{p-1}(\varepsilon^{p-1}\int_a^b |u''(t)|^p\,dt+\frac{(18)^p}{\varepsilon^{p+1}}\int_a^b |u(t)|^p\,dt).\end{aligned}$$

Integrating this with respect to x over the interval (a,b) gives

$$\int_a^b |u'(x)|^p\,dx = 2^{p-1}(\varepsilon^p\int_a^b |u''(t)|^p\,dt+\frac{(18)^p}{\varepsilon^p}\int_a^b |u(t)|^p\,dt).$$

We now subdivide R into intervals of length ε and obtain by adding all of these inequalities that

$$\int_{-\infty}^{\infty} |u'(x)|^p\,dx \leq 2^{p-1}(\varepsilon^p\int_{-\infty}^{\infty} |u''(t)|^p\,dt+\frac{(18)^p}{\varepsilon^p}\int_{-\infty}^{\infty} |u(t)|^p\,dt) \qquad (11)$$

Suppose now that $u \in C_0^\infty(R^n)$. Then we can apply (11) to u regarded as a function of x_i and integrate with respect to the remaining variables to obtain

$$\int_{R^n} |\frac{\partial u}{\partial x_i}|^p \, dx \le 2^{p-1}(\varepsilon^p \int_{R^n} |\frac{\partial^2 u}{\partial x_i^2}|^p \, dx + \frac{(18)^p}{\varepsilon^p} \int_{R^n} |u|^p \, dx)$$

Taking the pth root of this and using $(A^p + B^p)^{1/p} \le A + B$, we obtain (10). (Actually, we don't quite obtain (10). We actually obtain the inequality (10) for 2ε instead of ε. But since ε is an arbitrary positive constant, (10) holds). Finally, (as usual) to obtain the result for $u \in W_0^\infty(\Omega)$, we take a sequence of functions in C_0^∞ converging to u.

COROLLARY 27 *The interpolation inequality stated in Theorem 26 also applies to members of* $W^{k,p}(\Omega)$, *provided that* Ω *is a bounded* C^2 *domain (or any other domain for which there is a bounded extension operator* $E: W^{2,p}(\Omega) \to W^{2,p}(R^n)$. *Here the constant C may also depend on p and* Ω.

PROOF Because of the extension operator, an inequality of the form (10) holds for functions in $W^{2,p}(\Omega)$. The full result follows by induction from this case.

6. The Spaces $H^k(\Omega)$ and $H_0^k(\Omega)$.

Definitions $H_0^k(\Omega) = W_0^{k,2}(\Omega)$ and $H^k(\Omega) = W^{k,2}(\Omega)$. These spaces are Hilbert spaces with inner product

$$(u,v)_k = \sum_{|\beta| \le k} \int D^\beta u(x) D^\beta \bar{v}(x) \, dx \, .$$

If $\Omega = R^n$, we get a very useful representation of such functions in terms of the Fourier-Plancherel Transform. Recall that for functions $f \in L^2(R^n)$, we define the Fourier-Plancherel transform of f as

$$\hat{f}(\xi) = \lim_{R \to \infty} (2\pi)^{-n/2} \int_{\|x\| \le R} e^{-ix.\xi} f(x) \, dx \, .$$

The limit exists in the topology of $L^2(R^n)$, $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$, and f can be recovered by using the inversion formula

$$f(x) = \lim_{R\to\infty} (2\pi)^{-n/2} \int_{\|\xi\|\le R} e^{ix.\xi} \hat{f}(\xi)\, d\xi .$$

Again, the limit here exists in the topology of $L^2(R^n)$. The reason for the limits in these formulas is that the integrands are not necessarily in $L^1(R^n)$. Clearly, the Fourier-Plancherel transform is an isometric (i.e. norms are equal) isomorphism (i.e. bounded linear mapping with a bounded inverse) from $L^2(R^n)$ onto $L^2(R^n)$.

Integration by parts shows that for $f \in C_0^\infty(R^n)$, the transform of $\dfrac{\partial f}{\partial x_j}$ is $i\xi_j \hat{f}(\xi)$. From this, we see by induction that the transform of $D^\beta f$ is $(i\xi)^\beta \hat{f}(\xi)$. If now we let $f \in H^k(R^n)$ and take a sequence of C_0^∞ functions converging to f, we find that $(i\xi)^\beta \hat{f}(\xi)$ is in $L^2(R^n)$ for all $|\beta| \le k$ and the transform of $D^\beta f$ is $(i\xi)^\beta \hat{f}(\xi)$. Thus, we see that if $f \in H^k(R^n)$ then $\hat{f} \in \hat{H}^k(R^n)$, where

$$\hat{H}^k(R^n) = \{g \in L^2(R^n) : (1+\|\xi\|)^k g(\xi) \in L^2(R^n)\}.$$

It is easy to see that C_0^∞ is dense in $\hat{H}^k$ and if g is in C_0^∞ then g is the transform of an infinitely differentiable, rapidly decaying function f (a function in the Schwarz class, to be precise). Taking limits, we see that if g is in $\hat{H}^k$ then g is the transform of a function f belonging to H^k. Further, if we define an inner product on $\hat{H}^k$ as

$$(u,v)_{\hat{H}^k} = \sum_{|\beta|\le k} \int \xi^{2\beta} u(\xi) \bar{v}(\xi)\, d\xi ,$$

we find that $\|f\|_k = \|\hat{f}\|_{\hat{H}^k}$. Thus, the Fourier-Plancherel transform is an isometric isomorphism from H^k onto $\hat{H}^k$. Questions about functions in H^k are thus transformed into equivalent (and often simpler) questions about functions in $\hat{H}^k$.

Problem 12 Consider the initial value problem for the wave equation

$$\frac{d^2u}{dt^2} = \Delta u, \; u(0) = f \in H^k, \; \frac{du}{dt}(0) = g \in H^{k-1}.$$

(We think of u as being a function of t taking values in H^k). Construct a candidate u for a solution using Fourier transforms.

a) Show that u is a continuous H^k-valued function of t.

b) Show that u is a continuously differentiable H^{k-1}-valued function of t.

c) Show that $\frac{du}{dt}$ is a continuously differentiable H^{k-2}-valued function of t and that $\frac{d^2u}{dt^2} = \Delta u$ in H^{k-2}.

d) How large does k have to be in order for u to be a classical (i.e. C^2) solution.

Hints: Clearly it suffices to answer the equivalent questions about $\hat{u}$. Use the Dominated Convergence Theorem to help you answer a), b), c). For d), use the Sobolev Imbedding Theorem.

7. Trace Theorems.

In PDE Theory, one often needs to know how functions behave on boundaries of domains. If f is a function defined on a domain Ω, we call the restriction of f to $\partial\Omega$ the *trace of f*. If all we know about f is that it is in some L^p space, then the trace of f is not well-defined because $\partial\Omega$ has measure zero. However, if $kp > n$ and Ω is a bounded C^1 domain in R^n, then we know by Corollary 18 that functions in $W^{k,p}(\Omega)$ are continuous on $\overline{\Omega}$ and thus they have well-defined traces that are bounded functions. In this section, we concern ourselves with the important case $kp < n$.

In the following results, a vector x in R^n is denoted by $x = (x', x_n)$, where x' belongs to R^{n-1}.

LEMMA 27 *If $u \in W^{1,1}(R^n)$, then for every $\zeta \in R$, the function $v(x') = u(x', \zeta)$ is in $L^1(R^{n-1})$, and*

$$\|v\|_{L^1(R^{n-1})} \leq \|u\|_{L^1(R^n)} + \|D_n u\|_{L^1(R^n)}.$$

Remark. One needs to be careful when talking about traces of equivalence classes of functions. The trace certainly exists for $u \in C_0^\infty(R^n)$. For $u \in W^{1,1}(R^n)$, we know that we can find a sequence of functions in $C_0^\infty(R^n)$ that converges to u. The norm inequality asserted in the lemma shows that the sequence of traces of these functions converges in $L^1(R^{n-1})$. It is in this sense that the trace of u exists in $L^1(R^{n-1})$.

PROOF It suffices to prove the result for the case $\zeta = 0$ and $u \in C_0^\infty(R^n)$. By the Mean Value Theorem for integrals

$$\int_0^1 \int_{R^{n-1}} |u(x', x_n)|\, dx' dx_n = \int_{R^{n-1}} |u(x', \sigma)|\, dx'$$

for some $\sigma \in [0,1]$. But

$$\begin{aligned} |u(x',0)| &= |u(x',\sigma) - \int_0^\sigma D_n u(x',t)\, dt| \\ &\le |u(x',\sigma)| + \int_0^1 |D_n u(x',t)|\, dt. \end{aligned}$$

Integrating this over R^{n-1} gives

$$\begin{aligned} \|v\|_{L^1(R^{n-1})} &\le \int_{R^{n-1}} |u(x',\sigma)|\, dx' + \int_{R^{n-1}} \int_0^1 |D_n u(x',t)|\, dt dx' \\ &= \int_0^1 \int_{R^{n-1}} |u(x',\sigma)|\, dx' dt + \int_{R^{n-1}} \int_0^1 |D_n u(x',t)|\, dt dx'. \end{aligned}$$

This completes the proof of the lemma.

LEMMA 28 *If* $u \in W^{1,p}(R^n)$ *where* $p < n$, *then for every* $\zeta \in R$, *the function* $v(x') = u(x', \zeta)$ *is in* $L^r(R^{n-1})$, *where*

$$r = \frac{(n-1)p}{n-p} = 1 + \frac{n(p-1)}{n-p}$$

and there is a constant C depending on only n and p such that

$$\|v\|_{L^r(R^{n-1})} \le C\, |u|_{1,p}^{R^n}.$$

PROOF We can assume that $p > 1$ because the $p = 1$ case is dealt with in the previous lemma. We first show that if $u \in W^{1,p}(R^n)$ then $w = |u|^r \in W^{1,1}(R^n)$ and

$$\|w\|_{L^1} \le const.\, \|Du\|_{L^p}^{r-1} \|u\|_{L^p}, \quad \|D_i w\|_{L^1} \le const.\, \|Du\|_{L^p}^{r}. \tag{12}$$

It suffices to prove this result for the case $u \in C_0^\infty(R^n)$. Let $q = p/(p-1)$. Then $(r-1)q = np/(n-p)$, so by the Sobolev Imbedding Theorem (Th. 15),

$$\| \, |u|^{r-1} \, \|_{L^q}^q \leq const. \|Du\|_{L^p}^{np/(n-p)}$$

and combining this with Hölder's Inequality, we get the first of (12):

$$\|w\|_{L^1} = \int |u|^r \, dx = \int |u|^{r-1} |u| \, dx \leq \|u\|_{L^p} \, \| \, |u|^{r-1} \|_{L^q} \leq const. \| Du\|_{L^p}^{r-1} \|u\|_{L^p} .$$

Since $D_i w = \pm r |u|^{r-1} D_i u$, we obtain the second of (12):

$$\|D_i w\|_{L^1} = r \| \, |u|^{r-1} \|_{L^q} \|D_i u\|_{L^p} \leq const. \|Du\|_{L^p}^r .$$

We now apply Lemma 27 to w and immediately obtain the inequality

$$\begin{aligned} \|v\|_{L^r(R^{n-1})} &\leq const. (\| Du\|_{L^p}^{r-1} \|u\|_{L^p} + \| Du\|_{L^p}^r)^{1/r} \\ &\leq const. (\| Du\|_{L^p}^{1-1/r} \|u\|_{L^p}^{1/r} + \|Du\|_{L^p}) \\ &\leq const. (\|u\|_{L^p} + \|Du\|_{L^p}). \end{aligned}$$

LEMMA 29 *If $u \in W^{k,p}(R^n)$ where $kp < n$, then for every $\zeta \in R$, the function $v(x') = u(x', \zeta)$ is in $L^r(R^{n-1})$, where*

$$r = \frac{(n-1)p}{n-kp}$$

and there is a constant C depending on only n, k and p such that

$$\|v\|_{L^r(R^{n-1})} \leq C \, |u|_{k,p}^{R^n} .$$

PROOF By Sobolev's Imbedding Theorem (Th. 15) applied to the first order derivatives of u, we have $u \in W^{1, np/(n-(k-1)p)}(R^n)$. Now apply Lemma 28.

Reminder: Parametrized Surface Integrals.

If $X(u) = (x_1(u_1,u_2),\ x_2(u_1,u_2),\ x_3(u_1,u_2))$ is a parametrization for a smooth surface S in R^3, it is well-known from elementary calculus that one may integrate functions defined on S using the formula

$$\int_S f(x)\, dS = \iint_\Xi f \circ X(u)\, K(u)\, du_1 du_2\,,$$

where Ξ is the domain of X and

$$K(u) = \left\| \frac{\partial X}{\partial u_1} \times \frac{\partial X}{\partial u_2} \right\| = \left(\left(\frac{\partial(x_3,x_2)}{\partial(u_1,u_2)}\right)^2 + \left(\frac{\partial(x_1,x_3)}{\partial(u_1,u_2)}\right)^2 + \left(\frac{\partial(x_1,x_2)}{\partial(u_1,u_2)}\right)^2 \right)^{1/2}.$$

Differential Geometry yields a generalization of this formula. Suppose now that Ξ is a domain in R^{n-1} and that $X:\Xi \to R^n$ is a parametrization for a smooth hypersurface S. Then surface integrals over S may be calculated using

$$\int_S f(x)\, dS = \int_\Xi f \circ X(u)\, K(u)\, du\,,$$

where

$$K(u) = \left(\sum_{k=1}^n \left(\frac{\partial(x_1,x_2,\ldots,\hat{x}_k,\ldots,x_n)}{\partial(u_1,u_2,\ldots,u_n)} \right)^2 \right)^{1/2}.$$

Here the $\hat{x}_k$ notation means that the x_k term does not appear.

THEOREM 30 *Suppose that Ω is bounded and is of class C^k. If $u \in W^{k,p}(\Omega)$ where $kp < n$, then the restriction v of u to $\partial\Omega$ is in $L^r(\partial\Omega)$, where*

$$r = \frac{(n-1)p}{n-kp}$$

and there is a constant C depending on only n, k and p and Ω such that

$$\|v\|_{L^r(\partial\Omega)} \le C\, |u|_{k,p}^{\Omega}.$$

Remark. See the remark following the statement of Lemma 27 for clarification of the phrase "restriction v of u to $\partial\Omega$". The same remark applies because we know that an extension operator $E: W^{k,p}(\Omega) \to W^{k,p}(R^n)$ exists, and thus the restriction to Ω of functions in $C_0^\infty(R^n)$ is dense in $W^{k,p}(\Omega)$.

PROOF Let $E: W^{k,p}(\Omega) \to W^{k,p}(R^n)$ be the extension operator of Theorem 11. Since any $u \in W^{k,p}(\Omega)$ is associated with an element $Eu \in W^{k,p}(R^n)$, we might as well just study the properties of the trace on $\partial\Omega$ of $C_0^\infty(R^n)$ functions.

Let Ω_j and ψ_j be as in the definition of C^k domains. Since $\partial\Omega$ is compact, we might as well assume that there is a finite number of the Ω_j, $1 \le j \le N$, covering $\partial\Omega$. Let θ_j $1 \le j \le N$, be a partition of unity for $\partial\Omega$ subordinate to this cover. If $u \in C_0^\infty(R^n)$, then $(\theta_j u) \circ \psi_j^{-1} \in C_0^k(B)$ and we can extend $(\theta_j u) \circ \psi_j^{-1}$ to be in $C_0^k(R^n)$ by defining the function to be zero outside *B*. By Lemma 29, the trace w_j of $(\theta_j u) \circ \psi_j^{-1}$ on the hyperplane $P: y_n = 0$ satisfies

$$\| w_j \|_{L^r(P)} \le C |(\theta_j u) \circ \psi_j^{-1}|_{k,p}^B \le C_j |u|_{k,p}^{R^n},$$

where C depends on only *n*, *p*, and *k* and C_j is independent of u.. $X_j(y) = \psi_j^{-1}(y_1, \ldots, y_{n-1}, 0)$ is a parametrization for the hypersurface $S_j = (\partial\Omega) \cap \Omega_j$ and we may estimate the trace $v_j = w_j \circ \psi_j$ of $\theta_j u$ on this hypersurface using this parametrization (see the "reminder" preceding the statement of the theorem).

$$\int_{S_j} |v_j(x)|^r \, dS = \int_{P \cap B} |w_j(y)|^r \, K_j(y) \, dy \le R_j \int_{P \cap B} |w_j|^r \, dy,$$

where $R_j = \max(K_j)$. Comparing this to the preceding inequality, we see that there is a constant M_j independent of *u* such that

$$\| v_j \|_{L^r(S_j)} \le M_j |u|_{k,p}^{R^n}.$$

The function *v* satisfies a similar inequality because $v = \sum v_j$. Finally, it is clear that the result holds for arbitrary $u \in W^{k,p}(\Omega)$ (see the remark following the statement of the theorem).

Problem 13 Modify the proof of Lemma 27 to show that if $u \in W^{1,p}(R^n)$, then for every $\zeta \in R$, the function $v(x') = u(x', \zeta)$ is in $L^p(R^{n-1})$, and there exists a constant K depending only on n and p such that $\|v\|_{L^p(R^{n-1})} \le K|u|_{1,p}^{R^n}$.

Problem 14 Deduce from the previous problem and Lemma 28 that the function v of Lemma 28 belongs to $L^q(R^{n-1})$ for all q satisfying $p \le q \le r$.

Appendix: Some Spaces of Continuous Functions.

Here, we define the spaces of continuous functions that appear in these notes. Caution: Notation and definitions of such function spaces vary from text to text. Recall that we stated that Ω is a domain in R^n. The connectedness of Ω is not needed in the following definitions, so we need only assume that Ω is an open subset of R^n.

1. $C(\Omega)$ is the set of functions continuous in Ω.
2. $C(\overline{\Omega})$ is the set of functions continuous in $\overline{\Omega}$.
3. $C^k(\Omega)$ is the set of functions which have derivatives of order $\le k$ that are continuous in Ω.
4. $C^k(\overline{\Omega})$ is the set of functions in $C(\overline{\Omega})$ which have derivatives in Ω of order $\le k$ that can be extended to be members of $C(\overline{\Omega})$.
5. $C^\infty(\Omega)$ is the set of functions in $C^k(\Omega)$ for all k.
6. $C^\infty(\overline{\Omega})$ is the set of functions in $C^k(\overline{\Omega})$ for all k.
7. $C_0(\Omega)$ is the set of functions in $C(\Omega)$ that have supports that are compact subsets of Ω (recall that the support of a function is the closure of the set on which the function fails to vanish). Since Ω is open, such functions necessarily vanish in a neighborhood of the boundary of Ω.
8. $C_0^k(\Omega)$ is the set of functions in $C^k(\Omega)$ that have supports that are compact subsets of Ω.
9. $C_0^\infty(\Omega)$ is the set of functions in $C_0^k(\Omega)$ for all k.
10. $C_B(\Omega)$ is the set of bounded functions in $C(\Omega)$. This is a Banach space when equipped with the "sup norm".
11. $C_B(\overline{\Omega})$ is the set of bounded functions in $C(\overline{\Omega})$. This is a Banach space when equipped with the "sup norm". If Ω is bounded, this space coincides with $C(\overline{\Omega})$.

12. $C^k_B(\Omega)$ is the set of functions in $C_B(\Omega)$ with derivatives of order $\leq k$ belonging to $C_B(\Omega)$. This is a Banach space if we define the norm of a member *f* of this space as $\sup_{|\beta|\leq k,\, x\in\Omega} |D^\beta f(x)|$.

13. $C^k_B(\overline{\Omega})$ is the set of functions in both $C^k_B(\Omega)$ and $C^k(\overline{\Omega})$. This is a Banach space, equipped with the same norm as in (12). If Ω is bounded, this space coincides with $C^k(\overline{\Omega})$.

14. $C^{k,\alpha}(\overline{\Omega})$, where $0<\alpha\leq 1$, is the set of functions in $C^k_B(\overline{\Omega})$ that have derivatives of order $\leq k$ that are uniformly Hölder continuous with exponent α. $C^{k,\alpha}(\overline{\Omega})$ is a Banach space with norm

$$\| f\|_{C^{k,\alpha}} = \sup_{|\beta|\leq k,\, x\in\Omega} |D^\beta f(x)| + [f]_{k,\alpha},$$

where
$$[f]_{k,\alpha} = \sup_{x,y\in\Omega,\, x\neq y,\, |\beta|=k} \frac{|D^\beta f(x) - D^\beta f(y)|}{\|x-y\|^\alpha}.$$

References

The results stated in these notes appear in most texts on Sobolev spaces, including those listed below. However, there are many different proofs of the results. For this reason, the key lemmas and theorems that appear in these notes are listed below with a reference to the source which has a proof that most resembles the proof in these notes.

Th. 1	[Fr], Th. 6.1
L. 2	[Fr], Th. 6.2
Th. 3	[Fr], Th. 6.2
L. 4	[Zi], L. 2.3.1
Th. 5	[Fr], Th. 6.3; [Ad], Th. 3.16.
Th. 6	[Ad], Th. 3.18.
Th. 7	[Ad], Th. 3.35. (see [Zi], Th. 2.2.2 for Lipschitz changes of variables).
Th. 11	[Ad], Th. 4.26; [G.T.], Th. 7.25.
Th. 12	[Fr], Th. 9.1
Th. 14	[Fr], Th. 9.2
Th. 15	[G.T.], Th. 7.10
L. 19	[Ad], Th. 1.31

Th. 22	[Ru], Appendix A4
Th. 23	[G.T.], Th. 7.22
Th. 26	[G.T.], Th. 7.27; [Ad], Th. 4.13

There is obviously no room here for a complete bibliography. For a more complete list of references, the reader should refer to the bibliographies of the texts listed below.

Ad. R. A. Adams, *Sobolev Spaces*, Academic Press, 1975.

Fr. A. Friedman, *Partial Differential Equations*, Krieger, 1983.

G.T. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1983.

Ma. V. G. Maz'ja, *Sobolev Spaces*, Springer-Verlag, 1985.

Ru. W. Rudin, *Functional Analysis*, MacGraw-Hill, 1973.

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