PROJECT DETAILS: Nonlinear Elastic Stuctures

0.1 Introduction

The traditional assumptions for beam models are that the displacement, slope and curvature of the beam are all small. However, there are situations in which the displacement and slope of the beam are not small, while the curvature still is small (e.g. a long, slender cantilevered beam or rod). In such situations, the strains are still small enough for the linear theory of elasticity to apply, but the actual equations of motion of the beams are nonlinear for geometrical reasons.

The purpose of the proposed research is primarily to investigate the boundary feedback stabilization of such beams. Together with this, an analysis, both theoretical and numerical, of the equations of motion will be done.

Over the last several years, there has been a great deal of interest in the control of beams, partly because of the important applications in industry. This interest has resulted in so much work in both the engineering and mathematical literature that it is impossible to give a comprehensive bibliography here. Instead, we mention two of the earliest works on the subject by Chen et al. [1, 2]. Very little of this vast literature concerns beam models other than the usual linear models. The reasons for this are obvious. First, the assumptions of small displacement and slope are often well justified. Second, one has the power of linear operator theory at one's disposal. A notable exception to this rule is the work done by Lagnese and Leugering [4, 5] and Lagnese [6], in which the authors investigate the controllability of beams that can undergo an intermediate range of motions. The range is intermediate in the sense that the motions are large enough to cause a nonlinear coupling between the longitudinal and transverse deflections of the beam, however assumptions of small slope and displacement are still made. We mention this model in more detail below.

There are two reasons for which the Project Leader believes that this study is important. The first is that situations can arise where the linear theory does not hold. The second is that even when the linear theory ought to be a good approximation, one should check, especially for control problems, that the actual nonlinearities physically present do not induce instabilities.

The discussion below is set out as follows. First, in Section 0.2, we discuss a simple model of a beam that can undergo large deflections. Then, in Section 0.3, we state and prove a boundary feedback stabilization result concerning the same model. We give these details partly to convince the reader that such problems can indeed be analyzed. In Section 0.4, we discuss other beam models. Finally, in Section 0.5, some goals of the project are discussed in more detail.

0.2 Modelling Large Deflections of a Beam

The traditional assumptions for beam models are that the displacement, slope and curvature of the beam are all small. However, there are situations in which the displacement and slope of the beam are not small, while the curvature still is small (e.g. a long cantilevered beam). Here we develop a model for such beams essentially by removing the approximations that are based on small displacement and slope in the Euler-Bernoulli Model.

The equation of motion of an Euler-Bernoulli Beam is

$$\rho\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 w}{\partial x^2})=0,$$

where x is a coordinate along the axis of the beam, $w(x,t)$ is the transverse displacement of the beam at position x and time t, E is Young's Modulus of Elasticity, ρ is the mass per unit length of the beam, and I is a moment function depending on the cross-section of the beam. As mentioned above, the model is derived with the assumptions that w and $\frac{\partial w}{\partial x}$ are small.

The total energy of an Euler-Bernoulli Beam is E_p+E_k, where the elastic potential energy E_p and kinetic energy E_k are given by the expressions

$$E_p=\frac{1}{2}\int_0^L EI\,(\frac{\partial^2 w}{\partial x^2})^2\,dx,$$

$$E_k = \frac{1}{2}\int_0^L \rho\,(\frac{\partial w}{\partial t})^2\,dx.$$

The term $\frac{\partial^2 w}{\partial x^2}$ in E_p occurs because it is an approximation to the curvature of the beam. The actual curvature is

$$\kappa = \frac{\partial^2 w}{\partial x^2}/(1+(\frac{\partial w}{\partial x})^2)^{3/2},$$

so this approximation is valid only for beams which remain close to horizontal. E_k also contains an approximation in that only the vertical component $\frac{\partial w}{\partial t}$ of the velocity is assumed to be significant. Again, this seems to be a reasonable approximation for beams that remain close to horizontal.

We now consider a model that avoids these approximations. For simplicity, we consider here the case where the motion of the beam is confined to a plane. We choose a reference end of the beam and let $r(s,t) = (x(s,t), y(s,t))$ be a parametrization at time t of the centerline of the beam in terms of the arclength s from the reference end. Since s denotes arclength, we must have

$$\frac{\partial r}{\partial s}.\frac{\partial r}{\partial s} = 1. \tag{1}$$

The curvature of the beam is conveniently computed from this parametrization as

$$\kappa^2 = \frac{\partial^2 r}{\partial s^2}.\frac{\partial^2 r}{\partial s^2}.$$

As in the Euler-Bernouilli model, we assume that there is no longitudinal stretching or contraction of the beam. (This assumption is satisfactory for a long, thin beam of which the stretching is negligible compared to the transverse displacement. We mention later a model that takes into account the stretching). Thus each value of the arclength function s corresponds uniquely to a material point of the centerline of the beam and the velocity of such a point is $\frac{\partial r}{\partial t}$. These considerations show that we can write the energies in the form

$$E_p = \frac{1}{2}\int_0^L EI\,\frac{\partial^2 r}{\partial s^2}.\frac{\partial^2 r}{\partial s^2}\,ds, \tag{2}$$

$$E_k = \frac{1}{2}\int_0^L \rho\,\frac{\partial r}{\partial t}.\frac{\partial r}{\partial t}\,ds. \tag{3}$$

One could also include a kinetic energy term due to the rotational motion of a beam element. This will be mentioned later.

One can derive the equations of motion of the beam by applying the Principle of Virtual Work, which states that the first variation of $\int L\,dt$ vanishes, where $L = E_k - E_p$. However, we must take into account the fact that r satisfies the constraint Eq. (1) and one way of doing this is to add a Lagrange multiplier term

$$\frac{1}{2}\int_0^L \tau(s,t)(1 - \frac{\partial r}{\partial s}.\frac{\partial r}{\partial s})\,ds$$

to L. Application of the Principle of Virtual Work then leads immediately to the equation (we assume for simplicity that ρ and EI are constant, but this is not necessary)

$$\rho\frac{\partial^2 r}{\partial t^2} + EI\frac{\partial^4 r}{\partial s^4} = \frac{\partial}{\partial s}\tau\frac{\partial r}{\partial s}. \tag{4}$$

The extra unknown τ is to be found by stipulating that the vector valued function r must satisfy both Eq. (1) and Eq. (4). Eqs. (1) and (4) thus comprise a system of nonlinear equations for the beam motion.

It is easy to see that τ represents the tension force in the beam. Further, the unit tangent vector $T = \frac{\partial r}{\partial s}$ and the unit normal vector N are related by the equations $\frac{\partial T}{\partial s} = \kappa N$, $\frac{\partial N}{\partial s} = -\kappa T$ so the fourth order term in Eq. (4) represents the rate of change with respect to arclength of

$EI(\frac{\partial \kappa}{\partial s}N - \kappa^2 T)$, which is to be interpreted as being the net force on cross-sections of the beam due to bending.

We let $\mathcal{E}(t) = E_p(t) + E_k(t)$ denote the total energy of the beam and suppose that r and τ are classical solutions of Eqs. (1) and (4). Differentiating the energy and integrating by parts yields (using Eq. (4) and the fact that $\frac{\partial^2 r}{\partial s \partial t}$ is perpendicular to $\frac{\partial r}{\partial s}$ because of Eq. (1))

$$\dot{\mathcal{E}}(t) = [EIr_{ss}.\dot{r}_s + (\tau r_s - EIr_{sss}).\dot{r}]_{s=0}^{s=L}. \tag{5}$$

Here dots denote derivatives with respect to t and subscripts are used for derivatives with respect to s. The term EIr_{ss} represents the torque on a cross-section of the beam, and the term $\tau r_s - EIr_{sss}$ represents the net force on a cross-section.

Consider a cantilevered beam that is horizontally clamped at the end $s = 0$ which coincides with the origin and is free at the end $s = L$. It is not difficult to see that in this case the following boundary conditions must be satisfied.

$$r(0,t) = (0,0), \qquad r_s(0,t) = (1,0), \tag{6}$$

$$r_{ss}(L,t) = (0,0), \quad EIr_{sss}(L,t) = \tau(L,t)r_s(L,t). \tag{7}$$

In this case, we of course have conservation of energy: $\dot{\mathcal{E}}(t) = 0$. Also, the tangential component of the second of Eqs. (7) shows that $\tau = -\kappa^2$ at $s = L$.

0.3 Uniform Exponential Stabilization of the Nonlinear Beam

If a force F is now applied to the end $s = L$ of the cantilevered beam then the boundary condition in the second of Eqs. (7) becomes (we scale variables so that $\rho = EI = 1$)

$$\tau(L,t)r_s(L,t) - r_{sss}(L,t) = F.$$

It is clear from Eq. (5) that if we choose a force $F = -\gamma\dot{r}(L,t)$, where $\gamma > 0$, then

$$\dot{\mathcal{E}}(t) = -\gamma\dot{r}^2|_{s=L}. \tag{8}$$

Remark. The linear feedback law chosen here can be replaced by a nonlinear one satisfying conditions similar to the conditions of the nonlinear feedback law of [5]. Indeed, it is more realistic to assume that a feedback law is nonlinear because nonlinear effects are invariably present. However, to simplify this discussion, we assume a linear law in the following theorem and mention that stabilization by nonlinear feedback also works.

Theorem *Let $\gamma > 0$ and suppose that r and τ are classical solutions of the equations*

$$\frac{\partial^2 r}{\partial t^2} + \frac{\partial^4 r}{\partial s^4} = \frac{\partial}{\partial s}\tau\frac{\partial r}{\partial s}, \tag{9}$$

$$\frac{\partial r}{\partial s}.\frac{\partial r}{\partial s} = 1, \tag{10}$$

$$r_{ss}(L,t) = r(0,t) = (0,0), \quad r_s(0,t) = (1,0), \tag{11}$$

$$\tau(L,t)r_s(L,t) - r_{sss}(L,t) = -\gamma\dot{r}(L,t). \tag{12}$$

Then there are constants C and ω that depend only on L and γ such that

$$\mathcal{E}(t) \le Ce^{-\omega t}\mathcal{E}(0).$$

Proof. We prove the result using a Lyapunov Functional

$$\mathcal{F}_\epsilon(t) = \mathcal{E}(t) + \epsilon\rho(t), \tag{13}$$

where

$$\rho(t) = \int_0^L \dot{r}.(sr_s - r)\, ds. \tag{14}$$

(One might wonder if the integrand in the expression defining $\rho(t)$ could be replaced by $sr_s.\dot{r}$ which is more similar to the corresponding term in Lyapunov Functionals often used for scalar beam equations (see [1] and [4] for example). However, this does not work and the form of $\rho(t)$ chosen here is required to take care of both the constraint terms and the vector boundary conditions).

After differentiating and the usual integrations by parts, we find

$$\dot{\mathcal{E}}(t) = -\gamma \dot{r}^2|_{s=L},$$

$$\dot{\rho}(t) = -\gamma \dot{r}.(sr_s - r) + \frac{1}{2} s\dot{r}^2|_{s=L} - \frac{1}{2}\int_0^L r_{ss}.r_{ss} + 3\dot{r}.\dot{r}\, ds.$$

Further,

$$||sr_s - r|| \leq \int_0^L s||r_{ss}||\, ds \leq (\frac{L^3}{3}\int_0^L ||r_{ss}||^2\, ds)^{\frac{1}{2}}.$$

From these statements, one easily deduces that if $0 < \epsilon < 6\gamma/(3 + 2\gamma^2 L^3)$ then

$$\dot{\mathcal{F}}_\epsilon(t) \leq -\frac{\epsilon}{2}\mathcal{E}(t). \tag{15}$$

It is also easy to deduce that $|\rho(t)| \leq L^2\mathcal{E}(t)$ and thus

$$(1 - \epsilon L^2)\mathcal{E}(t) \leq \mathcal{F}_\epsilon(t) \leq (1 + \epsilon L^2)\mathcal{E}(t). \tag{16}$$

From (15) and (16), we obtain

$$\dot{\mathcal{F}}_\epsilon(t) \leq -\frac{\epsilon}{2(1+\epsilon L^2)}\mathcal{F}_\epsilon(t),$$

integration of which yields

$$\mathcal{F}_\epsilon(t) \leq \mathcal{F}_\epsilon(0)e^{-\omega t},$$

where $\omega = \frac{\epsilon}{2(1+\epsilon L^2)}$. Thus, by (16) we obtain

$$\mathcal{E}(t) \leq \frac{1+\epsilon L^2}{1-\epsilon L^2}\mathcal{E}(0)e^{-\omega t},$$

which proves the Theorem.

0.4 Some Other Models Describing Large Deflections of Beams.

The Rayleigh Model improves on the Euler-Bernoulli Model by taking into account the rotational motion of the beam elements. Suppose that the moment of inertia of a small element of length δx is $J\delta x$. Incorporating this into the Euler-Bernoulli Model yields the following equation for the motion of the beam.

$$\rho\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}(J\frac{\partial^3 w}{\partial t^2 \partial x}) + \frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 w}{\partial x^2}) = 0.$$

We can easily include the rotational effects in our vector model as well. If we let θ denote the angle that the unit tangent vector T makes with the x-axis then $\frac{\partial T}{\partial t} = -\frac{\partial \theta}{\partial t}N$. Thus $(\frac{\partial \theta}{\partial t})^2 = \frac{\partial T}{\partial t}.\frac{\partial T}{\partial t} = \frac{\partial^2 r}{\partial t \partial s}.\frac{\partial^2 r}{\partial t \partial s}$. This results in the following modification of the kinetic energy equation (3)

$$E_k = \frac{1}{2}\int_0^L \rho\frac{\partial r}{\partial t}.\frac{\partial r}{\partial t} + J\frac{\partial^2 r}{\partial t \partial s}.\frac{\partial^2 r}{\partial t \partial s}\, ds. \tag{17}$$

Once again, application of the Principle of Virtual Work yields the equation of motion. (Here we have not assumed that the coefficients are constant).

$$\rho\frac{\partial^2 r}{\partial t^2} - \frac{\partial}{\partial s}(J\frac{\partial^3 r}{\partial t^2 \partial s}) + \frac{\partial^2}{\partial s^2}(EI\frac{\partial^2 r}{\partial s^2}) = \frac{\partial}{\partial s}\tau\frac{\partial r}{\partial s}. \tag{18}$$

This equation must also be solved in conjunction with (1).

Another phenomenon that one can take into account is the longitudinal stretching and contractions of a beam. The boundary stabilization problem for two such models is discussed in [4, 5]. In [5], which deals with the more accurate model, $w(x,t)$ again denotes the transverse deflection of the beam at reference point x and time t, while the corresponding longitudinal position of the same point is $x + u(x,t)$. As in the Euler-Bernoulli model, the curvature is still approximated by w_{xx}. One argues that the potential energy density due to longitudinal stretching is $\frac{1}{2}EA(s_x - 1)^2$, where s denotes the arclength and A is the cross-sectional area of the beam. In [5], the authors make the approximation $s_x = u_x + \frac{1}{2}w_x^2$, which is often used in engineering literature. This leads to the following expression for the total elastic potential energy.

$$U = \frac{1}{2}\int_0^L EA(u_x + \frac{1}{2}w_x^2)^2 + EIw_{xx}^2\, dx.$$

The resulting equations of motion are accurate for an intermediate range of motions, but, because of the approximations made, they are not valid for the beam motions considered here.

We can incorporate the effects of longitudinal strain into our model as follows. Let x denote the arclength function for the *unstretched* beam. The actual arclength function for the moving beam is

$$s(x,t) = \int_0^x ||r_\sigma(\sigma,t)||\, d\sigma,$$

and the amount by which the portion $(0,x)$ of the beam is stretched is $s(x,t) - x$. Thus we arrive at a potential energy due to stretching

$$E_s = \frac{1}{2}\int_0^L EA(\frac{\partial}{\partial x}(s(x,t) - x))^2\, dx = \frac{1}{2}\int_0^L EA(||\frac{\partial r}{\partial x}|| - 1)^2\, dx.$$

Since x is not the arclength function for the moving beam, the expression for the curvature of the beam is more complicated. It is given by $\kappa^2 = (|r_{xx}|^2|r_x|^2 - (r_{xx}.r_x)^2)/|r_x|^6$. Thus the equation of motion is obtained by applying the Principle of Virtual Work to the Lagrangian $E_k - E_p$, where E_k is given by (3) or (17) with x in place of s and E_p is given by

$$E_p = \frac{1}{2}\int_0^L EI(|r_{xx}|^2|r_x|^2 - (r_{xx}.r_x)^2)/|r_x|^6 + EA(|r_x| - 1)^2\, dx. \tag{19}$$

One might wonder if the more complex curvature expression appearing in (19) can be approximated by the more simple expression (2), especially if the ratio A/I is large. The resulting equation of motion is

$$\rho\frac{\partial^2 r}{\partial t^2} + \frac{\partial^2}{\partial s^2}(EI\frac{\partial^2 r}{\partial s^2}) - \frac{\partial}{\partial s}(EA\, p(\frac{\partial r}{\partial s})) = 0, \tag{20}$$

where $p: R^2 \to R^2$ is given by

$$p(\xi) = (1 - ||\xi||^{-1})\xi.$$

The problem with this is that it does not reduce to the wave equation for purely longitudinal motion. Instead, it gives a singular perturbation of the wave equation. However, this equation does provide an avenue for the study of Eqn. (4), for instead of solving (4) subject to the constraint (1), one could study (20) as $A/I \to \infty$. It is interesting to note that the cantilevered beam which has (20) as its equation of motion can be uniformly exponentially stabilized in a fashion similar to the stabilization of the beam in the previous section. The proof of this fact relies on the same Lyapunov Functional (13).

0.5 What Needs to be Done.

0.5.1 Existence and Uniqueness of Solutions.

A major task of the project is to investigate the conditions under which classical solutions and finite energy solutions Eqs. (4), (18), and the equations associated with (19) exist. Preliminary investigations indicate that the Theory of Nonlinear Semigroups is applicable. This approach, which begins with a change of variables to make the energy look as if it comes from an innerproduct in a Hilbert space, is one of the approaches that the Project Leader intends to pursue. The advantage of the nonlinear semigroup theory is that results follow directly from an investigation of the operators appearing in the evolution equations. However, for the problems to be studied here, even showing that there exists a change of variables that associates the energy with an inner product is an intriguing mathematical problem.

There are at least two other paths to follow here. One is to prove that limits exist of Galerkin approximations involving eigenfunctions of the linearized equations. Another is to consider a finite element subspace of the solution space (e.g. of the finite energy space), calculate the potential and kinetic energies for elements of this subspace, write down the Lagrange Equations (a system of nonlinear ode's) for these energies, and finally show that the solutions of these ode's converge in some sense to solutions of the corresponding partial differential equation. The advantages of this second approach are twofold. One is that algorithms for the numerical solution of the equations can be developed at the same time. The second is that the constraint condition (1) is easier to implement.

0.5.2 Numerical Analysis and Numerical Experiments.

It is intended that the numerical analysis project is to be done in connection with the "Existence/Uniqueness" project as described above. Once this is set up, it will be a useful tool for doing numerical experiments on various aspects of the beam motion and control problems.

0.5.3 Other Beam Models

There are several other linear beam models, each designed to model different phenomena associated with beams. One of the Project Leader's goals is to modify these to obtain satisfactory models for large deflections of beams. Among these are the Timoshenko model which allows for shearing of the beam elements, and various models associated with internal friction, such as the Kelvin-Voight Model and what Russell calls the Shear Diffusion Model (see, Russell [10, 11] and Rogers [9] for a discussion of these models). Realistic models of beams should incorporate the natural damping that occurs. Indeed, Russell [11] remarks that such damping needs to be utilized in control problems.

0.5.4 The Study of Various Control Problems.

The control problem of Section 0.3 is a prototype of the control problems that should be investigated here. Stabilization of the other beam models discussed should be studied and different control laws should be considered. For example, one might expect that the damping force $-\gamma\dot{r}(L,t)$ in the control law (12) could be replaced by a force that acts only in the direction perpendicular to the beam (e.g. $-\gamma\dot{r}(L,t).N(L,t)N(L,t)$).

More complicated systems should also be investigated. As an example, consider two beams clamped end to end ([1] involves such an investigation for Euler-Bernoulli beams and [3] involves a more complex network of Timoshenko beams). A preliminary analysis shows that if the beams (assumed to have different material parameters) are parallel at the point where they are connected then the cantilevered system is exponentially stabilized by a control on the free end as in Section 0.3. However, if the beams are not parallel then the control steers the beams into a neighborhood of their equilibrium state, the size of the neighborhood depending on the angle between the tangent vectors of the beams where they are joined.

0.5.5 Motion in 3 Space Dimensions.

So far, we have been looking at beam motion confined to a plane. For motion in 3 space dimensions, the main modifications of the equations of motion are to allow for the twisting of the beam about its centerline and for the bending parameters depending on the direction of bending. All of the issues concerning motion in a plane have their invariably more complicated counterparts for motion in space. There is much work to do here!

0.5.6 Analysis and Control of Strings.

Under the same assumptions that are made in the derivation of the Euler-Bernoulli Equation, i.e. small displacement and slope, one easily shows that the motion of a string is governed by the wave equation

$$\rho\frac{\partial^2 w}{\partial t^2} - \tau\frac{\partial^2 w}{\partial x^2} = 0,$$

where τ is the constant tension in the string and, as in the beam model, w is the transverse displacement and ρ is the mass per unit length of the string. Situations arise for which the assumptions of small slope and displacement are not valid. The equations describing such motions are similar to our equations for beams, except for the fact that the curvature terms are absent. Thus, the motion of a string that is not stretchable is governed by the constraint (1) and a modified version of (4)

$$\frac{\partial r}{\partial s}.\frac{\partial r}{\partial s} = 1,$$

$$\rho\frac{\partial^2 r}{\partial t^2} - \frac{\partial}{\partial s}\tau\frac{\partial r}{\partial s} = 0.$$

As for the beam equations, one must seek functions r and τ that satisfy these equations. Although the second equation is deceptively similar to the original wave equation, the system is highly nonlinear. If τ ever becomes negative, then the equation will even change type.

Similarly, the equation governing the motion of an elastic (i.e. stretchable) string is

$$\rho\frac{\partial^2 r}{\partial t^2} - \frac{\partial}{\partial x}(\lambda\, p(\frac{\partial r}{\partial x})) = 0,$$

where $p : R^2 \to R^2$ is given by

$$p(\xi) = (1 - ||\xi||^{-1})\xi,$$

and λ is an elastic parameter for the string.

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