Bifurcations of generalised Julia sets near the complex quadratic family

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Abstract: We consider a two-dimensional noninvertible map that was introduced by Bamon, Kiwi and Rivera in 2006 as a model of wild Lorenz-like chaos [1, 2]. The map acts on the plane by opening up the critical point to a disk and wrapping the plane twice around it; points inside the disk have no preimage. The bounding critical circle and its images, together with the critical point and its preimages, form the *critical set*. In a specific parameter regime the map is a nonanalytic perturbation of the complex quadratic family. As parameters are varied away from the complex quadratic family the dynamics on the plane initially stay qualitatively the same. On the other hand, saddle points and their *stable and unstable sets* then appear as new ingredients of the dynamics. The stable, unstable and critical sets interact with the *generalised Julia set* [3], leading to the (dis)appearance of chaotic attractors and to dramatic changes in the topology of the generalised Julia set. In particular, we find generalised Julia sets in the form of *Cantor bouquets*, *Cantor tangles* and *Cantor cheeses*.

1 Introduction

We study the dynamics of the two-dimensional noninvertible family of maps [1, 2]

$$\begin{aligned} f : \mathbb{C}\backslash\{0\} &\to \mathbb{C}, \\ z &\mapsto (1-\lambda+\lambda|z|^2)\left(\frac{z}{|z|}\right)^2 + c, \end{aligned} \tag{1}$$

with $c \in \mathbb{C}$ and $\lambda \in [0,1]$. For $\lambda = 1$, the family (1) reduces to the well-known *complex quadratic family*

$$\begin{aligned} f_1 : \mathbb{C} &\to \mathbb{C}, \\ z &\mapsto z^2 + c, \end{aligned} \tag{2}$$

where $c \in \mathbb{C}$ is the only parameter. The map (2) is analytic, whereas (1) is nonanalytic for $\lambda \in [0,1)$. We are interested in the question of whether, or in which form, the well-known dynamics of the complex quadratic family (2) influences the dynamics of (1) for other values of $\lambda \in [0,1]$. In particular, we want to know what elements of the dynamics of (2) survive for $\lambda < 1$, and what additional dynamical features may be found in (1).

The complex quadratic family (2) wraps the plane twice around the origin and translates by c, while the map (1) first opens up the origin to a circle with radius $1-\lambda$, wraps the plane around this circle twice and translates it by c. In particular, (1) is only defined on the punctured plane $\mathbb{C}\backslash\{0\}$, and only the points outside the circle around c with radius $1-\lambda$ have two preimages. We call the origin the *critical point* and the circle around c with radius $1-\lambda$ the *critical circle* of (1). For $\lambda = 1$ in (1) the critical circle has radius 0 and coincides with the *critical value* c of (2). The backward iterates of the critical point and the forward iterates of the critical circle play a special role in the organisation of the dynamics of (1), and we call them the *backward critical set* and the *forward critical set*, respectively. Together, they form the *critical set*.

As the complex quadratic family (2) is analytic, it admits only attracting and repelling fixed and periodic points, whereas (1) also allows for the existence of saddle points, their stable and unstable sets and chaotic attractors. The *stable and unstable sets* of a saddle point consist of points that converge to this point under forward iteration, or that have a sequence of preimages converging to it, respectively.

The main ingredient of the dynamics of (2) is the *Julia set* [4]. It can be defined as the boundary of the basin of attraction of infinity, and we extend this definition to the map (1). The connectivity of the Julia set in (2) is governed by a *fundamental dichotomy*: it is connected if the orbit of the critical value c is bounded, and it is totally disconnected if this orbit goes to infinity. When considering the family (1) with $\lambda < 1$, the orbit of c is replaced by all orbits in the forward critical set, which allows for an intermediate case, where only some orbits in the forward critical set stay bounded.

2 Global transition for fixed $c = 0.1$ and decreasing λ

We now study the changes in the phase portrait of map (1) for fixed $c = 0.1$ when $\lambda \in [0, 1]$ is decreased from $\lambda = 1$; the corresponding phase portraits are shown in Figure 1. Panel (a) shows the phase portrait for $\lambda = 1$, that is, of the complex quadratic map (2), which is defined for all $z \in \mathbb{C}$. The map has an attracting fixed point (blue triangle) and a repelling fixed point (red square) on the real line. The Julia set (black) is the boundary of the basin of attraction of infinity (grey). The grey scale corresponds to sets with different escape times to a neighbourhood of infinity. The critical point (green dot) at the origin has infinitely many preimages (green dots). At $\lambda = 1$, the critical circle of (1) has radius $1-\lambda = 0$ and coincides with the critical value c (green dot) of (2). Panel (b) shows the phase portrait at $\lambda = 0.95$; the critical circle (green circle) has opened up its radius to 0.05, but everything else stays qualitatively the same. Panel (c) shows the phase portrait at $\lambda = 0.93$; the attracting fixed point has split up into two attracting fixed points (blue triangles) and one saddle fixed point (black cross) in a *pitchfork bifurcation*. The saddle fixed point has a stable set (blue curves) and an unstable set (red curves). The stable set has infinitely many branches, which have the points in the backward critical set as branch points and which end on the Julia set. The unstable set consists of two branches, which connect the saddle fixed point to the two attracting fixed points. Panel (d) shows the phase portrait at $\lambda = 0.9$, where the critical point enters the critical circle in a *forward-backward critical tangency* [2] and its preimages in the backward critical set disappear accordingly. The unstable set of the saddle fixed point meets the critical point and its closure becomes a chaotic attractor. Panel (e) shows the phase portrait at $\lambda = 0.7791$; the critical point lies inside the critical circle and two saddle period-two points (black crosses), their stable set (cyan curves) and their unstable set (purple curves) have appeared. The saddle and repelling fixed points on the real line meet each other in a *saddle-node bifurcation*. Panel (f) shows the phase portrait at $\lambda = 0.6$; the repelling fixed points,

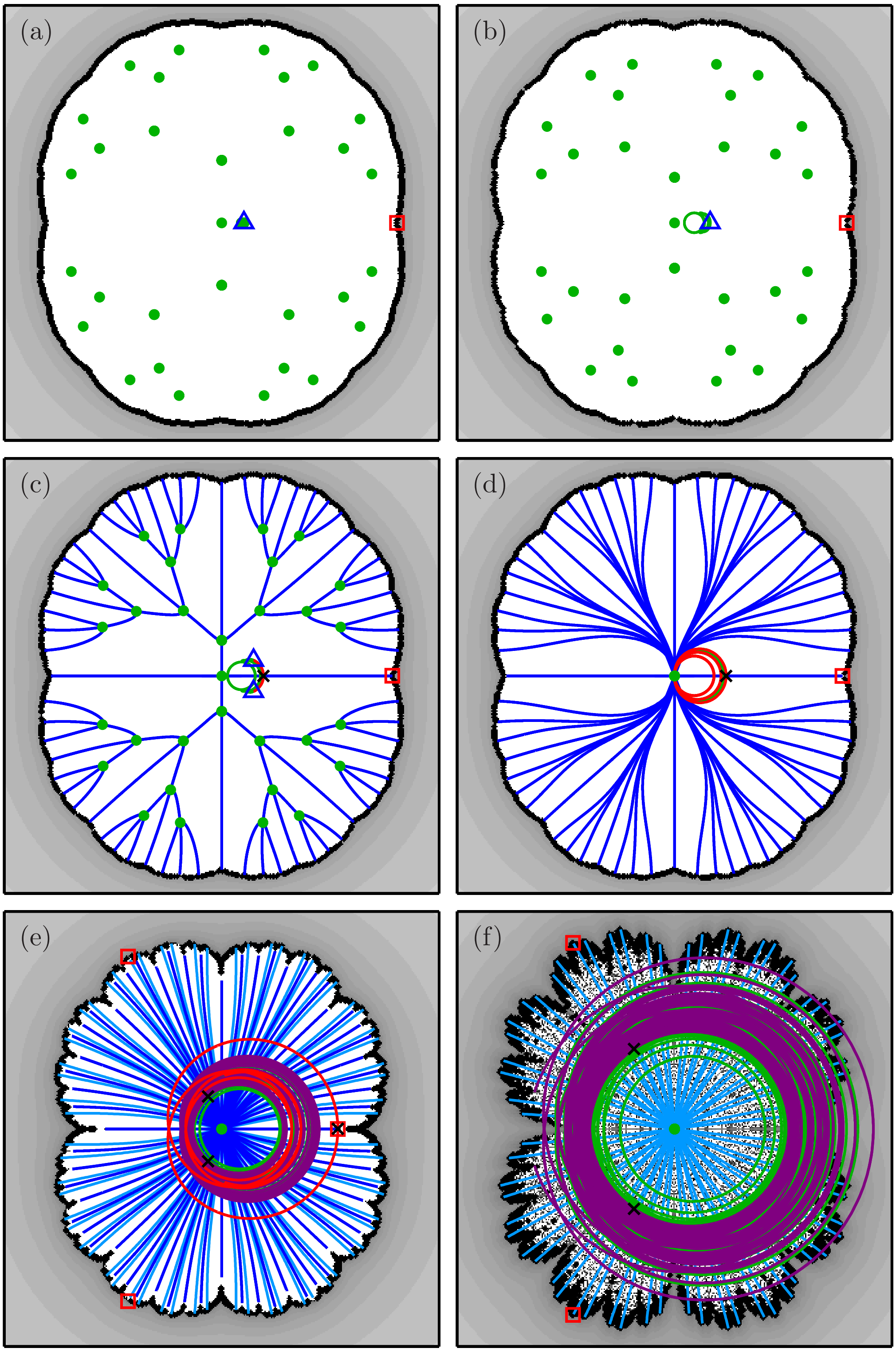

Figure 1: The transition of the phase portrait of (1) for $c = 0.1$ in $[-1.1, 1.1] \times [-1.1, 1.1]$; shown are the Julia set (black), the critical set (green), the fixed points (black cross, red square and blue triangles), the period-two points (black crosses and red squares), their stable sets (dark and light blue) and their unstable sets (red and purple). Panels (a)–(f) are for $\lambda = 1$, $\lambda = 0.95$, $\lambda = 0.93$, $\lambda = 0.9$, $\lambda = 0.7791$, and $\lambda = 0.6$, respectively.

the saddle fixed point, its stable and unstable sets and the chaotic attractor have disappeared. The Julia set now coincides with the closure of the stable set of the period-two saddle points and forms a so-called *Cantor bouquet* [5]. This is an infinite union of arcs that emanate from one point, such that the end points of these arcs are dense in the set. This set is locally connected only at the point of connection of the arcs, which is the critical point in this case. Furthermore, the set of end points of the arcs, together with the point of connection of the arcs, is a connected set, whereas the set without the critical point is totally disconnected; such a point is called an *explosion point*.

3 Conclusions

We find three mechanisms for creating or destroying saddle points and chaotic attractors in the transition of the phase portrait of the map (1) for fixed $c = 0.1$ and decreasing λ from $\lambda = 1$, namely, a pitchfork bifurcation, a forward-backward critical tangency and a saddle-node bifurcation, at which the Julia set becomes a Cantor bouquet. Pitchfork and saddle-node bifurcations also occur in other maps, but the (dis)appearance of saddle points and chaotic attractors in a forward-backward critical tangency and the creation of a Cantor bouquet Julia set in a saddle-node bifurcation are new and specific to this type of map [3]. For other values of c, we find two additional types of interesting Julia sets, which we call *Cantor tangle* and *Cantor cheese* [3]. They resemble the Cantor bouquet, but contain a dense set of explosion points; see [3] for more details.

Overall, our numerical investigation enables us to extend the fundamental dichotomy in the complex quadratic family (2) to a *trichotomy* in the family (1): The topology of the generalised Julia set of (2) persists for $\lambda < 1$ if the orbits in the forward-critical set all stay bounded or all go to infinity; in the "intermediate case" of some bounded and some unbounded orbits, the Julia set is either a Cantor bouquet, a Cantor tangle or a Cantor cheese.

References

[1] R. Bamon, J. Kiwi, and J. Rivera, *Wild Lorenz-like attractors*, Preprint, 2006, arXiv:math0508045v2

[2] S. Hittmeyer, B. Krauskopf, and H.M. Osinga, *Interacting global invariant sets in a planar map model of wild chaos*, SIAM J. Appl. Dyn. Syst., 12(3), 2013

[3] S. Hittmeyer, B. Krauskopf, and H.M. Osinga, *Interactions of the Julia set with critical and (un)stable sets in an angle-doubling map on* $\mathbb{C}\setminus\{0\}$, Preprint, 2013

[4] P. Blanchard, *Complex analytic dynamics on the Riemann sphere*, Bull. Amer. Math. Soc., 11(1), 1984

[5] J.M. Aarts, and L.G. Oversteegen, *The geometry of Julia sets*, Trans. Amer. Math. Soc., 338(2), 1993