Mathematics of Proportional Representation

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This talk is based on the paper

“Proportional Representation and Strategic Voters”

written jointly with Shaun White.

This paper is to appear in

“Journal of Theoretical Politics”.

View this paper and my other research on my Social Sciences Research Network author page:

http://ssrn.com/author=331772
Aims of this paper

• This paper was initially motivated by a desire to explain the behaviour of voters at the New Zealand general election held September 17th, 2005.

• Some NZ voters voted insincerely even though their doing so could have cost their most preferred party seats.

• We analyse the election and present two models that account for the behaviour observed in the election.

• We investigate opportunities for strategic voting under proportional representation (PR), other than those that emerge due to rounding.
Electoral system of New Zealand

- In 1993 New Zealand adopted mixed member proportional system (MMP) after many years of first-past-the-post (FPP).
- Every NZer has a party vote and an electoral vote. Party vote determines the proportionality of the House. Electoral vote determines the local representative.
- Parties that receive either
  - 5% of the party vote or
  - an electoral seat
are entitled to share 120 seats in the House proportionally to their party vote.
- The seats are allocated using the Sainte-Lague formula: firstly to electoral MPs, then parties fill their remaining quotas from their party lists.
The 2005 NZ general election

- The election took place September 17th, 2005.
- The two opinion polls closest to the election gave the following results (party vote):

<table>
<thead>
<tr>
<th>Poll</th>
<th>Date</th>
<th>Labour</th>
<th>Nats</th>
<th>NZ First</th>
<th>Greens</th>
</tr>
</thead>
<tbody>
<tr>
<td>TVNZ</td>
<td>15/09</td>
<td>38%</td>
<td>41%</td>
<td>5.5%</td>
<td>5.1%</td>
</tr>
<tr>
<td>Herald</td>
<td>16/09</td>
<td>44.6%</td>
<td>37.4%</td>
<td>4.5%</td>
<td>4.6%</td>
</tr>
</tbody>
</table>

- The margin of error was ±3%.
- The Green party were not expected to win an electoral seat, and NZ First were expected to win at most one.
- As it turned out neither the Green party nor NZ First won an electoral seat but both passed the threshold.
The 2005 NZ general election

- Anecdotal evidence (reports to the authors) has suggested that some voters with preferences
  
  \[
  \text{Labour} > \text{Greens} > \ldots
  \]

  may have cast their vote for the Greens.

- Greens received 5.3% of the party vote while Labour got 41.1%.

- National received 39.1% and NZ First 5.7%.

- Hypothetically, about 0.4% of the electorate who intended to vote Labour voted strategically for the Greens.
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- Thus we assume that the Greens would receive 4.9% of the vote if everybody voted sincerely and Labour would receive 41.5%.
The seats and the Shapley-Shubik power indices

<table>
<thead>
<tr>
<th>Party</th>
<th>Hypothesised</th>
<th></th>
<th></th>
<th>Actual</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Seats</td>
<td>SS</td>
<td></td>
<td>Seats</td>
<td>SS</td>
<td></td>
</tr>
<tr>
<td>Labour</td>
<td>54</td>
<td>0.414</td>
<td></td>
<td>50</td>
<td>0.324</td>
<td></td>
</tr>
<tr>
<td>National</td>
<td>50</td>
<td>0.214</td>
<td></td>
<td>48</td>
<td>0.262</td>
<td></td>
</tr>
<tr>
<td>NZ First</td>
<td>7</td>
<td>0.214</td>
<td></td>
<td>7</td>
<td>0.143</td>
<td></td>
</tr>
<tr>
<td>Green Party</td>
<td>0</td>
<td>0.0</td>
<td></td>
<td>6</td>
<td>0.110</td>
<td></td>
</tr>
<tr>
<td>Maori Party</td>
<td>4</td>
<td>0.081</td>
<td></td>
<td>4</td>
<td>0.076</td>
<td></td>
</tr>
<tr>
<td>United Future</td>
<td>3</td>
<td>0.048</td>
<td></td>
<td>3</td>
<td>0.043</td>
<td></td>
</tr>
<tr>
<td>ACT</td>
<td>2</td>
<td>0.014</td>
<td></td>
<td>2</td>
<td>0.029</td>
<td></td>
</tr>
<tr>
<td>Progressive</td>
<td>1</td>
<td>0.014</td>
<td></td>
<td>1</td>
<td>0.014</td>
<td></td>
</tr>
</tbody>
</table>

Labour and Greens could not form a government and, as a result, the strange bedfellows phenomenon had occurred with parties coloured purple forming the government.
Seat and power maximising voters

- A seat maximiser derives utility from the allocation of seats. If there are $m$ parties, she has a vector of utilities $u_1, \ldots, u_m$, where $u_i$ is the utility of a seat won by the $i$th party. Her total utility is

$$U(s_1, \ldots, s_m) = u_1 s_1 + u_2 s_2 + \ldots + u_m s_m,$$

where $s_i$ is the number of seats of the $i$th party.
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\]

where \( s_i \) is the number of seats of the \( i \)th party.

• A power maximiser’s utility is derived from the power of parties measured by some power index. The total utility of the parliament will in this case be

\[
U(s_1, \ldots, s_m) = u_1 p_1 + u_2 p_2 + \ldots + u_m p_m,
\]

where \( p_i \) is the power of the \( i \)th party.
Seat and power maximising voters

- A seat maximising voter, who prefers Labour to the Greens, would prefer the actual outcome to the hypothesised one if
  \[ u(\text{Greens}) > 0.667 \cdot u(\text{Labour}). \]

- A power maximising voter, who prefers Labour to the Greens, would prefer the actual outcome to the hypothesised one if
  \[ u(\text{Greens}) > 0.826 \cdot u(\text{Labour}). \]
Sainte-Lague formula in action

Let $\alpha = \frac{u(\text{Greens})}{u(\text{Labour})}$ for a power maximising voter.

The following graph shows for which values of $\alpha$ voting for the Greens become profitable for a Labour voter:

The graph for seat maximising voters is similar.
A little history

The subject of manipulability of proportional representation (PR) beyond rounding, is a contentious issue in political science literature.

- Duverger (1954) dismissed the possibility of strategic voting in PR.
- He was criticised for that by Leys (1959) and Sartori (1968) who believed that “the wasted vote logic must be applicable to certain kinds of PR systems.”
- Bowler and Lanoue (1992) considered that “under proportional representation ... voting sincerely is a dominant strategy”
- They were criticised for that by Cox (1997), who believed that the manipulability of PR is implied by the GS theorem.
Main assumptions

• The world with the rounding is far too complicated. Voters are unlikely to make these calculations.

• From now on we ignore rounding and allow fractional seats.

• To understand incentives for strategic voting we restrict ourselves with just three parties which will be A, B, C.
The results may be represented by a point $\mathbf{x}$ of the simplex $S^{m-1}$:

$$\mathbf{X} = (x_1, \ldots, x_m), \quad x_1 + \ldots + x_m = 1,$$

where the homogeneous barycentric coordinate $x_i$ is the fraction of the vote the $i^{th}$ party has won. In case of three parties:

![Diagram of a triangle with vertices A, B, and C, and a point X inside the triangle labeled with homogeneous coordinates $x_1$, $x_2$, and $x_3$. The point X represents a proportional distribution of votes among the three parties.]
Geometric representation of parliaments

A parliament is also represented by a point $x$ of the simplex $S^{m-1}$:

$$X = (x_1, \ldots, x_m), \quad x_1 + \ldots + x_m = 1,$$

where the homogeneous barycentric coordinate $x_i$ is the fraction of the seats the $i^{th}$ party has won.

The parliament is a mixture of parties!
A seat allocation rule is any mapping

\[ F_a : S^{m-1} \rightarrow S^{m-1} \].

Given a vector of scores \( sc \in S^{m-1} \), a seat allocation rule determines the distribution of seats in parliament \( x = (x_1, \ldots, x_m) \) by \( x = F_a(sc) \).

Main examples of \( F_a \):
- Identity seat allocation rule (no threshold),
- Threshold seat allocation rule.
For the vector of normalised scores $x$ the second party is below the threshold, so: $F_a(z) = z$, $F_a(x) = y$. 
How can the score be influenced?

If a group of like-minded voters with a sincere preference $L : A > B > C$ change it to $L' : B > A > C$, this causes a horizontal shift:

The score of $C$ remains the same, the score of $B$ increases and the score of $A$ decreases.
Types of Manipulability

We have ordinal preferences, say $A > B > C$, and also utilities $u_1 > u_2 > u_3$. Utilities can vary but as long as they are ordered as shown, voters will be in the same ordinal type.

Suppose voters of this type vote strategically submitting their preferences as $B > A > C$ (that is vote for $B$ instead of $A$) and they manage to change the result of the election.

We say that the system provides voters with strong incentive to manipulate if all voters of this type benefit from the misrepresentation and weak incentive if some voters of this type will benefit and some will not.

The voting rule is micro-manipulable if an arbitrary small percentage of the population can manipulate.
Theorem

For PR without threshold a seat maximiser has no incentives to manipulate.

This is probably what Bowler and Lanoue (1992) had in mind.
Seat maximisers under PR with a threshold

Theorem

PR with a threshold is always micro-manipulable by seat maximising voters but never strongly.

When scores are approaching \( x \) on the threshold, parliaments tend to \( y \).
Indices of voting power for $m = 3$ parties

Let the parties be $A$, $B$, and $C$. Regardless of the voting power index used, *four regions emerge*, where the vectors of indices of power for parliaments will be constant:

\[
\begin{array}{ccc}
A & K & B \\
M & L & C \\
(1, 0, 0) & \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) & (0, 1, 0) \\
(0, 0, 1) & & \end{array}
\]
Power maximisers and pure PR

Theorem

Let the parliament choosing rule $F$ be pure PR. Then $F$ is manipulable by power maximisers but never strongly.

Only uncertainty averse voters will be better off.
Strong manipulability of PR with threshold

Theorem

Let the parliament choosing rule be PR with a threshold. Then the rule is strongly manipulable by power maximising voters.

The horizontal arrow takes us from \((0, 0, 1)\) to \((1/3, 1/3, 1/3)\) which is profitable for all voters of \(A > B > C\) type.
Undershooting phenomenon

Under PR with a threshold, manipulating uncertainty averse voters may undershoot.

From $X$ uncertainty averse voters would manipulate to get from $(1/3, 1/3, 1/3)$ to the region $(0, 1, 0)$ but they may undershoot and end up in $(0, 0, 1)$. 
Conclusion

• In this paper we formalised the problem of strategic voting under systems of proportional representation and this allowed to bring clarity into the problem.

• We show that the incentives to manipulate may depend on the voter's attitude towards post-election uncertainty. An important distinction between weak and strong incentives to manipulate is made.

• An important distinction is made between seat maximising voters and power maximising voters. Their opportunities to manipulate are described for PR with and without threshold.
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Reflections on MMP. The threshold

The main purpose of a threshold is to prevent very small parties from entering the parliament. Hence the main task of a designer is to determine what "very small" means.

Parties like the Green Party, which are important to the fabric of the society and consistently get a support of approximately 5% of the electorate cannot be considered "very small."

By adopting (instead of designing) uncritically a voting system from Germany, New Zealand has got an electoral system which significantly distorts the society's support for minor political parties.

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Reflections on MMP. One-man parties

• Under MMP a party can also be eligible for list seats if it wins at least one constituency seat (three in Germany).
• This rule allows one-man parties to enter the parliament and practically deprives the threshold rule of any bite.
• At the election 28.71% of voters gave their electorate vote and their party vote to different parties. This figure is high enough to suggest that a reasonable amount of insincere voting went on (e.g. Epsom).
• Voters in some constituencies (Epsom) get disproportionately large voting power.
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Reflections on MMP. General ills of PR

• While we know how to distribute seats fairly, we don't know how to distribute power fairly. As a result small parties have disproportionately large power.
• Party boffins, not voters, determine who gets on the party list. List MPs are not accountable to the voters.
• Parliament gets a bunch of MPs who would be otherwise unelectable (e.g. porn stars in Italy).
• Urgent amendments to the current MMP are in order.
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Thank you for your attention!