Simple Games and Ideal Secret Sharing Schemes

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Plan for the Talk

• Introduction to Secret Sharing
• Introduction to Simple Games
• Composition of Games
• Main Result: Classification Weighted Ideal Secret Sharing Schemes
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- Introduction to Simple Games
- Composition of Games
- Main Result: Classification Weighted Ideal Secret Sharing Schemes

“Good friends, good books, and a sleepy conscience: this is the ideal life.” — Mark Twain
Back in the USSR

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The secret (the launch code) was shared between three officials so that each pair of them could access the secret but no single agent could.
The secret combination opening the vault key must be distributed among bank employees. The bank policy requires the presence of three employees in opening the vault, but at least one of them must be a departmental manager.
If a significant sum of money is being transferred, an approval may require:

- approval of two vice-presidents, or
- three senior tellers; or
- a vice-president and two senior tellers.
Cloud computing

Cloud storage and cloud computing provides us with new security challenges.
Shamir’s idea of storing sensitive data

For security data can be stored on several servers so that if some servers are compromised the data cannot be stolen and can be recovered from the remaining servers.
A secret sharing scheme ‘divides’ the secret $S$ into ‘shares’—one for every user—in such a way that:

- $S$ can be easily reconstructed by any authorised coalition of users, but
- an unauthorised coalition of users cannot determine $S$. 
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The concept of authorised coalition must be formalised.
Access structure

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Definition
An access structure is a pair $G = (P, W)$, where $W$ is a subset of the power set $2^P$, different from $\emptyset$, which satisfies the monotonicity condition:

$$\text{if } X \in W \text{ and } X \subset Y \subseteq P, \text{ then } Y \in W.$$  

Coalitions from $W$ are called authorised. We also denote

$$L = 2^P \setminus W$$

and call coalitions from $L$ unauthorised.
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and call coalitions from $L$ **unauthorised**.

The access structure is a simple game.
In the nuke briefcases game, if the set of agents is $P = \{1, 2, 3\}$, then the access structure is

$$W = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

In a threshold access structure or k-out-of-n access structure a coalition is authorised if it contains at least $k$ agents.
If in the bank transfert game there are two vice-presidents $v_1, v_2$ and three senior tellers $t_1, t_2, t_3$, then the set of minimal authorised coalitions is

\[
\{\{v_1, v_2\}, \{t_1, t_2, t_3\}, \{v_1, t_1, t_2\}, \ldots, \{v_2, t_2, t_3\}\}.
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Access Structure. Example 2

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The access structure is fully defined by the set of minimal authorised coalitions.
Non-authorised coalition. Example

This is an old style of unauthorised coalition. These days the bad guys are armed with laptops.
Shamir’s Scheme

Here is a pictorial interpretation of 3-out-of 4 scheme.

Any three would know the whole polynomial including \( c \).
Ideal Secret Sharing Schemes

In situations like cloud storage the length of shares may represent a significant problem.

A secret sharing scheme is called ideal if it is

- **perfect** (revealing no information about the secret to non-authorised coalitions) and
- the size of the domain of secrets is the same as the domain of shares (it cannot be smaller in perfect schemes).
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Problem

*Characterise access structures that can carry an ideal secret sharing scheme.*
Example of Simple Game: UN Security Council

The 15 member UN Security Council consists of five permanent and 10 non-permanent countries. A passage requires:

- approval of at least nine countries,
- subject to a veto by any one of the permanent members.
Weighted Majority Games

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**Definition**

A simple game $G$ is called a weighted majority game if there exists a weight function $w : P \rightarrow \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of all non-negative reals, and a real number $q$, called the quota, such that

$$X \in W \iff \sum_{i \in X} w_i \geq q.$$
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Such game is denoted $[q; w_1, \ldots, w_n]$. 
Weightedness of Games in Examples

UN Security Council game is weighted:

\[
39, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1
\]

Opening the vault game is not weighted:

\[
(m_1, t_1, t_2), (m_2, t_3, t_4); (m_1, m_2), (t_1, t_2, t_3, t_4)
\]

is a trading transform, which is a certificate of nonweightedness.
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Comparing seniority of players

Given a game $G$ we may also define a relation $\succeq_G$ on $P$ by setting $i \succeq_G j$ if for every set $X \subseteq P$ not containing $i$ and $j$

$$X \cup \{j\} \in W \implies X \cup \{i\} \in W.$$

It is known as Isbell’s desirability relation.
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We also define $i \succ_G j$ as $i \succeq_G j$ but not $j \succeq_G i$. 

Compressing information about the game

Many players in a game have equal status. Identifying equivalent players we get a multiset of players:

\[ P = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}. \]

A game with \( m \) equivalence classes is called \textit{m-partite}. 
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We have suppressed at this point the information about winning coalitions of those games.
Minimal and shift-minimal winning coalitions

Due to monotonic property the set of winning coalitions $W$ is completely defined by the set

$$W^\text{min} = \{ X \in W \mid \text{every proper subset of } X \text{ is losing} \}.$$
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The set of shift-minimal coalitions

$$W^{s\text{min}} = \{ X \in W^{\text{min}} \mid \text{every shift of } X \text{ is losing} \}.$$  

fully determines a complete game.
Compact presentation of shift-minimal winning coalitions

• UN Security Council game (has $10^4$ winning coalitions):
  $\{1, 5, 2, 10\}$, the type of shift-minimal winning coalitions is $\{1, 5, 2, 4\}$.

• Opening vault game:
  $\{1, n, 1\}, \{2, n, 2\}$, the type of shift-minimal winning coalition is $\{1, 2, 2\}$.
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- Opening vault game:
  \[ \{1^{n_1}, 2^{n_2}\} , \text{the type of shift-minimal winning coalition is } \{1, 2^2\} . \]
Composition of games (example)

The most general type of compositions of simple games was introduced by Shapley (1962).

We can take, for example, a unanimity game as a higher level game, i.e., both organisations must approve the decision.

Within each organisation we may use simple majority. This is how the European Union works.
Composition of games (formal definition)

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Definition
Let $G = (P_G, W_G)$ and $H = (P_H, W_H)$ be two games defined on disjoint sets of players and $g \in P_G$. We define the composition game $C = G \circ g H$ over $g$ by defining $P_C = (P_G \setminus \{g\}) \cup P_H$ and

$$W_C = \{ X \subseteq P_C \mid X_G \in W_G \text{ or } X_G \cup \{g\} \in W_G \text{ and } X_H \in W_H \},$$

where $X_G = X \cap P_G$ and $X_H = X \cap P_H$. 
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where $X_G = X \cap P_G$ and $X_H = X \cap P_H$.

It expresses the idea that a collective member may be a player in the game.
Proposition

Let $G, H, K$ be three games defined on the disjoint set of players and $g \in P_G$, $h \in P_H$. Then

$$(G \circ_g H) \circ_h K \cong G \circ_g (H \circ_h K),$$

that is the two products are isomorphic.
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Definition
A game $G$ is said to be indecomposable if there does not exist two games $H$ and $K$ and $h \in P_H$ such that $\min(|H|, |K|) > 1$ and $G \cong H \circ_h K$. 

Example
Both UN Security Council game and Opening Vault game are indecomposable.
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Since all $n$ players are equivalent, there exist $k$ such that it takes $k$ or more players to win. Such a game is called $k$-out-of-$n$ game, denoted $H_{n,k}$. 
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The game $H_{n,n}$ is special and is called the unanimity game on $n$ players. We will denote it as $U_n$. Only $U_2$ is indecomposable.
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The game \( H_{n,1} \) does not have a name in the literature. We will call it anti-unanimity game and denote \( A_n \). Only \( A_2 \) is indecomposable.
Lemma

Let $G, H$ be two games on disjoint sets of players and $H$ is neither an anonymity nor an anti-unanimity game.

- If for two elements $g, g' \in P_G$ we have $g \succ g'$ (and $g'$ is not a dummy), then $G \circ g H$ is not complete;

- If $G$ and $H$ are complete and $g \in P_G$ is a member of the weakest desirability class of $G$, then $G \circ g H$ is complete.

So, in the future we will compose complete games only through a player from the least desirable class.
The semigroup of complete games

Theorem (Freeman-Slinko, 2012)

Let $\mathcal{G}$ be the set of all complete games. Then $\mathcal{G}$, equipped with the operation of composition, is a semigroup with identity. Every $G \in \mathcal{G}$ can be expressed uniquely as a product of indecomposable games in this semigroup.
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Corollary

Every weighted simple game can be expressed uniquely as a product of indecomposable weighted simple games.
Counterexample

Do weighted games form a subsemigroup? No.
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Example

Let $G$ be defined on $P_G = \{1^2, 2^4\}$ and $H$ on $P_H = \{3^3\}$ with weighted voting representations

$$[7; 3, 3, 2, 2, 2, 2] \quad \text{and} \quad H = [2; 1, 1, 1],$$

respectively. Let $g \in G$ be one of the players of weight 2. Then the composition $G \circ g \ H$ is not weighted:

$$\left(\{1, 2, 3^2\}, \{1, 2, 3^2\}; \{1^2, 3\}, \{2^2, 3^3\}\right).$$

as this is a certificate of nonweightedness.
Two Major Theorems

Theorem (Beimel-Tassa-Weinreb, 2008)

Every ideal weighted game is a composition of indecomposable ideal weighted games.

Theorem (Farràs-Padró, 2010)

Any indecomposable ideal weighted game belongs to one of the seven following types:

- One-partite: \( k\)-out-of-\( n\) games - type \( H \);
- Bipartite: types \( B_1, B_2, B_3 \);
- Tripartite: types \( T_1, T_2, T_3 \);

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Theorem (Hameed, Slinko, 2013)

$G$ is an ideal weighted simple game if and only if

$$G = H_1 \circ \ldots \circ H_s \circ I \circ A_n \quad (s \geq 0);$$

where $H_i$ is an indecomposable 1-partite game.

Also, $I$ is an indecomposable game of types $B_1$, $B_2$, $B_3$, $T_1$, $T_3$, and $A_n$ is the anti-unanimity game on $n$ players.

Moreover, $A_n$ can be present only if $I$ is either absent or it is of type $B_2$.

This decomposition is unique.
The full paper is on ArXiv:

http://arxiv.org/abs/1308.3763

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Thank you for your attention!