Electoral Equilibria under Scoring Voting Rules

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Introduction

“For several fundamental reasons, it is particularly appropriate that we should include the analysis of incentives in political institutions as an essential part of the domain of modern economics.”

Roger Myerson, Schumpeter Lecture (1999)
The three main reasons:

- markets and politics are substantively interconnected systems
- failures of the political system can affect people’s welfare at least as much as failures of the market systems
- there are logical similarities between political competition and market competition
Downs’ thesis

Downs suggested that the famous Hotelling (1929) “Main Street” market competition model can be also used to analyse political competition.

“... my central hypothesis: political parties in a democracy formulate policy strictly as a means of gaining votes. ”

Anthony Downs (1957)

We should recognize though that each voter’s welfare depends on the policies of the candidate who wins the election, not the candidate to whom he gives his vote.
The Hotelling’s spatial model

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![Diagram showing ideal positions $x_1$ and $x_2$.]
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![Diagram showing ideal positions and candidates](image)

- There are \(m\) candidates. A *profile* is an \(m\) vector \(x = (x_1, \ldots, x_m) \in [0, 1]^m\) that specifies each candidate’s position: \(x_i\) is candidate \(i\)’s position.
Ideological spectrum in New Zealand

One of the main assumptions of this model is that the ideological spectrum is one-dimensional.
Stigler (1972) also argued that “political and economic motives will be similar and best modeled by maximization of market or vote share.”

We assume that parties choose their positions on the ideological spectrum in order to maximize their share of the votes.
Questions to answer

In the political context, the main questions are:

- Do equilibria situations exist?
- Will the candidates cluster together, advocating identical or similar policy positions, or will they adopt diverse positions that appeal to different groups of voters?
- Which characteristics of the voting rule force rational candidates to adopt the position of the median voter and which give an incentive to diversify?
- How do the optimal strategies depend on the voting system in use?
The key is the investigation of profiles (vectors of candidate positions) that are in *Nash equilibrium.*
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Two kinds of Nash equilibria exist:

- A *convergent* Nash equilibrium (CNE) occurs when all candidates adopt the same ideological position.
- A *non-convergent* Nash equilibrium (NCNE) is when not all candidate positions are the same.
Most of the literature on competitive determinants of political policy positions has focused on just one electoral system: plurality voting.

- For $m = 2$ we have CNE (Hotelling 1929) — known as Principle of Minimum Differentiation.
- For $m = 3$ no Nash equilibria exist;
- For $m = 4, 5$ there is unique NCNE;
- For $m \geq 6$ there are infinitely many NCNE (Eaton and Lipsey, 1975).
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Broadening the options

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• One such alternative voting rule is famous Borda rule. In an $m$-candidate election under Borda a candidate gets $m - i$ points each time she is ranked $i$th best. The candidate with most points wins.

• This rule belongs to a large class of rules called positional scoring rules.
Positional scoring rule

- Each voter ranks the candidates. The candidate ranked $i$-th receives $s_i$ points.

- Specified by an $m$-vector $s = (s_1, s_2, \ldots, s_m)$ of scores with $\bar{s} = \frac{1}{m} \sum_{i=1}^{m} s_i$ being the average score.

- Require that $s_1 \geq \cdots \geq s_m$ and $s_1 > s_m$, i.e., the scores are nonincreasing and first is better than last. For example:
  - Plurality: $s = (1, 0, 0, \ldots, 0, 0)$
  - Borda: $s = (m-1, m-2, \ldots, 1, 0)$
  - Antiplurality: $s = (1, 1, 1, \ldots, 1, 0)$
  - $k$-Approval: $s = (1, 1, \ldots, 1, 0, \ldots, 0)$.
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The candidates' overall scores are then calculated by integrating across all voters.
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Positional scoring rules with ties

- If two or more candidates occupy the same policy position, the voters will be indifferent between them.

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- A candidate ranked by a voter in an indifference group that is ranked from \( i+1 \)-th to \( j \)-th in his ranking receives
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- For example, if Borda rule is used:

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The score of a candidate positioned at $x^1$ would be

$$\frac{s_1 + s_2}{2} \frac{x_1 + x_2}{2} + \frac{s_2 + s_3}{2} \frac{x_3 - x_2}{2} + \frac{s_4 + s_5}{2} \left(1 - \frac{x_1 + x_3}{2}\right).$$
Economic interpretation of positional scoring rule

Hotelling’s “Main Street” model originally stipulated that the customers buy always from the closest retailer. We may modify it as follows.

- The issue space is a road through an urban area, with customers distributed along it.
- Firms choosing locations so as to maximise their share of the market.
- $s = (s_1, \ldots, s_m)$ is a vector of probabilities, $s_i$ being the probability a customer buys from the $i$-th nearest firm.
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Economics question

- Plurality rule cannot explain for \( m > 2 \) the tendency of firms to cluster together.
- Can we explain this tendency by a more general model?
Convergent equilibria

**Theorem** (Cox, 1987). For $m$ candidates and scoring rule $s$, a profile $x = (x^*, \ldots, x^*)$ is a CNE if and only if

$$c(s, m) \leq x^* \leq 1 - c(s, m),$$

(1)

where $c(s, m) = \frac{s_1 - \bar{s}}{s_1 - s_m}$ is the *c-value* (with $\bar{s} = \frac{1}{m} \sum_{i=1}^{m} s_i$).

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- If $c(s, m) > 1/2$ (best-rewarding rule$^1$), the inequality (1) cannot hold. So no CNE exist.
- If $c(s, m) \leq 1/2$ (worst-punishing or intermediate rule), any $x^*$ in $[c(s, m), 1 - c(s, m)]$ is a CNE.

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**Note.** Borda is intermediate.

\(^1\)Terminology of R. Myerson
The blue interval is the set of valid equilibrium platforms.

The size of the blue interval depends on the rule. The stronger the incentive to place first, the smaller it is.

If \( c(s, m) > 1/2 \) then there is no blue interval – no CNE exist.
Non-convergent equilibria

- What about equilibria in which not all candidates adopt the same platform?
Non-convergent equilibria

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• **Theorem (Cox).** In a three-candidate election under scoring rule \( s = (s_1, s_2, s_3) \):
  
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• If $m = 4$, can we characterize the rules for which NCNE exist? This is the first challenge.
The four-candidate case

**Theorem (C.-S., 2011).** In a four-candidate election under scoring rule $s = (s_1, s_2, s_3, s_4)$, NCNE exist iff both the following conditions are satisfied:

- a) $s_1 > s_2 = s_3$;
- b) $c(s, 4) > 1/2$.

Moreover, the NCNE is unique and symmetric with two candidates at

$$x_1 = \frac{s_1}{4(s_1 - s_2)}$$

and two at

$$x_2 = 1 - x_1.$$
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The distance between the pairs of candidates in NCNE depends on the rule. As $c(s, 4) \rightarrow 1/2$, the positions converge to the half-way point.

When $c(s, 4)$ falls below $1/2$, only CNE exist, given by the previous theorem.

If $c(s, 4) > 1/2$ but $s_2 \neq s_3$ then no NE of either kind exist.
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The five-candidate case

**Theorem (C.-S., 2011).** In a five-candidate election under scoring rule $s = (s_1, s_2, s_3, s_4, s_5)$, NCNE exist iff both the following conditions are satisfied:

- a) $s_1 > s_2 = s_3 = s_4$;
- b) $c(s, 5) > 1/2$.

Moreover, the NCNE is unique and symmetric, with equilibrium profile $x = ((x^1, 2), (1/2, 1), (x^2, 2))$, where

$$x^1 = \frac{1}{6} \left( \frac{s_1 + s_2}{s_1 - s_2} \right) \quad \text{and} \quad x^3 = 1 - x^1.$$
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**Note.** For both \( m = 4 \) and \( m = 5 \) CNE and NCNE cannot coexist together.
The general case

What about the general case of $m$ candidates?

- For $m \geq 6$, a complete characterisation of scoring rules admitting NCNE is not known.
- What can we say about particular classes of scoring rules?
- Is the dichotomy holds for $m \geq 6$?

**Theorem (C.-S., 2011).**

Given a scoring rule $s$, in a Nash equilibrium the number of occupied positions $q$ satisfies

$$q \geq \left\lceil \frac{1}{2} \left( 1 - c(s, m) \right) \right\rceil.$$  

E.g., if $c(s, m) > \frac{3}{4}$, then $q \geq 3$. For the plurality $c(s, m) > 1 - \frac{1}{m}$ so we have at least $\frac{m}{2}$ occupied positions.
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Repeated highest scores

**Theorem. (C.-S. 2011)** Given a scoring rule $s$, let $1 \leq k \leq m - 1$ be such that $s_1 = \cdots = s_k > s_{k+1}$. Then a necessary condition for NCNE is $\min(n_1, n_q) > k$. 

**Corollary.** (C.-S. 2011) If $s$ is a scoring rule such that $s_1 = \cdots = s_k > s_{k+1}$ for some $k \geq \lfloor m/2 \rfloor$, then $s$ allows no NCNE.

**Example.** If $m$ is odd, consider $k$-approval with $k = \frac{m-1}{2}$. That is, $s = (1, \ldots, 1, 0, \ldots, 0)$, where the first $k$ positions are ones. Then $c(s, m) = 1 - \frac{1}{m} \frac{m-1}{2} = 1 + \frac{1}{2} m > 1 + \frac{1}{2}$. So the rule is best-rewarding but by Corollary it has no NCNE.
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$$c(s, m) = 1 - \frac{1}{m} \left( \frac{m - 1}{2} \right) = \frac{1}{2} + \frac{1}{2m} > \frac{1}{2}.$$ 

So the rule is best-rewarding but by Corollary it has no NCNE.
Concave scores

We say that the score vector \( s = (s_1, \ldots, s_m) \) is concave if

\[
s_1 - s_2 \geq s_2 - s_3 \geq \cdots \geq s_{n-1} - s_n \geq s_n - s_m.
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Such rules are best rewarding or intermediate: \( c(s, m) \geq 1/2 \).
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**Theorem. (C.-S. 2011)** Let \( s \) be a scoring rule \( s = (s_1, \ldots, s_n, s_{n+1}, \ldots, s_m) \), with

\[
s_{n+1} = s_{n+2} = \cdots = s_m
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for some \( 1 \leq n < m \). If \( s \) is concave then there are no NCNE, unless the subrule \( s' = (s_1, \ldots, s_n, s_{n+1}) \) is Borda and \( n + 1 \leq \lfloor m/2 \rfloor \) (i.e., more than half the scores are constant).
Theorem. Suppose $m$ is even and $s$ is a scoring rule with $c$-value $c(s, m) \leq 1/2$ and $s_{m/2} < \bar{s}$. Then any profile $x = ((x_1, m/2), (x_2, m/2))$ satisfying

$$\frac{1}{2} - \frac{\bar{s} - s_{m/2}}{s_1 - s_{m/2}} \leq x_1 < \frac{1}{2} \quad \text{and} \quad x_2 = 1 - x_1$$

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Example. Let $m = 6$. Suppose $s = (2, 2, 1, 1, 1, 0)$. We have $c(s, m) = 5/12 < 1/2$, so the rule is indeed worst-punishing. Also, $s_3 = 1 < 7/6 = \bar{s}$. So any profile with half the candidates one each side such that $1/3 \leq x_1 < 1/2$ and $x_2 = 1 - x_1$ is a NCNE.
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Dichotomy fails. Both types of equilibria can coexist together!
Multipositional NCNE

Theorem. Let there be $m = qr$ candidates, $q \geq 2$. Let $s = (s_1, \ldots, s_{r-1}, 0, |0, \ldots, 0, |\ldots|0, \ldots, 0)$ be a scoring rule. Divide the interval into $q$ equally sized subintervals. Then the profile in which $r$ candidates locate at the half-way point of each subinterval is a NCNE if and only if $c(s', r) \leq 1/2$, where $s' = (s_1, \ldots, s_{r-1}, 0)$.

Example: $m = 9$ candidates and $q = r = 3$. NCNE for rules: 2-approval, $s = (1, 1, 0, 0, 0, 0, 0, 0, 0)$ or $(2, 1, 0, 0, 0, 0, 0, 0, 0)$. 
Multipositional NCNE

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**Theorem.** Let there be \( m = qr \) candidates, \( q \geq 2 \). Let 

\[
s = (s_1, \ldots, s_{r-1}, 0, \underbrace{0, \ldots, 0}_{r}, \cdots, \underbrace{0, \ldots, 0}_{r})
\]

be a scoring rule. Divide the interval into \( q \) equally sized subintervals. Then the profile in which \( r \) candidates locate at the half-way point of each subinterval is a NCNE if and only if \( c(s', r) \leq 1/2 \), where \( s' = (s_1, \ldots, s_{r-1}, 0) \).

**Example:** \( m = 9 \) candidates and \( q = r = 3 \). NCNE for rules: 2-approval, \( s = (1, 1, 0, 0, 0, 0, 0, 0, 0) \) or \((2, 1, 0, 0, 0, 0, 0, 0, 0)\).
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Finding a NCNE reduces to finding a CNE on each subinterval.
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And this is not an equilibrium since $s'' = (1, 0, 0)$ is best-rewarding:
Firms locations revisited

- If consumers predominately buy from the nearest firm, say if the city is large, we get NCNE – firms congregate at multiple locations spread through the city.
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• Our results can be seen a confirmation for the Principle of Local Clustering.

• **Principle of Local Clustering** (Eaton-Lipsey, 1975). When a new firm enters a market, or when an existing firm relocates, there is a strong tendency for that firm to locate as close as possible to another firm. This behaviour tends to create local clusters of firms in many equilibrium and disequilibrium situations.
Conclusions

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- Concave score vectors produce rules without NCNE with small number of well-described exceptions.
- In NCNE candidates spread along the issue space grouped into clusters.
- Plurality, $s = (1, 0, \ldots, 0)$ does not explain the Principle of Local Clustering but more general scoring rules do.
Future Research

What is next?

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- Suppose a rule does not have any type of equilibria. Will the clustering effect still be observed to some extent?