Proportional Representation and Strategic Voters*

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Abstract. The goal of this paper is to examine the incentives to vote insincerely, other than those created by rounding, faced by voters in the systems of proportional representation (PR). We rigorously investigate two models of voter behaviour. The first model assumes that a voter is primarily interested in the distribution of seats in the post-election parliament (seat maximiser) while the second considers a voter who is concerned with the distribution of power in it (power maximiser). We show that under pure PR seat maximisers do not have any incentives to manipulate, which justifies the Bowler and Lanoue (1992) claim, and that such incentives for seat maximisers appear with the introduction of a threshold.

We show that, even in the absence of a threshold, there will always exist circumstances where a power maximiser would have an incentive to vote insincerely. We demonstrate that her incentives to make an insincere vote depend on her attitude toward uncertainty. The introduction of a threshold creates new and stronger opportunities for strategic voters regardless their attitude towards uncertainty. Finally we discuss the overshooting/undershooting phenomenon, when either too many or too few like-minded voters attempt to manipulate.

We use two models to explain voters' behaviour at the most recent (2005) New Zealand general election and demonstrate that rounding creates not only incentives but also disincentives for strategic voting.

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Keywords: parliament choosing rule, proportional representation, power index, strategic voting, manipulability, overshooting, undershooting.

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1 Introduction

The subject of the manipulability of proportional representation (PR) has long been discussed in political science literature but often in a non-rigorous way. As a result, a diverse range of opinions has been expressed. Duverger (1954) dismissed the possibility of strategic voting in PR and was criticised for that by Leys (1959) and Sartori (1968) who believed that the wasted vote logic must be applicable to certain kinds of PR systems. Bowler and Lanoue (1992) considered that "under proportional representation ... voting sincerely is a dominant strategy" and were criticised for that by Cox (1997, footnote on p. 270) who thought that political scientists do not pay a sufficient attention to consequences of the classical Gibbard-Sattethwaite (GS) theorem (Gibbard, 1973; Sattethwaite, 1975). These disagreements have arisen because there has not yet appeared a model rigorous enough and general enough to answer the questions "Which systems of PR are manipulable?" and "For those systems that are manipulable, how and by which type of voters can they be manipulated?" In this paper we present such a model and answer these questions.

A prime example of successful formalisation of election in a single member constituency or election to a single position is the concept of a social choice function and the aforementioned GS theorem that states that every non-dictatorial social choice function is manipulable. Contrary to a widespread belief, the GS theorem is not directly applicable in the PR framework. It applies to social choice functions which map preferences of agents expressed as linear orders over a finite set of alternatives onto this set of alternatives. To any profile of linear orders such a social choice function assigns just a single alternative. In a system of proportional representation the set of alternatives is the set of competing political parties (or party lists) and the corresponding choice function, which we call a *parliament choosing rule*, maps the set of voters' profiles onto the set of parliaments, which are "mixtures" of alternatives but not alternatives themselves. Indeed, if three parties A, B, C participated in an election and won 50%, 30% and 20% of the seats, respectively, then the resulting parliament can be expressed as $0.5A + 0.3B + 0.2C$ or simply $(0.5, 0.3, 0.2)$. The fact that the values of the choice function are mixtures and not alternatives makes a profound difference since, for each agent, there are now a range of ways in which she can extend her order on the set of parties to an order on the set of parliaments and the way this extension is done determines the type of this voter. We show that, indirectly, the GS theorem can be applied to the PR framework, but for that we had to assume that the voters belong to a type which we call "daft" voters. We will give a precise definition later, here we just mention that being daft is not a realistic model of voters' behaviour. In reality several other types of voters might exist and it is important to classify these types and to investigate for which of those types PR is manipulable.

The concept of manipulability for parliament choosing rules is richer than for social choice functions. An important contribution of this paper is the distinction between weak manipulability and strong manipulability. We say that the opportunity of manipulation is weak if agents' willingness to undertake this manipulative move is determined not only by their preference order over the alternatives but also by their attitude towards uncertainty.

One obvious way to extend preferences from the set of parties to the set of parliaments is to use the lexicographic order or to introduce utilities. If the first method is used, an agent with preferences $A > B > C$ and lexicographic extension of them to the set of parliaments will prefer the parliament (α, β, γ) to the parliament $(\alpha', \beta', \gamma')$ if and only if $\alpha > \alpha'$ or $\alpha = \alpha'$ but $\beta > \beta'$. Such an agent will be called a *lexicographic seat maximiser*. She is primarily concerned with the number of seats her favorite party secures in the parliament. Alternatively, with the introduction of utilities u_1, u_2, u_3, the agent will prefer the first parliament to the second if $u_1\alpha + u_2\beta + u_3\gamma > u_1\alpha' + u_2\beta' + u_3\gamma'$. Such an agent cares not only about the number of seats of her favourite party but also for the number of seats of several other parties. Such an agent will be called a *weighted seat maximiser*.

In this paper we allow parties to hold fractional seats to exclude any need for rounding. In practical implementations of PR fractional seats are not allowed and a technique of rounding off will need to be applied (see e.g., Saari, chapter 4). Cox and Shugart (1996) demonstrated that this need to "round off" can render PR manipulable even for lexicographic seat maximisers. The mechanism of the manipulation is as follows. If a party is in a position where receiving a few more or a few less votes will not alter the number of seats it will take, then some of that party's supporters may peel off, and attempt to influence the distribution of the remaining seats. Rounding is an important issue, but its consequences, as we mentioned earlier, have been analysed elsewhere. By excluding it from consideration we can focus directly on other sources of manipulative behaviour.

We show that when a threshold - a rule that a party must secure at least a certain percentage of the vote in order to reach parliament - is not used in PR, then seat maximisers will have no incentives to manipulate. This is may be what Bowler and Lanoue had in mind when they formulated their claim. When a threshold is in operation some seat maximisers will always have incentives to manipulate however this manipulability is never strong.

Our next model revolves around the following observations. A parliamentary election held under a method of PR will determine how parliamentary seats are to be allocated. Once the election is over, and all the results are in, a government formation process begins. We will model that process with a simple game, where the voting weight of each parliamentary party is given by the number of seats it has won. The solution of the game will be a government, a set of ministers, a set of policies to be enacted, and so on. Under PR, a small change in the way ballots are cast will result in a small change in the voting weights of the parties in the post-election government formation game. But there do exist circumstances where a small change in the way ballots are cast can effect a significant change in certain other facets of the government formation game, in particular in the voting powers of the parties and in the set of feasible solutions. To illustrate, suppose three parties contest an election held under a method of PR, and let the vector $\mathbf{x} = (x_1, x_2, x_3)$ denote the resultant parliament, where the ith coordinate gives the fraction of seats won by the ith party. Compare the parliaments $\mathbf{x}_1 = (0.49, 0.49, 0.03)$, $\mathbf{x}_2 = (0.5, 0.49, 0.02)$, and $\mathbf{x}_3 = (0.51, 0.49, 0.01)$. Suppose that in the parliament a strict majority is sufficient to pass any motion. We contend that there is no reason to a priori believe that either the government-formation game or its solution would be significantly different were the post-election parliament to be $\mathbf{x}_1$ rather than $\mathbf{x}_2$. We contend that a comparison of $\mathbf{x}_2$ and $\mathbf{x}_3$ does not lead to the same conclusion: if $\mathbf{x}_2$ is the election result, then any two of the three parties could form a coalition government (as could all three); if $\mathbf{x}_3$ is the election result, the first party would likely form a government alone.

We consider a model of a sophisticated voter who is aware that a post-election government formation game will take place. We will assume that this voter, first and foremost, concern with the distribution of power in the post-election parliament, this is why we call them *power maximisers*. We show that, irrespective of which parliament choosing rule is adopted, there will always exist circumstances where a power maximising voter would have an incentive to vote insincerely even for the PR without threshold, in which case the manipulability is weak. We show that the introduction of a threshold makes a parliament choosing rule strongly manipulable by power maximising voters.

The suggestion that thresholds can create opportunities for strategic voting under the systems of PR is not new (see, for example, Roberts (1988) and Cox (1997, p.197). What is new is the characterisation of opportunities for strategic voting presented by thresholds for each of the classes of voters that have just been discussed. Our main results are obtained for the case $m = 3$. In fact, this is the main case. It is clear that if we are able to demonstrate manipulability of a parliament choosing rule for $m = 3$ parties, it will be manipulable for any $m \geq 3$. Austin-Smith and Banks (1988), and Baron and Diemeier (2001) also assume $m = 3$.

In the last section we use our model to explain voter behaviour at the most recent New Zealand general election. We also investigate how the incentives to manipulate, described earlier, interrelate to the rounding which cannot be ignored in a real situation. In particular, we consider how the method of rounding New Zealand uses, based on the Saint-Lague formula, influences incentives to vote strategically. We show that the rounding may actually be considered as deterrent to manipulation since it requires a degree of coordination from manipulators in order to avoid under and overshooting. In game-theoretic terms would be manipulators have to play a coordination game the outcome of which is by no means certain even if the profile is manipulable.

It may be worthwhile to note that the so-called choose-k rules (Brams and Fishburn, 2002) which always pick a k-element subset of the set of alternatives are not suitable for modelling PR since a parliament cannot be identified with the set of parties represented in it.

The main contributions of this paper are as follows. We rigorously define a parliament choosing rule and show which consequences regarding manipulability of this rule can be extracted from the GS theorem. We introduced two realistic types of voters: seat maximisers and power maximisers and two degrees of manipulability for a parliament choosing rule, that is weak and strong manipulability. We believe we are the first to demonstrate that pure PR is manipulable even in the absence of rounding and that the incentives to vote strategically might depend on the attitude of a particular voter to post-election uncertainty. We raise the issue of under and overshooting and show that it can serve as deterrent of manipulation. Inspired by Saari (1994), we take a geometric approach and use graphs with baricentric coordinates to represent parliaments and possible manipulating attempts. This is not entirely new as Jones (2005) takes a similar approach to investigate voting power paradoxes.

A few words about related literature are in order. In a narrower framework of spatial voting strategic voting under PR has been studied extensively. Austin-Smith and Banks (1988), and more recently Baron and Diermeier (2001), and De Sinopoli and Iannantuoni (2005) have constructed multi-stage spatial models of political systems that incorporate proportional representation. In these models voters (i) have preferences over the set of policies that governments might pursue but (ii) do not necessarily vote for the party to which they are ideologically closest. Voters might support parties expousing views more extreme than their own in a bid to counteract votes from other voters whose opinions lie on the opposite side of the policy spectrum. In all these papers it is assumed that, at the ballot box, voters can indicate a preference for just one party (i.e. the positional scoring rule is plurality). The essential difference between their and our approaches is that in the former the corresponding choice functions map profiles of policies into policies completely avoiding the government-formation game. Karp et al (2002) modelled split voting in NZ, but, unlike us, they assume voters use their party vote sincerely and their electorate vote strategically.

Below, Sections 2 (Parliament choosing rules), 3 (The government formation game and voting power indices) introduces the main objects of the investigation, their properties and representations. Section 4 (Manipulability of parliament choosing rules. What does the GS theorem imply?) introduces the concept of manipulability for parliament choosing rules and investigates consequences of the GS theorem. Section 5 (Strategic opportunities for seat maximising voters) and Section 6 (Strategic opportunities for power maximising voters) contain the main theoretical results. Section 7 (Overshooting) discusses overshooting and undershooting. Section 8 uses the results of previous sections to explain voter behaviour at the most recent New Zealand general election, and Section 9 concludes.

2 Parliament Choosing Rules

We model a parliamentary election. We assume that a parliamentary body is to be elected, that the body contains a fixed number k of seats, and that m political parties are competing for those seats. We assume n voters are eligible to vote, and all do.

Voters have preferences on the set of political parties $\mathcal{A} = \{a_1, \ldots, a_m\}$. Every voter has a favourite party, a second favourite, and so on. No voter is indifferent between any two parties. Every voter's preferences can then be represented as a linear order on $\mathcal{A}$. Let $\mathcal{L}(\mathcal{A})$ be the set of all possible linear orders. The Cartesian product $\mathcal{L}(\mathcal{A})^n$ will then represent preferences of the n-voter society. Elements of this Cartesian product are called *n-profiles* or simply *profiles*. The collection of all ballot papers will also be a profile. At the ballot box, voters do not necessarily rank the parties in the order of their sincere preference.

We assume each voter forms an expectation of what will transpire at the election. We follow Cox and Shugart and assume that these expectations "are publicly generated - by, for example, polls and newspapers' analysis" of the parties' prospects - "so that diversity of opinion in the electorate is minimised" (1996, page 303).

As we mentioned earlier, in theoretical part of this paper we will ignore rounding and allow fractional seats. We presume that every party decides on a party list before the election, i.e. ranks its candidates in a certain order with no ties. After the fractions of the seats each party has won is known, the composition of the parliament is decided on the basis of those party lists. If a party is allowed to have s MPs then the first s candidates from the party list become MPs.

The result of the election will be a parliament. Any parliament can be represented by a point in the simplex

$$S^{m-1} = \left\{ (x_1, \ldots, x_m) \mid \sum_{i=1}^{m} x_i = 1 \right\},$$

where x_i is the fraction of the seats the ith party wins at the election. For example, when $m = 3$ every parliament $\mathbf{x} = (x_1, x_2, x_3)$ can be represented by a point X of the triangle S^2, whose barycentric co-ordinates are x_1, x_2 and x_3. In this case we normally name parties A, B, C instead of a_1, a_2, a_3.

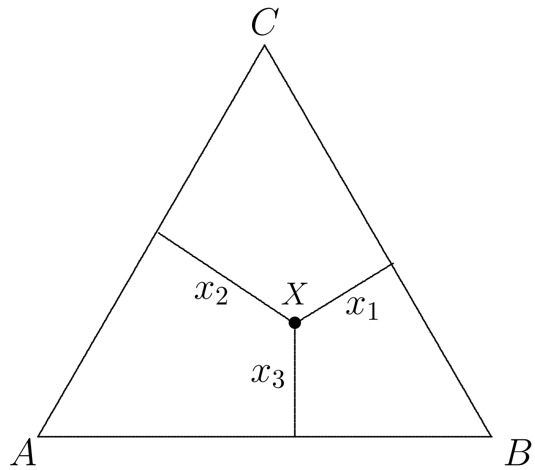

Figure 1: Triangle of Parliaments

A parliament choosing rule is employed to calculate the distribution of seats in the parliament. It has two ingredients: a *score function* and a *seat allocation rule*, both of which will be defined below. The idea is that, firstly, the votes for each party are being counted and the support for each party is being evaluated by the score function and, secondly, the seat allocation rule allocates seats to the parties on the basis of the scores calculated by the score function.

Let us define the score function first. Given a profile $R = (R_1, \ldots, R_n)$ and a set of alternatives $\mathcal{A}$, a score function assigns to each $a_i \in \mathcal{A}$ a real number. The greater this number, the better a_i is supposed to have done. There are a wide variety of score functions (McCabe-Dansted and Slinko(2006) has a comprehensive list of them). In this paper we will work with *normalised positional score functions*.

Let $w_1 \geq w_2 \geq \ldots \geq w_m = 0$ be m real numbers which we shall refer to as *weights*, and let $\mathbf{w} = (w_1, \ldots, w_m)$. Let $\mathbf{v}(a) = (i_1(a), \ldots, i_m(a))$, where $i_\ell(a)$ indicates the number of voters that stated that they rank party a ℓth best. Then, given a profile $R = (R_1, \ldots, R_n)$, the positional score of party a is given by:

$$sc_{\mathbf{w}}(R, a) = \mathbf{w} \cdot \mathbf{v}(a) = w_1 i_1(a) + \ldots + w_m i_m(a).$$

Well known vectors of weights and their respective scores include:

- the Plurality score $sc_{\mathbf{p}}(R, a)$, where $\mathbf{p} = (1, 0, \ldots, 0)$,
- the Borda score $sc_{\mathbf{b}}(R, a)$, where $\mathbf{b} = (m-1, m-2, \ldots, 1, 0)$,
- the Antiplurality score $sc_{\mathbf{a}}(R, a)$, where $\mathbf{a} = (1, \ldots, 1, 0)$.

The *vector of normalised positional scores* is given by

$$\mathbf{sc_w}(R) = \frac{1}{\sum_{i=1}^{m} sc_{\mathbf{w}}(R, a_i)} \left(sc_{\mathbf{w}}(R, a_1), sc_{\mathbf{w}}(R, a_2), \ldots, sc_{\mathbf{w}}(R, a_m) \right).$$

Clearly, $\mathbf{sc_w}(R) \in S^{m-1}$. We can now give the following two definitions.

Definition 1. *A* normalised positional score function *is a mapping*

$$F_s\colon \bigcup_{n=1}^{\infty} \mathcal{L}(\mathcal{A})^n \to S^{m-1},$$

which assigns to every profile its vector of normalised positional scores for some fixed vector of weights $\mathbf{w}$.

Like parliaments, normalised positional scores can similarly be represented on the Triangle of Scores. However, unlike parliaments, scores might not fill the whole triangle. Suppose, in the case of three parties, that the normalised positional score function is defined by the vector of weights $\mathbf{w} = (w_1, w_2, w_3)$ with $w_1 \geq w_2 \geq w_3$. We may assume that the weights has been normalised so that $w_1 + w_2 + w_3 = 1$ and $w_3 = 0$.

Let $\mathbf{w}_1 = \mathbf{w}$ and let $\mathbf{w}_2, \ldots, \mathbf{w}_6$ be the vectors obtained from $\mathbf{w}$ by permutations of coordinates and let $W_1, \ldots, W_6$ be the points of the triangle whose barycentric coordinates in the Triangle of Scores are equal to the coordinates of $\mathbf{w}_1, \ldots, \mathbf{w}_6$, respectively.

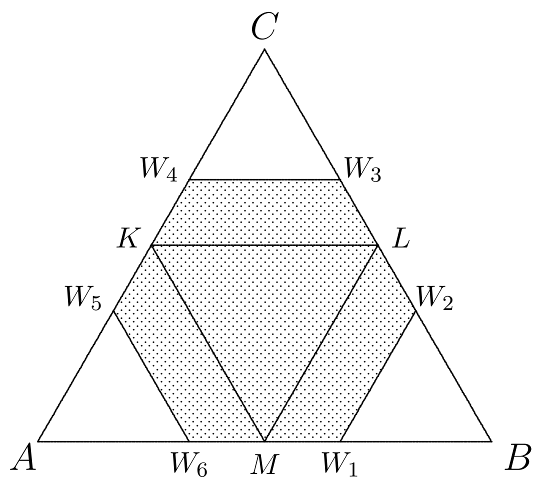

Figure 2: Hexagon of Scores

For example, W_1 will have coordinates (w_1, w_2, w_3) and correspond to the score of the unanimous profile, when all voters prefer A to B to C. The six points $W_1, \ldots, W_6$ will be on the boundary of the Triangle of Scores since one of their coordinates is zero. Any vector of scores $\mathbf{s} = (s_1, s_2, s_3)$ will be an affine linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_6$:

$$\mathbf{s} = \alpha_1 \mathbf{w}_1 + \ldots + \alpha_6 \mathbf{w}_6, \tag{1}$$

where $\alpha_1 + \ldots + \alpha_6 = 1$. It is easy to see that the point S, corresponding to $\mathbf{s}$, will always lie in the convex hull of $W_1, \ldots, W_6$ which will be called the Hexagon of Scores. In Figure 2 we show the hexagon of Borda scores. In this case $\mathbf{w} = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$. For the plurality rule, $\mathbf{w} = (1, 0, 0)$ and the hexagon degenerates into the whole triangle ABC. For antiplurality rule $\mathbf{w} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$, and the hexagon degenerates into the triangle KLM.

In reality, only the rational points of the Hexagon of scores can be realised as scores. The size of the society needed will depend on the least common multiple of the denominators of the coefficients of $\mathbf{w}_1, \dots, \mathbf{w}_6$. Therefore the points representing scores are everywhere dense in this hexagon. To see this it is sufficient to replace all α_i in (1) with their sufficiently close rational approximations. It is important to note that any point inside KLM can be always approximated by normalised scores while points outside KLM may not be approximated by scores for some $\mathbf{w}$.

Definition 2. *A* seat allocation rule *is any mapping*

$$F_a \colon S^{m-1} \to S^{m-1}.$$

Given a vector of scores $\mathbf{sc} \in S^{m-1}$, a seat allocation rule determines the distribution of seats in parliament $F_a(\mathbf{sc})$. There are two main examples of such rule.

Example 1 (Identity seat allocating rule)**.** *F_a is the identity function, i.e., $F_a(\mathbf{x}) = \mathbf{x}$.*

For the next example, we fix a *threshold*, which is a positive real number ϵ such that $0 < \epsilon \leq 1/m$. We define a *threshold function* $\delta_\epsilon \colon [0,1] \to [0,1]$ so that

$$\delta_\epsilon(x) = \begin{cases} 0 & \text{if } x < \epsilon, \\ x & \text{if } x \geq \epsilon. \end{cases}$$

Example 2 (Threshold seat allocating rule)**.** *Let ϵ be a positive real number such that $0 < \epsilon \leq 1/m$. Suppose $\mathbf{x} \in S^{m-1}$. Then we define $y_i = \delta_\epsilon(x_i)$ and $z_i = y_i / \sum_{i=1}^m y_i$. We now set $F_a(\mathbf{x}) = \mathbf{z}$, where $\mathbf{z} = (z_1, \dots, z_m)$. The restriction $\epsilon \leq 1/m$ guarantees that F_a is always defined.*

We are now ready to give the main definition of this section.

Definition 3. *A* parliament choosing rule *is a composition $F = F_a \circ F_s$ of a score function and a seat allocation rule:*

$$F_a \circ F_s \colon \bigcup_{n=1}^{\infty} \mathcal{L}(\mathcal{A})^n \to S^{m-1}.$$

If the identity seat allocating rule is employed, we shall refer to the parliament choosing rule as *pure proportional representation*. If a threshold seat allocating rule is employed, we shall refer to the parliament choosing rule as *proportional representation with a threshold*.

In practice only the plurality score have been used in systems of proportional representation. However PR with other scoring rules have been considered theoretically. Potthoff and Brams (1998), for example, suggested combining PR and approval voting scores. Van der Hoot et al (2006) define "a scoring seat allocation rule" (Definition 9) which is a similar to our parliament choosing rule but less general as it does not admit PR with a threshold.

We have defined normalised positional scores and parliament choosing rules for societies of arbitrary cardinalities. This is a natural requirement since the number of voters is constantly changing. In these circumstances parliament choosing rules must be applicable no matter how large or small the number of voters n is.

Note that there is a significant difference between parliament choosing rules and choose-k rules (see Brams and Fishburn (2002) and the references therein). A choose-k rule picks a k-element subset of the set of alternatives, which is clearly inappropriate in our context when the parties and not the candidates are the alternatives. A parliament choosing rule reveals not only which parties win parliamentary seats, but also how many seats each of them wins.

At the end of this section we consider how a voter may influence the outcome of the election in terms of scores. Here we consider the three -party elections only. Let us consider a voter with preference $A > B > C$ and suppose that, if she votes sincerely, the outcome of the election — in terms of scores — will correspond to the point X of hexagon of scores, shown on Figure 3.

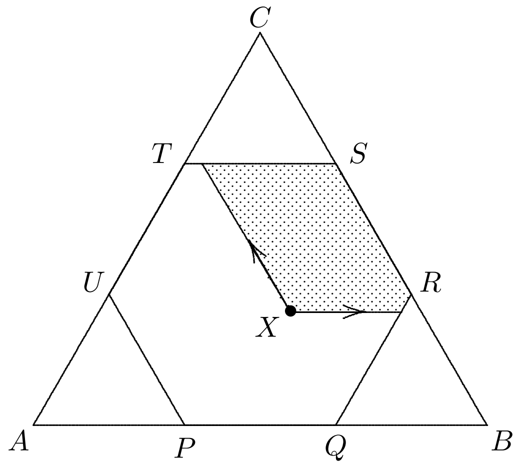

Figure 3: Possible directions of change under a manipulation attempt

Irrespective of the positional scoring rule, by voting insincerely she cannot improve the score of A, nor worsen the score of C. If she votes insincerely, she will expect the vector of scores to fall in the shaded area. By insincerely reporting her preferences to be $B > A > C$, she will move the vector of scores horizontally east. This she can do so long as the score function is not antiplurality. By insincerely reporting $A > C > B$, she moves the vector of scores north west, parallel to BC, and this is possible except in the event the score function is plurality.

3 The Government Formation Game and Voting Power Indices

Choosing a parliament is effectively a fair division problem. It might be thought desirable to allocate each political party a quantity of seats in direct proportion to its support in society. Suppose we do desire this, and suppose we accept that the support for a party can be measured by the score it is assigned by a score function at an election: then PR is an obvious choice for a parliament choosing rule.

But does PR provide a satisfactory solution to the fair division problem? For sure, each party gets a (roughly) "fair" share of parliamentary representation. However, once the election is over, a government has to be formed and a coalition arrangement may need to be negotiated. The political power of each player in the government formation game may not be proportional to either its score or its parliamentary representation. PR can divide seats up "fairly" but it is unlikely to divide power up "fairly." This may be a potential source of manipulative behaviour of some voters.

Later we will assume that some voters may be interested in the post-election distribution of parliamentary power. For this reason we need some tools for evaluating distributions of power. Indices of voting power are an obvious first choice for this purpose (Felsenthal and Machover, 1998). We will assume that the distribution of voting power in a parliament can be computed by a (normalised) *voting power index*. Given a parliament $\mathbf{x} = (x_1, \ldots, x_m)$, a voting power index P computes a vector of voting powers $P(\mathbf{x}) = (p_1, \ldots, p_m)$, where p_i denotes the proportion of voting power held by party a_i.

Before formally defining a voting power index we need the following standard definitions. A *weighted voting game* is a simple m-person game characterised by a non-negative real vector $(w_1, \ldots, w_m)$, where w_i represents the ith player's voting weight, and a quota q. The quota gives the minimum number of votes necessary to form a winning coalition. A coalition C is winning if $\sum_{i \in C} w_i > q$. Given a parliament $\mathbf{x} = (x_1, \ldots, x_m)$, the formation of the government is a weighted voting game with weights $x_1, \ldots, x_m$ and quota $\frac{1}{2}$ (we will assume throughout that any strict majority of votes is sufficient to pass any motion in parliament), i.e. the players are the parties and their weights are the proportion of parliamentary seats that they hold.

Let $M = \{1, 2, \ldots, m\}$ and let $v = (M, W)$ be a simple m-person game with $W \subseteq 2^M$ being the set of all winning coalitions. A coalition C is called a *minimal winning coalition* if $C \in W$ and $C \setminus \{i\} \notin W$ for all $i \in C$. A party is called a *dummy* if it does not belong to any minimal winning coalition.

Definition 4. *Any mapping $P \colon S^{m-1} \to S^{m-1}$ is called a voting power index if the following conditions hold. Let $\mathbf{x}$ be a parliament and suppose $P(\mathbf{x}) = (p_1(\mathbf{x}), \ldots, p_m(\mathbf{x}))$, then*

PI1. If the ith party is a dummy in parliament $\mathbf{x}$, then $p_i(\mathbf{x}) = 0$,

PI2. If the set of minimal winning coalitions of parliament $\mathbf{x}$ is the same as the set of minimal winning coalitions of the parliament $\mathbf{y}$, then $P(\mathbf{x}) = P(\mathbf{y})$.

This definition follows Holler and Packel's definition of a power index for games (Holler and Packel, 1983). Allingham (1975) requires also a monotonicity condition. However the Deegan-Packel (1979) index and the Public Good Index (Holler and Packel, 1983) do not satisfy the monotonicity requirement and we do not postulate it. Non-monotonic indices have their justification in Riker's "size principle" (Riker, 1962, p.47), which says that "... participants create coalitions just as large as they believe will ensure winning and no larger."

Perhaps the best known voting power indices are the Banzhaf (Bz) and Shapley-Shubik (S-S) indices (Banzhaf, 1965; Brams, 1975; Shapley and Shubik (1954). These indices count, in different ways, how many times a player is critical for some winning coalition. According to Felsenthal and Machover (1998, p.9), these two indices "have, by and large, been accepted as valid measures of a priori voting power. Some authors have a preference for one or another of these two indices; many regard them as equally valid. Although other indices have been proposed — ... — none has achieved anything like general recognition as a valid index."

It is worth pointing out that more seats do not necessarily translate into more power. For instance, compare the parliaments $(x_1, x_2, x_3) = (0.98, 0.01, 0.01)$ and $(y_1, y_2, y_3) = (0.51, 0.48, 0.01)$; party B has no more power in the second than in the first.

4 Manipulability of Parliament Choosing Rules. What does the GS Theorem Imply?

Since a parliament choosing rule F associates, with every profile R, a parliament $F(R) \in S^{m-1}$, to be strategic a voter must be able to compare any two parliaments from S^{m-1}. Such a voter will have to have an order $\succsim$ on S^{m-1} and we require that this order is consistent with the voters order on $\mathcal{A}$. As usual, the strict preference relation of $\succsim$ will be denoted by $\succ$ and the indifference by $\sim$. To explain what consistency means we define vectors $\mathbf{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ whose only nonzero coordinate is a 1 positioned in the jth place. Such a vector corresponds to the parliament where all seats are occupied by members of party a_j.

Definition 5. *Let L be a voter's linear order on the set $\mathcal{A}$ of political parties:*

$$a_{i_1} > a_{i_2} > \ldots > a_{i_m}.$$

We say that an order $\succsim$ on S^{m-1} is consistent with L if

$$\mathbf{e}_{i_1} \succ \mathbf{e}_{i_2} \succ \ldots \succ \mathbf{e}_{i_m},$$

i.e. this voter prefers the parliament where all the seats belongs to a_{i_1} to the parliament where all the seats belong to a_{i_2}, etc.

The voter's order $\succsim$ on parliaments contains more information about her preferences than the order that she submits in the election. However it is not possible to elicit this order at the polling booth. We will refer to $\succsim$ as to the *type* of this voter. Types will be combined into *classes* of voters which are arbitrary sets of types. Two most important classes of voters will be the class of seat maximisers and the class of power maximisers. They will be defined later.

Definition 6 (Manipulability)**.** *Let R be a profile such that $R_{i_1} = \ldots = R_{i_k} = L$ for some group of indices $I = \{i_1, \ldots, i_k\}$ and linear order L on $\mathcal{A}$. A parliament choosing rule F is said to be manipulable at R by a group of k voters of type $\succsim$, if $\succsim$ is consistent with L and there exists a linear order L' on $\mathcal{A}$ such that for the profile R', which results when $R_{i_1}, \ldots, R_{i_k}$ in R are replaced with L',*

$$F(R') \succ F(R).$$

The rule F is said to be manipulable if for every $n > 1$ there exists a profile $R \in \mathcal{L}(\mathcal{A})^n$ which is manipulable by a group of voters of a certain type.

What the above definition says is that if R is a profile of sincere preferences, voters in positions $i_1, \ldots, i_k$ happen to be of type $\succsim$ and they may vote insincerely by submitting the linear order L', ceteris paribus, and all be better off, then this will render the profile R manipulable.

Definition 7 (Micro-manipulability)**.** *Let F be a manipulable parliament choosing rule. Let k_n be the smallest number for which there exists a profile in $\mathcal{L}(\mathcal{A})^n$ which is manipulable by a group of k_n voters. The rule F is said to be micro-manipulable if the ratio k_n/n tends to 0.*

Micro-manipulability of the rule, for which the size of society is not fixed, is the analogue of individually manipulable rule for the societies of a fixed size. This term was coined by Donald Saari (1994) and we refer the reader to him for more justification of the concept. Roughly speaking, F is micro-manipulable if, as $n \to \infty$, the manipulating group may consist of an arbitrary small fraction of the society. And although, to the best of our knowledge, there are no rigorous results to this extent, it usually happens that if F is micro-manipulable, then for all $n > 1$ there exists an n-profile which is manipulable by a single voter. We will illustrate this with an example in the next section.

Example 3 (Daft voter)**.** *A daft voter's only concern is which party has a majority of seats in the parliament. Thus she will be indifferent between any two parliaments where the same party has a majority of seats. Suppose now that in a parliament* $\mathbf{x}$ *parties* $a_{i_1}, \ldots, a_{i_k}$ *hold an equal number of seats, while every other party has a lesser number. Suppose* $i_1 < \ldots < i_k$*. Then we assume that* $\mathbf{x} \sim \mathbf{e}_{i_1}$*. The order* $\succsim$ *on* S^{m-1} *of a daft voter will have only* m *indifference classes:* $[\mathbf{e}_1], \ldots, [\mathbf{e}_m]$*, where* $[\mathbf{z}]$ *is the indifference class containing* $\mathbf{z}$*. For example, for* $m = 3$ *they will be as shown on Figure 4, with segments* KO *and* OM *belonging to* $[\mathbf{e}_1]$ *and the segment* LO *without point* O *belonging to* $[\mathbf{e}_2]$*.*

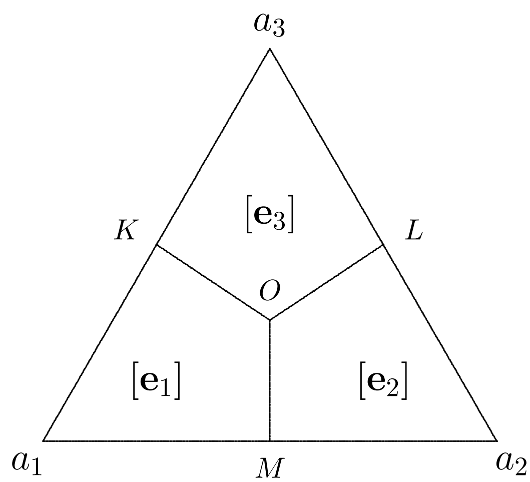

Figure 4: *Equivalence classes in* S^2 *for a daft voter*

We have introduced daft voters not because this is a realistic model but to spell out clearly the implications of the GS theorem for PR. There have been several informal attempts to relate it to PR (see, for example, Cox (1997, p.11) but not a formal one. Here we show what can be deduced from the GS theorem formally.

Theorem 1. *Let* $m \geq 3$ *and* F *be a parliament choosing rule. Then, for all* $n > 1$*, at a certain* n*-profile* F *can be manipulated by a single daft voter.*

Proof. Let us restrict our parliament choosing rules to profiles from $\mathcal{L}(\mathcal{A})^n$. Let $F\colon \mathcal{L}(\mathcal{A})^n \to S^{m-1}$ be a parliament choosing rule and let $\pi\colon S^{m-1} \to \mathcal{A}$ be the mapping that maps any parliament to the party which has most seats in it. If in parliament $\mathbf{x}$ there are several parties $a_{i_1}, \ldots, a_{i_k}$, with $i_1 < \ldots < i_k$, which have an equal number of seats and all other parties have less seats, then we set $\pi(\mathbf{x}) = a_{i_1}$. Let us consider the composition $f = \pi \circ F$. This composition is a social choice function $f\colon \mathcal{L}(\mathcal{A}) \to \mathcal{A}$. Since it is obviously non-dicatatorial and can take all $m \geq 3$ values, the GS theorem is applicable to it. There therefore exists an n-profile at which f is manipulable by a single voter, say the ith voter. This implies that if the ith voter is daft she can manipulate F at the same profile. $\square$

A daft voter is not a realistic model of a real voter. A daft voter is indifferent between the parliaments $(1,0,0)$ and $(1/3,1/3,1/3)$ which is absurd. Surely real voters are more sophisticated than that. More realistic models will be considered in subsequent sections.

When we consider manipulability by voters of a certain class, we have two grades of manipulability, one is stronger than another.

Definition 8 (Weak and Strong Manipulability)**.** *Let R be a profile such that $R_{i_1} = \ldots = R_{i_k} = L$ for some group of indices $I = \{i_1, \ldots, i_k\}$ and linear order L on $\mathcal{A}$. A parliament choosing rule F is said to be weakly manipulable at R by voters of class $\mathcal{C}$ if there exists a linear order L' on $\mathcal{A}$ and a type $\succsim$ of voters, belonging to $\mathcal{C}$, such that for the profile R', which results when $R_{i_1}, \ldots, R_{i_k}$ in R are replaced with L',*

$$F(R') \succ F(R). \tag{2}$$

F is said to be strongly manipulable at R by voters of class $\mathcal{C}$ if (2) holds for every type in $\mathcal{C}$.

The difference between the two concepts of manipulability is as follows. As we will see later, it is possible, for example, that a parliament choosing rule can be manipulated at a certain profile, but only by uncertainty averse voters. It is also possible that a certain profile can be manipulated only by uncertainty seekers. A parliament choosing rule is strongly manipulable at a particular profile only if all voters having a particular preference order on $\mathcal{A}$ have identical incentives to vote insincerely.

5 Strategic Opportunities for Seat Maximising Voters

In this section we consider the behaviour of voters who are concerned about the post-election distribution of parliamentary seats.

Example 4 (Weighted seat maximiser)**.** *This voter has a vector of utilities $\mathbf{u} = (u_1, \ldots, u_m)$, where u_i is the utility of one seat in the parliament that is held by the ith party a_i. Given the parliament $\mathbf{x} = (x_1, \ldots, x_m)$ the total utility of $\mathbf{x}$ for this voter will be*

$$u(\mathbf{x}) = \mathbf{u} \cdot \mathbf{x} = x_1 u_1 + x_2 u_2 + \ldots + x_m u_m.$$

A voter of this type has $\mathbf{x} \succeq \mathbf{y}$ if and only if $u(\mathbf{x}) \geq u(\mathbf{y})$.

Example 5 (Lexicographic seat maximiser)**.** *A voter that we describe in this example has lexicographic preference order $\succsim$ over S^{m-1}, that is, she prefers parliament $\mathbf{x} = (x_1, \ldots, x_m)$ to parliament $\mathbf{y} = (y_1, \ldots, y_m)$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$, etc.*

We call voters of these two classes *seat maximisers.* Let us show that in the absence of threshold the Bowler and Lanoue claim is true for seat maximising voters.

Theorem 2. *For pure PR a profile cannot be manipulated by a group of seat maximising voters.*

Proof. Consider a group of voters $v_{i_1}, \ldots, v_{i_k}$ with common utility vector $\mathbf{u} = (u_1, \ldots, u_m)$. Let $I = \{i_1, \ldots, i_k\}$. Assume without loss of generality that

$$L : a_1 > a_2 > \ldots > a_m, \quad L' : a_{j_1} > a_{j_2} > \ldots > a_{j_m}.$$

be, respectively, these voters' sincere and insincere preferences with R and R' be the corresponding profiles before and after the manipulation attempt when these voters change their preferences from L to L'. Due to our assumption about L we have $u_1 > u_2 > \ldots u_m$. For lexicographic seat maximisers the statement is obvious so let us consider weighted seat maximisers. Let $\mathbf{w} = (w_1, \ldots, w_m)$ be the vectors of weights used to define the score function for this rule. Let also $\mathbf{x} = F_s(R) = \mathbf{sc_w}(R)$ and $\mathbf{x}' = F_s(R') = \mathbf{sc_w}(R')$ be the corresponding parliaments. Since $\sum_{i=1}^{m} sc_{\mathbf{w}}(R, a_i) = \sum_{i=1}^{m} sc_{\mathbf{w}}(R', a_i)$, let us define their common value as S. Let $i \in I$. Then the change in the ith voter's utility will be:

$$u(\mathbf{x}) - u(\mathbf{x}') = \frac{1}{S}\left((u_1 w_1 + u_2 w_2 + \ldots + u_m w_m) - (u_{i_1} w_1 + u_{i_2} w_2 + \ldots + u_{i_m} w_m)\right).$$

Since $u_1 > u_2 > \ldots u_m$ and $u_1 > u_2 > \ldots u_m$, this will be positive by the rearrangement inequality (Hardy, Littlewood and Pólya, 1952, p 261). Therefore this group would be better off voting sincerely. $\square$

This looks very much like Bowler and Lanoue claim and may explain their assumptions on voting behaviour of individuals. The stark contrast between the two previous theorems is partly to blame for the existing confusion in the literature about manipulability of PR.

However, when PR with a threshold is used weighted seat maximisers can sometimes manipulate.

Theorem 3. *Let a parliament choosing rule F be PR with a threshold. Then the rule is micro-manipulable by seat maximising voters but never strongly.*

Proof. Firstly, we will assume that the scoring rule is not antiplurality, i.e., $\mathbf{w} \neq \mathbf{a}$. Since the vectors of scores are everywhere dense in the hexagon of scores, we may assume that all points of the hexagon are in fact scores. Let ϵ be the threshold and F_a be the corresponding seat allocation rule. Suppose that the voter's ranking of alternatives is $A > B > C$ and $\mathbf{u} = (u_1, u_2, u_3)$ is his vector of utilities. By Theorem 2 we can be sure that any successful manipulation can be achieved only by crossing the threshold. Let $\mathbf{x} = (x, \epsilon, z)$ be a vector of scores such that $x > \epsilon$ and $z > \epsilon$. Then $F_a(\mathbf{x}) = \mathbf{x}$ and the utility of this parliament $\mathbf{x}$ for the voter is $u(\mathbf{x}) = u_1 x + u_2 \epsilon$. Let us consider the parliament $\mathbf{y} = \left(\frac{x}{1-\epsilon}, 0, \frac{z}{1-\epsilon}\right)$. Then the utility of parliament $\mathbf{y}$ will be $u(\mathbf{y}) = \frac{x}{1-\epsilon} u_1$. Easy calculations show that for $u_2 > \frac{x}{1-\epsilon} u_1$ we have $u(\mathbf{x}) > u(\mathbf{y})$. Indeed, in this case $u(\mathbf{x}) = u_1 x + u_2 \epsilon > u_1 x + \epsilon \frac{x}{1-\epsilon} u_1 = \frac{x}{1-\epsilon} u_1 = u(\mathbf{y})$.

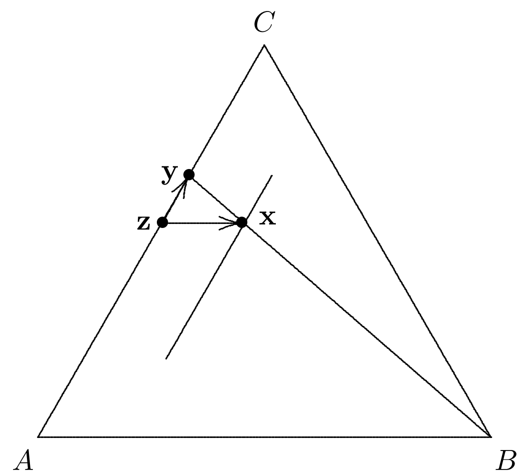

Figure 5: Manipulating move, when the scoring rule is not antiplurality.

Let us consider $z \in [0, 1]$ such that $\mathbf{x}$ and $\mathbf{z} = (1 - z, 0, z)$ lie on a horizontal line. When $\mathbf{z}$ approaches $\mathbf{x}$ along this line, $F_a(\mathbf{z}) \to \mathbf{y}$. This shows that we can choose a point $\mathbf{z}'$ arbitrary close to $\mathbf{x}$ moving from which to $\mathbf{x}$ will give us a jump in utility. If a small group of voters has preferences $A > B > C$ and utilities specified above, it will be beneficial for them to cross the threshold moving eastward. To make such a horizontal move they have to submit preferences $B > A > C$. By doing this they will move the score eastward unless the scoring rule is antiplurality. Thus F is micro-manipulable in this case. However, if $u_2 \leq \dfrac{x}{1-\epsilon} u_1$, this will not be a successful manipulation, hence the profile is not strongly manipulable. Similar considerations apply when crossing the threshold in the other direction.

If $\mathbf{w} = \mathbf{a}$, i.e., the rule is antiplurality, then the only possible direction of change is north-west. It appears that in this case, when the utility u_2 is small relative to u_1 it is beneficial to cross the threshold in the other direction as shown on Figure 6, depriving your second best party of the votes and forcing it out of parliament but getting more seats for your most preferred party. This can be done since $\dfrac{x}{1-\epsilon} u_1 > x u_1$.

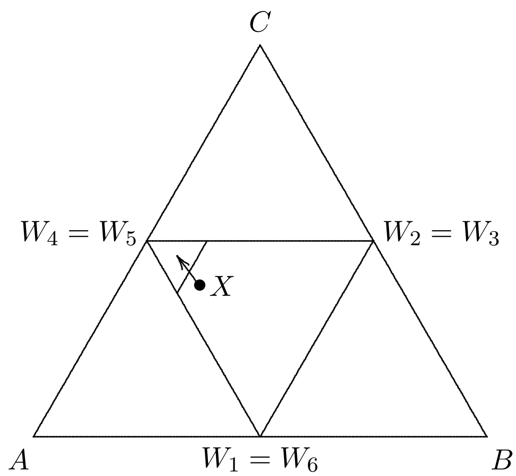

Figure 6: Manipulating move for antiplurality.

□

As we mentioned above, if a rule is micro-manipulable, it is usually possible to find a profile which is manipulable by a single voter. We give an example.

Example 6. *Assume 100 voters participate in an election. Suppose F is PR with a five percent threshold, and suppose the scoring rule is plurality. Suppose a weighted seat maximising voter has preferences $A > B > C$ and utility vector $\mathbf{u} = (10, 8, 0)$. Suppose that in the event all voters report truthfully the outcome of the election in terms of scores will be $\mathbf{sc}(R) = (0.48, 0.04, 0.48)$, which results in the outcome in terms of parliaments $F(R) = (0.5, 0, 0.5)$ with $u(F(R)) = 5$. If the voter submits $B > A > C$, then the vector of scores will be $\mathbf{sc}(R') = (0.47, 0.05, 0.48)$ and the parliament will be $F(R') = (0.47, 0.05, 0.48)$. The utility of the latter will then be $u(F(R')) = 4.7 + 0.4 = 5.1$, which is greater than that achievable with a sincere vote.*

6 Power Maximising Voters. Attitude Towards Uncertainty

We now model a voter whose primary concern is with the distribution of power in the post-election parliament.

Example 7 (Power maximiser)**.** *We assume that such a voter has a certain power index P in mind, which she uses to measure powers of parties and also a vector of utilities $\mathbf{u} = (u_1, \ldots, u_m)$, normalised so that $\min_j u_j = 0$. These utilities are ordered according to the preferences of this voter over $\mathcal{A}$, that is, this voter prefers a_i to a_j if and only if $u_i > u_j$. Given a parliament $\mathbf{x} = (x_1, \ldots, x_m)$ its total utility for this voter will be*

$$u(\mathbf{x}) = P(\mathbf{x}) \cdot \mathbf{u},$$

where $\cdot$ is the dot product in $\mathbb{R}^m$. Her preference order $\succsim$ on S^{m-1} is defined so that for two parliaments $\mathbf{x}, \mathbf{y}$

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow u(\mathbf{x}) \geq u(\mathbf{y}).$$

We will denote this type of voters as $\succsim_{P,\mathbf{u}}$. These types form the class $\mathcal{P}$ of power maximising voters.

Some power maximising voters may be further categorised according to their attitude to uncertainty. Fix a voter i, and set U_r equal to this voter's rth largest utility. We emphasise that since we assume that voters submit strict linear orders as their preferences, all utilities $u_1, \ldots, u_m$ are different, hence U_r is unambiguously defined and $U_1, \ldots, U_m$ is a permutation of $u_1, \ldots, u_m$ with $U_1 > U_2 > \ldots > U_m = 0$. We will say that the voter is *uncertainty averse* if

$$U_2 - U_1 \geq U_3 - U_2 \geq \ldots \geq U_m - U_{m-1}.$$

and *uncertainty seeking* if

$$U_2 - U_1 \leq U_3 - U_2 \leq \ldots \leq U_m - U_{m-1}.$$

For $m = 3$ a voter is either uncertainty averse or uncertainty seeking, but for $m > 3$ this is not the case. Here we concentrate on the case when $m = 3$; in this case a voter is uncertainty averse if $U_2 \geq \frac{1}{2}(U_1 + U_3)$ and uncertainty seeking if $U_2 \leq \frac{1}{2}(U_1 + U_3)$.

Example 8. *Consider the case where $m = 3$, and a power maximising voter with index of voting power P prefers A to B to C. In this case $U_i = u_i$ for $i = 1, 2, 3$. Suppose this voter is comparing two parliaments $\mathbf{x}$ and $\mathbf{y}$ with respective vectors of power indices $\mathbf{p} = P(\mathbf{x}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbf{q} = P(\mathbf{y}) = (0, 1, 0)$. As we have $u(\mathbf{x}) = \dfrac{1}{3}(U_1 + U_2 + U_3)$ and $u(\mathbf{y}) = U_2$ we have $\mathbf{x} \succsim \mathbf{y}$ if the voter is uncertainty seeking and $\mathbf{y} \succsim \mathbf{x}$ if she is uncertainty averse.*

In this example, the vector $\mathbf{p}$ corresponds to a post-election situation where none of the three parties has an outright majority, and a coalition government will need to be formed. If a voter anticipates, prior to the election, that $\mathbf{p}$ will be the outcome, then she may be uncertain about the composition of the next government. The vector $\mathbf{q}$ corresponds to a post-election situation where party B has total power, and can form a government by itself. A voter of the opinion that $\mathbf{q}$ will be the outcome of the election will have no doubt as to the composition of the next government. Thus the voter will rank $\mathbf{x}$ over $\mathbf{y}$ if she is uncertainty seeking, or $\mathbf{y}$ over $\mathbf{x}$ if she is uncertainty averse.

Let $P \colon S^{m-1} \to S^{m-1}$ be an index of voting power. Then the image $\operatorname{Im} P = P(S^{m-1})$ of P is the set of all possible vectors of voting power that might emerge when P is used to measure the distribution of power. This set is finite. We illustrate this in the proposition below for the case $m = 3$.

Proposition 1. *Let the three parties be A, B, C. Let us split S^2 into four equilateral triangles as shown below (K, L, M are the midpoints of the respective sides):*

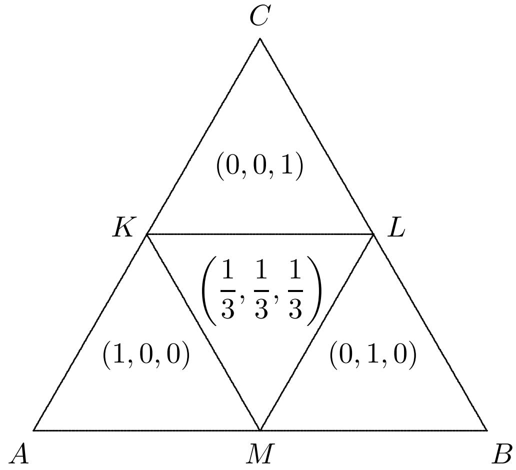

Figure 7: *Possible vectors of power indices.*

Then, irrespective of which power index P is used, the vectors of power indices inside those triangles are shown on Figure 7.

Proof. Irrespective of the choice of P, whenever the parliament $\mathbf{x}$ falls strictly inside one of the triangles AKM, BML, or CLK, then $P(\mathbf{x})$, will be $(1,0,0)$, $(0,1,0)$, or $(0,0,1)$, respectively, since two parties in this parliament will be dummies (condition PI1). Should the parliament $\mathbf{x}$ fall inside the inner triangle, then $P(\mathbf{x})$ will be $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ due to the symmetry (condition PI2). $\square$

If the parliament coincides with one of the vertices of the inner triangle M, K, and L, the vector of indices, regardless of the index, will be $(\frac{1}{2}, \frac{1}{2}, 0)$ or a permutation thereof. Should the parliament fall on the perimeter of the inner triangle (excluding points M, K, and L) the vector of power indices will depend on the index of voting power used by the voter. For example, the vector $P(\mathbf{x})$ of a voter who uses Banzhaf power index will be either $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ or some permutation thereof, and the vector $P(\mathbf{x})$ of a voter who uses Shapley-Shubik power index will be $(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$ or, again, some permutation thereof.

Example 9. *Consider again the case where $m = 3$ and a voter prefers A to B to C (denote this linear order by L). Then for the four parliaments $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{m}$, located on Figure 7 inside the triangles AKM, BML, CLK, KLM, respectively, all power maximisers whose type is consistent with L will prefer $\mathbf{x}$ to $\mathbf{y}$ to $\mathbf{z}$ and also $\mathbf{x}$ to $\mathbf{m}$ to $\mathbf{z}$. However, some of them will prefer $\mathbf{y}$ to $\mathbf{m}$ and some will prefer $\mathbf{m}$ to $\mathbf{y}$, depending on their attitude towards uncertainty.*

The difference between the two concepts of manipulability - weak and strong - in the class $\mathcal{P}$ is as follows. It is possible that a parliament choosing rule can be manipulated at a certain profile, but only by uncertainty averse voters. It is also possible that a certain profile can be manipulated only by uncertainty seekers. A parliament choosing rule is strongly manipulable in $\mathcal{P}$ at a particular profile only if all power maximising voters with some particular preference order on $\mathcal{A}$ have identical incentives to vote insincerely.

Theorem 4. *Let the parliament choosing rule F be pure PR with the normalised positional score function given by the vector of weights $\mathbf{w}$. Then the rule is always micro-manipulable by voters in $\mathcal{P}$ but never strongly. Moreover,*

1. *If* $\mathbf{w} = \mathbf{a}$*, i.e. for the antiplurality score, the rule is not manipulable by uncertainty averse voters.*

2. *If* $\mathbf{w} = \mathbf{p}$*, i.e. for the plurality score, the rule is not manipulable by uncertainty seeking voters.*

Proof. Suppose voters with preferences $A > B > C$ are comparing the outcome that would transpire if they vote truthfully with that that would arise if they voted untruthfully. They can move the vector of the scores in directions shown on Figure 3. Moving in those directions they cannot escape from the region inside KCL on Figure 8.

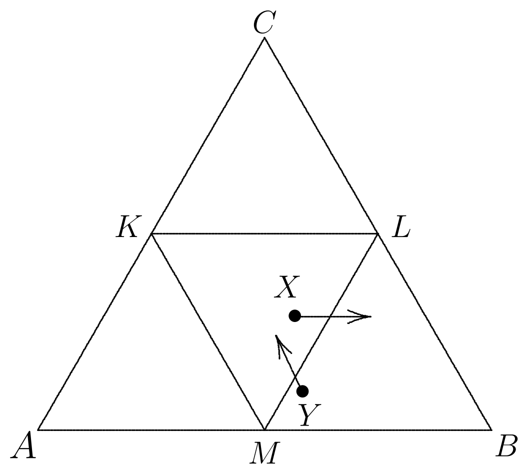

Figure 8: Weak manipulation under pure PR.

They would not wish to escape into KCL since the vector of power indices there is $(0, 0, 1)$ which has the lowest possible utility. Nor would they wish to escape out of AKM, where the vector of power indices is $(1, 0, 0)$ which gives them the highest possible utility. But if they were uncertainty averse, they would seek, by voting strategically, to move the expected vector of scores X from inside MKL, where the vector of power indices is $(1/3, 1/3, 1/3)$, or from on segment ML, to inside MLB, where power indices will be $(0, 1, 0)$. If they were uncertainty seeking, they would be keen to move the expected vector of scores Y the other way. In either case, if the vector of scores they expect to transpire if they vote sincerely is "close" enough to ML, and if the score function permits, an incentive to manipulate exists. It is not true, however, that all voters, whose type is consistent with $A > B > C$, will have an incentive to manipulate in the same fashion, hence the manipulative opportunities are not strong. $\square$

It is interesting to note that if a group of voters with preference $A > B > C$ expect that if they all vote sincerely the vector of scores will lie "in the vicinity of ML.", the uncertainty averse and uncertainty seeking members of this group would then attempt to manipulative against each other, even though they have identical preferences on the set of parties.

We now show that the introduction of a threshold creates opportunities for strong manipulation.

Theorem 5. *Let the parliament choosing rule F be proportional representation rule with a threshold with the normalised positional score function given by the vector of weights $\mathbf{w}$. Then the rule is strongly micro-manipulable by power maximisers iff $\mathbf{w} \neq \mathbf{a}$.*

Proof. Let us again use S^2 to represents vectors of normalised positional scores $\mathbf{sc_w}$. The introduction of a threshold changes the shape of the regions in which the vector of power indices, associated with the normalised score, is constant. The new regions are shown on Figure 9. The central region, in which the vector of power indices is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, becomes a hexagon. The three regions in which the vector of power indices is respectively equal to $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are no longer convex.

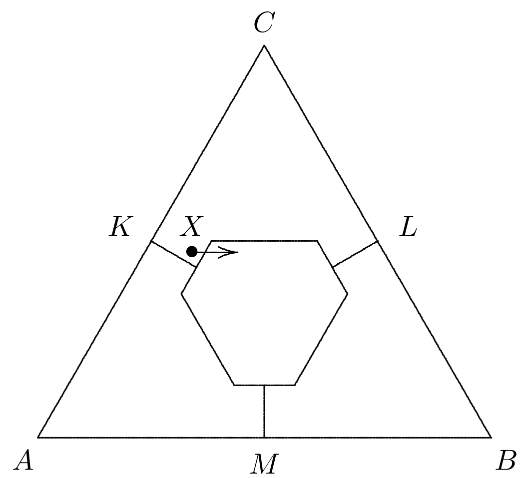

Figure 9: PR with a threshold. Possibility of strong manipulation

Suppose that a small group of voters with preference $A > B > C$ believe that if they vote sincerely the resulting normalised score will correspond to the point X (We note that X lies inside the triangle KLM and therefore it can be approximated by normalised positional scores for any vector $\mathbf{w}$). At this point, B does not score highly enough to overcome the threshold. If at the election this group insincerely state their preferences to be $B > A > C$, they may be able (so long as the score function is not antiplurality) to push B over the threshold, and move the expected vector of scores inside the hexagon. When this group votes truthfully, the vector of voting power is anticipated to be $(0, 0, 1)$. Untruthful voting could bring about the vector of voting power $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This is an unambiguously better prospect for all voters with preference $A > B > C$, regardless of their vector of utilities: hence the introduction of a threshold can create opportunities for strong manipulation. $\square$

7 Overshooting

For PR with a threshold manipulating voters can sometimes be in danger of overshooting and sometimes in danger of undershooting even if it is a micro-manipulation. More precisely, we will show that uncertainty seeking voters may undershoot and uncertainty averse voters may overshoot. For example, undershooting may occur with the sincere vector of scores X and overshooting may occur with the sincere vector of scores Y (see Figure 10).

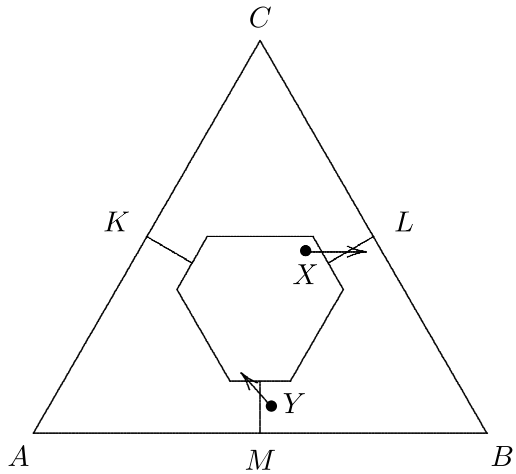

Figure 10: PR with a threshold. Overshooting and undershooting

We illustrate this with an example of undershooting roughly corresponding to the first situation.

Example 10. *Suppose the scoring rule is plurality and the threshold is 6%. Suppose in total we have 100 voters and the vector of scores corresponding to the parliament X is* $\frac{1}{100}(6, 46, 48)$ *with the parliament being the same as the score. If three voters who voted $A > B > C$ will vote $B > A > C$, then the resulting score will be* $\frac{1}{100}(3, 49, 48)$ *and the resulting parliament will be* $(0, 50.52, 49.48)$ *with the vector of power indices* $(0, 1, 0)$*. However if only one voter will change her preferences in this way, the resulting score will be* $\frac{1}{100}(5, 47, 48)$ *with the parliament* $(0, 49.47, 50.53)$ *and the vector of power indices* $(0, 0, 1)$*.*

From the first glance it might seem that in the absense of a threshold manipulation can be more difficult for uncertainty seeking voters than for uncertainty averse ones. It looks like the former might be in danger of overshooting and need some degree of coordination. In Figure 11 we see how an alleged micro manipulation attempt of voters with preferences $A > B > C$ can have a disastrous consequences if too many uncertainty seeking voters try to escape the region MLB into the region KLM reporting preferences $A > C > B$. It looks that they might end up in the region KLC and be much worse off compared to the sincere voting outcome (see Figure 11).

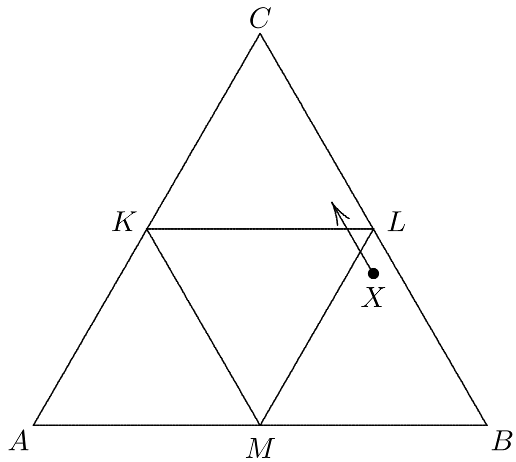

Figure 11: Pure PR. Hypothetical possibility of overshooting for uncertainty seeking voters

However, in reality this cannot happen. The reason is that the point X is very close to BC which means that the score of party A is very small. This in turn means that there will not be enough voters with preferences $A > B > C$ to manipulate by changing their reported preferences to $A > C > B$. This argument is easy to make rigorous, we leave it to the interested reader. Since the aforementioned move is the only suspect for overshooting in micro-manipulation attempts, we conclude that for pure PR no such overshooting is possible.

In the next section we will see that overshooting becomes a much more serious issue in the presence of rounding.

8 The 2005 New Zealand General Election

The NZ electoral system is mixed member proportional, with a 5% threshold. Voters have two votes - an electoral (district) vote, and a party vote. A first-past-the-post election is run in 69 electorates, with the winner of each electorate becoming an MP. Party votes are tallied nationally. The Saint-Lague formula is then applied to the party votes to determine how many seats in total each party is entitled to in the 120 seat parliament. If a party neither wins an electorate nor more than 5% of the party vote then it is excluded from consideration. If a party has fewer electorate wins than places in the highest 120 Saint-Lague quotients then its parliamentary representation is topped-up accordingly from the party list.[1]

The most recent New Zealand general election took place on September 17th 2005. The main political parties participating in the election were (in alphabetical order): ACT, Green Party, Labour, Maori Party, National, NZ First, Progressive, United Future.

Labour, Greens and Progressive can be considered as left parties; National, ACT and to some extent United Future are on the right side of the political spectrum.

[1]Sometimes a party wins more electoral seats than it is entitled to by its party vote, this creates an overhung parliament with more than 120 seats. In particular, this happened in 2005 when parliament with 121 seats resulted.

At the election 28.71% of voters gave their electorate vote and their party vote to different parties (New Zealand Election Results, `http://www.electionresults.govt.nz/`) (down from 39.04% in 2002). Many of these voters may have split because their first choice did not stand a candidate in their electorate. But the 28.71% figure is high enough to suggest a reasonable amount of insincere voting went on. In particular, anecdotal evidence (reports to the authors) has suggested that some voters with preferences

$$\text{Labour} > \text{Greens} > \ldots$$

may have cast their party vote for the Greens in attempt to obtain a more favourable for them power structure in the future parliament. We use our model of a power maximising voter to explain why.

The two opinion polls closest to the election gave the following results:

Poll	Date	Labour	National	NZ First	Greens
TVNZ Colmar Brunton	15 September	38%	41%	5.5%	5.1%
Herald Digipoll	16 September	44.6%	37.4%	4.5%	4.6%

Table 1: Party vote for the last two opinion polls

Results of previous polls are available on Wikipedia (cited August, 7, 2007). The Green Party were not expected to win an electorate seat, and NZ First were expected to win at most one. As it turned out, neither party won an electorate seat. Since Labour and Greens were not able to form a government, the strange-bedfellow phenomenon (Brams, Jones and Kilgour, 2002) occurred and Labour formed the government with NZ First, United Future and Progressive Party. Table 2, below, shows the actual election result. Also shown is what would have transpired had 0.4% of the electorate not given their party vote to the Greens and given them Labour, ceteris paribus.

	Actual			Hypothesised		
Party	Party Vote	Seats	SS	Party Vote	Seats	SS
Labour	41.10	50	0.324	41.50	54	0.414
National	39.10	48	0.262	39.10	50	0.214
NZ First	5.72	7	0.143	5.72	7	0.214
Green Party	5.30	6	0.110	4.90	0	0.0
Maori Party	2.12	4	0.076	2.12	4	0.081
United Future	2.67	3	0.043	2.67	3	0.048
ACT	1.51	2	0.029	1.51	2	0.014
Progressive	1.16	1	0.014	1.16	1	0.014

Table 2: Actual and hypothesised results of the NZ 2005 general election

Election results were obtained from `http://electionresults.govt.nz`. Alternative election scenarios can be investigated at `http://www.elections.org.nz/mmp.html`. Voting power indices were calculated at Leech, D. and Leech, R. (cited on August, 6, 2007).

Now let us make a calculation. Suppose that a group of voters at the 2005 general election behave as do voters in our model. Suppose that all members of this group were solely concerned with how Shapley-Shubik power will be distributed in the post-election parliament. Suppose that all members of this group rank the Labour party first, the Green Party second, and attribute zero or negligibly small utility to the powers of all the other parties contesting the election. Further suppose that each group member believes that the election outcome laid out in the right-hand-side of Table 2 is a distinct possibility. Such a supposition is not unreasonable, given the pre-election polls. Then members of this group may have an incentive to party vote Green. The existence and strength of such incentives will depend on each individual voter's utilities.

Define

$$\alpha_i = \frac{\mathbf{u}_i(\text{Greens})}{\mathbf{u}_i(\text{Labour})}$$

to be the ratio of utilities of the Greens and Labour calculated for the ith voter. To construct Figure 12 below we first fix the party votes obtained by all parties other than Labour and the Greens. We then allow the Greens' party vote to vary from 4.9% to 8.9% (and, necessarily, Labour's party vote to vary from 41.5% to 37.5%). For each possible Green party vote we show, on the vertical axis, the minimum value of α_i the ith voter must have in order to prefer the outcome arising from this Green party vote to the hypothesised outcome arising when the Greens secure 4.9%. For example, suppose the ith voter is comparing the outcome arising when the Greens win 5.3% of the party vote to the outcome arising when the Greens win 4.9% (i.e. he or she is comparing the parliament on the left-hand-side of Table 2 to that on the right-hand-side of Table 2). This voter prefers the former to the latter provided $\alpha_i > 0.826$.

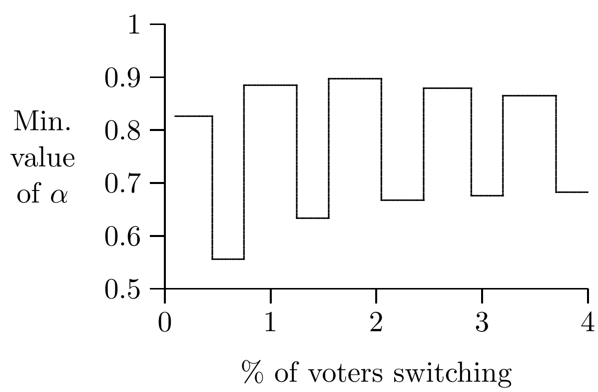

Figure 12 Graph of minimal utility ratio for which the manipulation is successful

The shape of the graph reflects the working of the Saint-Lague formula. Consider the situation, for example, when the Greens have 5.3% of the party vote (the actual election result). At this point, the 119th largest Saint-Lague quotient belongs to Labour, and the 120th largest to National. As the Greens' party vote increases (to the detriment of Labour's) past 5.35%, the Green party capture the 120th largest quotient from National. The Greens then win a 7th parliamentary seat, and National lose their 48th. As the Greens' party vote rises further, their 7th largest quotient eventually exceeds Labour's 50th largest. No seat changes hands, but the Greens then have the 119th largest quotient, and Labour the 120th largest. As the Greens' party vote increases past 5.65%, Labour's party vote decreases to the point where its 50th largest quotient falls below National's 48th largest. National's 48th seat is then restored at the expense of

Labour's 50th. The cycle then repeats itself as the Greens' party vote continues to increase.

The ith voter prefers a parliament with the Greens on between 6 and 11 seats, and with National on 47, to a parliament without the Greens, and with National on 50, provided they have $\alpha_i > 0.676$. This voter prefers a parliament with the Greens on between 6 and 11 seats, and with National on 48, to a parliament without the Greens, and with National on 50, provided that $\alpha_i > 0.897$.

The ith voter unreservedly prefers a parliament with a small number of Greens to a parliament without the Greens if she has $\alpha_i > 0.897$. Only those voters who value Green power nearly as highly as Labour power would meet this criterion. Such a voter would have a clear incentive to give their party vote to the Greens, despite their first preference being for Labour. If this voter party votes Green, she increases the likelihood that the Greens will reach parliament. If sufficiently many other group members feel and act the same way, then the Greens *will* enter parliament.

The ith voter with $0.676 < \alpha_i < 0.897$ prefers some parliaments where the Greens are present to those where the Greens are not, but not all. Such group members would not have an unambigious incentive to party vote Green unless they knew precisely how many other group members were also going to use their vote strategically. By voting strategically they would be at risk of both overshooting and undershooting.

We conjecture, then, that at the 2005 NZ general election certain voters with preference Labour $>$ Green $> \ldots$ felt they preferred the power configuration of a parliament with a small Green presence to that of a Green-less parliament, thought that polling data showed the Greens might not cross the threshold, and so party voted Green in order to increase the likelihood that the Greens would enter parliament. We do not suggest that these voters co-ordinated, nor that they had knowledge of the preferences or intentions of others beyond what was available from widely disseminated polling data. We conjecture that the Greens were polling so close to the threshold that these voters were not overly concerned about damaging Labour's prospects without improving the Greens showing (strategically undershooting). We conjecture that these voters were not worried about overshooting because they felt the proportion of the electorate that was (i) concerned about configurations of parliamentary power, (ii) had preference Labour $>$ Green $> \ldots$, and (iii) had $0.897 < \alpha$, was relatively small (in particular, less than 4.0%).

Also, It may be the case that, in practice, there are many kinds of voters - some voters may be concerned with the post-election distribution of voting power, others with the post-election distribution of seats, others with the policies to be pursued by the next government, etc. The calculations that this paper presents would obviously be too difficult for voters to do. However, the data presented shows that, acting on the intuitive level, they came very close to doing just that.

9 Conclusion

This paper has presented two new models of voter behaviour under methods of proportional representation: seat maximising and power maximising voters. We showed that seat maximising voters have no incentives to manipulate under pure PR but have such incentives under PR with a threshold. We showed that if voters are mindful of how voting power will be distributed in the post-election parliament, then incentives to vote insincerely will exist under any method of PR. We showed that attitudes to uncertainty may influence their incentives to vote insincerely. We demonstrated that introducing a threshold could encourage greater numbers of voters to vote strategically in the same manner and the incentives become stronger. We show that rounding can to a certain degree deter voters from manipulation since it may cause both undershooting and overshooting. Studying the most recent New Zealand general election we observe that, with two major minor parties having approximately 5% support in the society, the existing 5% threshold may be too high which results in a high level of insincere voting. A 3% threshold may be more appropriate.

Questions this paper raises that future research could address include: How do incentives to vote strategically vary with the choice of positional scoring rule? What if the scoring rule is not positional? The case $m > 3$ should be interesting with major power indices being different. Finally, the undershooting/overshooting phenomenon as deterrent of manipulation deserves a thorough investigation.

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