

Abstract

We find conditions on the order on subsets of a finite set which are necessary and sufficient for the relative ranking of any two subsets in this order to be determined by their extreme elements relative to an abstract convex geometry. It turns out that this question is closely related to the rationalisability of path independent choice functions by hyper-relations.

Orders on Subsets Rationalised by Abstract Convex Geometries

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1 Introduction

Let X be a finite set and \mathcal{A} the set of non-empty subsets of X . Suppose that elements of \mathcal{A} are interpreted as *acts* in an environment of uncertainty. Under this interpretation, an act is identified with its set of possible consequences.¹ Let \succsim be a complete and transitive preference order on \mathcal{A} . A classic literature, starting with Kannai and Peleg (1984) and surveyed by Barberà, Bossert and Pattanaik (2004), assumes that the singletons in \mathcal{A} are *linearly* ordered by \succsim and identifies further conditions on \succsim under which the relative ranking of any two subsets is determined by their “extreme” elements. In particular, defining

$$\begin{aligned}\bar{A} &= \{a \in A \mid \{a\} \succ \{b\} \text{ for all } b \in A \setminus \{a\}\} \\ \underline{A} &= \{a \in A \mid \{b\} \succ \{a\} \text{ for all } b \in A \setminus \{a\}\},\end{aligned}$$

Kannai and Peleg present simple axioms that imply $A \sim \bar{A} \cup \underline{A}$ for each $A \in \mathcal{A}$. While Kannai and Peleg’s axioms cannot be satisfied when $|X| \geq 6$, subsequent authors have weakened them so that suitable orders exist for any finite $|X|$ while preserving the property that $A \sim \bar{A} \cup \underline{A}$. See Bossert and Slinko (2006) for a recent contribution and related literature.

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¹It is natural to suppose that the acts being compared are not related through statewise dominance. Since we deal with *finite* sets, for which no particular structure is needed, the assumption that dominated acts have been removed is unproblematic.

The purpose of this note is to identify axioms under which preferences are determined by “extreme” outcomes for a more general notion of “extremeness”. In doing so, we shall illustrate the close connection between this problem and one from a quite different strand of the decision theory literature.

In the next section, we recall the notion of a *convex geometry* for a discrete set. This algebraic theory of convexity provides a natural and quite general definition of an “extreme point”. We wish to identify conditions on \succsim that are necessary and sufficient for the existence of a convex geometry such that the relative ranking of any two subsets is determined by their “extreme” elements with respect to this geometry.

In Section 3, we shall seek a solution to this problem in an unexpected quarter. We there recall some results from the theory of *choice functions*. For this purpose, we re-interpret the subsets of X as *opportunity sets*. A choice function is a mapping $C : 2^X \rightarrow 2^X$ that satisfies $C(A) \subseteq A$ for any A and $C(A) = \emptyset$ if and only if $A = \emptyset$. A choice function C satisfies *path independence* (Plott, 1973) if

$$C(A \cup B) = C(C(A) \cup B).$$

Aizerman and Malishevski (1981) and Nehring (1997) characterise the path independent choice functions in terms of “hyper-relations” on X . (We shall define this term precisely in Section 3.) Our main result turns out to be a corollary of the characterisation in Nehring (1997).

The link between these two seemingly unrelated problems is provided by the work of Koshevoy (1999), who showed that there is a one-to-one relationship between the set of extreme point operators for convex geometries and the set of path independent choice functions.

2 Abstract convex geometries

The abstract notion of a convex geometry is based on the idea of “closing” a set, in the sense of forming a convex hull. Recall (Edelman and Jamison, 1985) that $\mathcal{L} : 2^X \rightarrow 2^X$ is a *closure operator* if it satisfies, for all $A, B \subseteq X$:

$$(C1) \quad A \subseteq \mathcal{L}(A),$$

$$(C2) \quad A \subseteq B \text{ implies } \mathcal{L}(A) \subseteq \mathcal{L}(B),$$

$$(C3) \quad \mathcal{L}(\mathcal{L}(A)) = \mathcal{L}(A).$$

A closure operator \mathcal{L} is called a *convex geometry* if it satisfies, in addition, the *anti-exchange property*: for any set A satisfying $A = \mathcal{L}(A)$ and any $a, b \in X \setminus A$ such that $a \neq b$, if $b \in \mathcal{L}(A \cup \{a\})$ then $a \notin \mathcal{L}(A \cup \{b\})$.

If \mathcal{L} is a convex geometry, we may interpret $\mathcal{L}(A)$ as the *convex hull* of A . Sets of the form $\mathcal{L}(A)$ for some A are called *convex*. Given a convex geometry \mathcal{L} , its associated *extreme point operator* $\text{ex}: 2^X \rightarrow 2^X$ is defined as follows:

$$a \in \text{ex}(A) \quad \text{iff} \quad a \in A \quad \text{and} \quad a \notin \mathcal{L}(A \setminus \{a\}). \quad (1)$$

The following theorem will play a central role in proving our main result.

Theorem 1 (Koshevoy, 1999). *An extreme point operator for a convex geometry is a path independent choice function, and conversely, any path independent choice function is the extreme point operator for some convex geometry.*

Given a complete and transitive preference order \succsim on \mathcal{A} , we are interested in finding necessary and sufficient conditions for a ranking of subsets to depend only on its extreme points relative to some convex geometry. To be more precise, define the mappings $\text{Int}: 2^X \rightarrow 2^X$ and $\text{Ext}: 2^X \rightarrow 2^X$ as follows: $\text{Int}(\emptyset) = \text{Ext}(\emptyset) = \emptyset$ and

$$\begin{aligned} \text{Int}(A) &= \{a \in A \mid A \sim A \setminus \{a\}\}, \\ \text{Ext}(A) &= A \setminus \text{Int}(A) \end{aligned}$$

for any $A \in \mathcal{A}$. Thus, $\text{Ext}(A)$ contains those members of A whose individual removal would alter the ranking of the subset. We shall identify conditions under which Ext is the extreme point operator for some convex geometry and $A \sim \text{Ext}(A)$ for any $A \in \mathcal{A}$. For example, in the case of preferences satisfying the axioms of Kannai and Peleg,

$$A \sim \text{Ext}(A) = \underline{A} \cup \overline{A},$$

and Ext is the extreme point operator for the convex geometry with closure operator:²

$$\mathcal{L}(A) = \{x \in X \mid \overline{A} \succsim \{x\} \succsim \underline{A}\}.$$

3 Rationalising choice functions

Aizerman and Malishevski (1981) and Nehring (1997) study choice functions that are rationalised by *generalised binary relations* (or *hyper-relations*). A generalised binary

²The anti-exchange property of this operator is guaranteed by Kannai and Peleg's assumption that \succsim linearly orders the singletons. We do not require this assumption for our general result.

relation \sqsupseteq is a subset of $\mathcal{A} \times X$. We write $A \sqsupseteq a$ for $(A, a) \in \sqsupseteq$, and interpret $A \sqsupseteq a$ to mean that “choice a is unacceptable in the presence of $A \setminus \{a\}$ ”. The generalised binary relation \sqsupseteq is said to *rationalise*³ the choice function C if

$$C(A) = \{a \in A \mid \text{there is no } B \subseteq A \text{ such that } B \sqsupseteq a\}.$$

This generalises the idea of a choice function being rationalised by a binary relation on X . To motivate the generalisation, imagine that the elements of A are pure strategies in a given strategic-form game. Suppose the decision-maker regards a strategy as acceptable iff it is not strictly dominated by any mixed strategy that can be formed using the available pure strategies. Then we would specify $A \sqsupseteq a$ iff there is a mixture over A that strictly dominates a , and this generalised binary relation would rationalise her choice function. Note that the choice function may not be rationalisable by any binary relation on X , as there may exist some pure strategy in A that is not strictly dominated by any other pure strategy in A , yet is strictly dominated by some mixture over $A \setminus \{a\}$. (See also Nehring, 1997, Example 1.)

Following Nehring (1997),⁴ we shall impose the following “irreflexivity” and “monotonicity” conditions on \sqsupseteq :

$$A \sqsupseteq a \implies [A \setminus \{a\} \neq \emptyset \text{ and } A \setminus \{a\} \sqsupseteq a], \quad (\text{IRR})$$

$$A \sqsupseteq a \implies A \cup \{b\} \sqsupseteq a \quad (\text{MON})$$

for any $b \in X$. A generalised binary relation that satisfies (IRR) and (MON) will be called an *extended preference relation (EPR)*. Note that an EPR \sqsupseteq rationalises the choice function C iff

$$C(A) = A \setminus \{a \in A \mid A \sqsupseteq a\} \quad (2)$$

for each $A \in \mathcal{A}$. An *extended partial order (EPO)* is an EPR that satisfies the following “transitivity” condition:

$$[A \setminus \{a\} \sqsupseteq a \text{ and } A \sqsupseteq b] \implies A \setminus \{a\} \sqsupseteq b. \quad (\text{TRA})$$

This may be re-written as

$$\frac{[A \setminus \{a\} \sqsupseteq a \text{ and } (A \setminus \{a\}) \cup \{a\} \sqsupseteq b]}{\implies A \setminus \{a\} \sqsupseteq b}$$

³The terminology is Nehring’s. Aizerman and Malishevski say that C is “generated by a strongly dominant choice mechanism”. We have opted for the more “economical” nomenclature!

⁴We state a seemingly weaker version of (MON), but easily seen to be equivalent.

which shows it to be a special case of the Transitivity condition in Nehring (1997, p.410). In fact, it turns out that our (TRA), in the presence of (MON) and (IRR), is equivalent to Nehring's seemingly stronger version of transitivity.

The following result is a variation on Nehring (1997, Theorem 1(ii)).⁵ In particular, path independence may be shown to be equivalent to Nehring's conditions (α) and (η) .⁶ We provide a direct proof here for completeness.

Theorem 2. *If \sqsupseteq is an EPO, then (2) is a path independent choice function. Conversely, if C is a path independent choice function, then the generalised binary relation \sqsupseteq defined by*

$$A \sqsupseteq a \iff a \notin C(A \cup \{a\}) \quad (3)$$

is an EPO.

Proof. Suppose \sqsupseteq satisfies (IRR), (MON) and (TRA), and let C be defined by (2). It is obvious that $C(A) \subseteq A$ for any $A \subseteq X$. To see that $C(A) \neq \emptyset$ when $A \neq \emptyset$, suppose to the contrary that $C(A) = \emptyset$ for some non-empty $A = \{a_1, \dots, a_k\} \subseteq X$. Then $A \sqsupseteq a_i$ for each $i \in \{1, \dots, k\}$. Applying (TRA) successively, we see that

$$\begin{aligned} A \setminus \{a_1\} \sqsupseteq a_i \quad i \in \{2, \dots, k\}, \\ A \setminus \{a_1, a_2\} \sqsupseteq a_i \quad i \in \{3, \dots, k\}, \\ \dots \\ \{a_k\} \sqsupseteq a_k. \end{aligned}$$

which contradicts (IRR). It remains to show that C is path independent. For this, it suffices to verify the *Heritage (H)* and *Outcast (O)* conditions (Aizerman and Malishevski, 1981, Theorems 3 and 4; Danilov and Koshevoy, 2005, Proposition 1):

$$[a \in C(A) \text{ and } a \in B \subseteq A] \implies a \in C(B), \quad (\text{H})$$

$$a \notin C(A) \implies C(A \setminus \{a\}) = C(A). \quad (\text{O})$$

Axiom (MON) implies that C satisfies (H). Let us prove (O). Since (O) is trivial when $a \notin A$, suppose $a \in A$ and that $A \sqsupseteq a$. For any $b \in A \setminus \{a\}$,

$$A \setminus \{a\} \sqsupseteq b \implies A \sqsupseteq b$$

⁵The equivalence of (i) and (ii) in Nehring (1997, Theorem 2) is also closely related.

⁶Since Nehring does not include the condition $C(A) = \emptyset$ iff $A = \emptyset$ in the definition of a choice function, the first part of our Theorem 2 strengthens Nehring's version. For the second part, our argument does not require the full force of this additional property of choice functions – Nehring's assumption (ω_1) suffices.

by (MON). This shows $C(A \setminus \{a\}) \supseteq C(A)$. The converse we note that $A \setminus \{a\} \sqsupset a$ by (IRR) and by (TRA) we get

$$A \sqsupset b \implies A \setminus \{a\} \sqsupset b$$

which shows $C(A \setminus \{a\}) \subseteq C(A)$. So C satisfies (O). This proves the first part of the theorem.

For the converse, suppose that C is a path independent function and \sqsupset is defined as in (3). If $A \sqsupset a$, then we must have $A \setminus \{a\} \neq \emptyset$ since $a \notin C(\{a\})$ is excluded by the facts that $C(B) \subseteq B$ for any $B \subseteq X$ and $C(B) = \emptyset$ iff $B = \emptyset$. Also, since $A \cup \{a\} = (A \setminus \{a\}) \cup \{a\}$, $A \sqsupset a$ implies $A \setminus \{a\} \sqsupset a$. This proves (IRR). Next, suppose that $a \in C(A \cup \{a, b\})$. Then, by path independence,⁷

$$a \in C(C(A \cup \{a\}) \cup \{b\}).$$

Since $C(C(A \cup \{a\}) \cup \{b\}) \subseteq C(A \cup \{a\}) \cup \{b\}$, it follows that $a \in C(A \cup \{a\})$. This proves (MON). Finally, suppose that $A \setminus \{a\} \sqsupset a$ and $A \sqsupset b$. Since we already established that \sqsupset satisfies (MON), these imply

$$A \cup \{a, b\} \sqsupset a \text{ and } A \cup \{a, b\} \sqsupset b.$$

Equivalently

$$a \notin C(A \cup \{a, b\}) \text{ and } b \notin C(A \cup \{a, b\}).$$

Applying condition (O), which is a consequence of path independence, we have:

$$C(A \cup \{a, b\}) = C((A \setminus \{a\}) \cup \{b\})$$

The latter equality and $C(B) \subseteq B$ for any $B \subseteq X$ imply

$$b \notin C((A \setminus \{a\}) \cup \{b\}),$$

which is equivalent to $A \setminus \{a\} \sqsupset b$. This proves (TRA). □

3.1 The main result

We now return to the problem described at the end of Section 2. Consider the following two Axioms, which are assumed to hold for all $A \in \mathcal{A}$ and all $a, b \in A$:

Axiom 1 (A1). *If $A \sim A \setminus \{a\}$ and $A \sim A \setminus \{b\}$, then $A \sim A \setminus \{a, b\}$.*

⁷Indeed, as Nehring (1997, Fact 1) points out, the Heritage condition (H) suffices.

Axiom 2 (A2). *If $A \sim A \setminus \{a\}$, then $A \cup \{c\} \sim (A \setminus \{a\}) \cup \{c\}$ for any $c \in X$.*

We may now state the main result of the paper:

Theorem 3. *The following are equivalent:*

I. *Axioms (A1) and (A2) hold.*

II. *The function Ext derived from \succsim is the extreme point operator for some convex geometry and $A \sim \text{Ext}(A)$ for any $A \in \mathcal{A}$.*

To see the connection between Theorems 2 and 3, suppose that we are given a complete and transitive preference relation \succsim on \mathcal{A} . First extend \succsim to 2^X by specifying $\emptyset \sim \emptyset$ and $A \succ \emptyset$ for any $A \in \mathcal{A}$, and then define the generalised binary relation \sqsupset as follows:

$$A \sqsupset a \iff A \cup \{a\} \sim A \setminus \{a\}. \quad (4)$$

It is straightforward to observe that (IRR) is satisfied, while (MON) is equivalent to (A2) and (TRA) is equivalent to (A1). Moreover, if \sqsupset is an EPR and C is the choice function rationalised by \sqsupset , then

$$C(A) = A \setminus \{a \in A \mid a \in \text{Int}(A)\} = \text{Ext}(A).$$

Thus, the (I) \implies (II) part of Theorem 3, apart from the claim that $A \sim \text{Ext}(A)$ for all $A \in \mathcal{A}$, follows directly from Theorem 1 and the first part of Theorem 2.

Lemma 1. *Under (A1) and (A2), if $\text{Ext}(A) \subseteq B \subseteq A$, then $A \sim B$ and $\text{Ext}(A) = \text{Ext}(B)$. In particular, $A \sim \text{Ext}(A)$ for every $A \in \mathcal{A}$.*

Proof. Note that (A1) and (A2) together give:

$$a \in \text{Int}(A) \implies [b \in \text{Int}(A) \text{ iff } A \sim A \setminus \{a, b\}] \quad (5)$$

Let $\text{Int}(A) = \{a_1, a_2, \dots, a_k\}$ with $B = \text{Ext}(A) \cup \{a_i, a_{i+1}, \dots, a_k\}$ for some $i \in \{1, 2, \dots, k-1\}$. It follows from (5) that $\text{Int}(A \setminus \{a_1\}) = \{a_2, a_3, \dots, a_k\}$. Proceeding by induction we obtain the desired result. \square

Conversely, suppose that Ext is the extreme point operator for some convex geometry. Hence, it is a path independent choice function (Theorem 1). Since the generalised binary relation defined by (4) satisfies $A \sqsupset a$ iff $a \notin \text{Ext}(A \cup \{a\})$, it follows from the second part of Theorem 2 that \sqsupset is an EPO and hence that \succsim satisfies (A1) and (A2). This proves the (II) \implies (I) part of Theorem 3.

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