Isogeny Cryptography: Strengths, Weaknesses, and Challenges

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Plan

- Discrete logs and Diffie-Hellman
- Pollard rho and kangaroo
- Generalisations of Diffie-Hellman
- Isogenies
- SIDH/SIKE
- CSIDH

Please interrupt and ask questions.

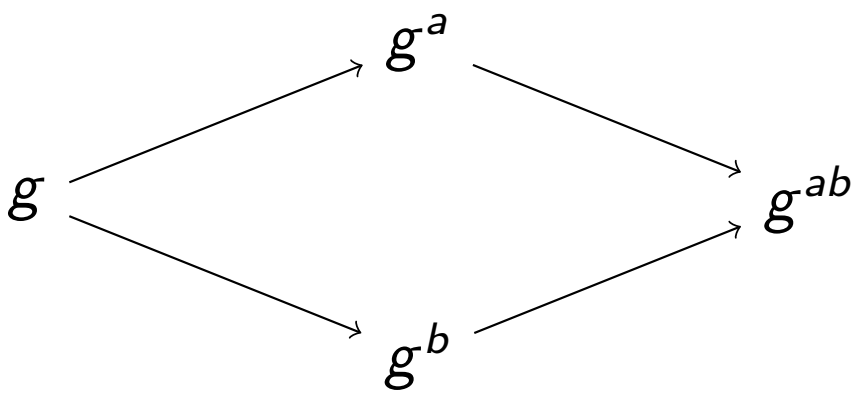
Discrete Logarithm Problem and Diffie-Hellman

Let G be a subgroup of $\mathbb{F}_q^*$ or $E(\mathbb{F}_q)$ of prime order.
Given $g \in G$ and $a \in \mathbb{N}$ it is **easy to compute** $h = g^a$.
Given $g \in G$ and $h = g^a$, it is hard to compute a.

Diffie-Hellman key exchange:

- Alice chooses a and sends $t_A = g^a$ to Bob.
- Bob chooses b and sends $t_B = g^b$ to Alice.
- Alice computes $t_B^a = g^{ab}$.
- Bob computes $t_A^b = g^{ab}$.

Diffie-Hellman key exchange



How hard is Discrete Logarithm Problem (DLP)?

- Let g have prime order r. Baby step giant step solves DLP in (deterministic) $O(\sqrt{r})$ time and space.
- Pollard rho solves DLP in (expected) $O(\sqrt{r})$ time and poly space.
- Idea: If can find integers a_1, a_2, b_1, b_2 such that

$$g^{a_1} h^{b_1} = g^{a_2} h^{b_2}.$$

  then solve the DLP as $a = (a_2 - a_1)(b_1 - b_2)^{-1} \pmod{r}$.
- Approach: Generate pseudorandom walks

$$x_i = g^{a_i} h^{b_i}$$

  such that (a_i, b_i) are known.
- **Pollard's big idea**: Use walks where the next step only depends on the current position.
- **Van Oorschot and Wiener's big idea**: Use distinguished points.

Pollard rho

Discrete logarithm "in an interval"

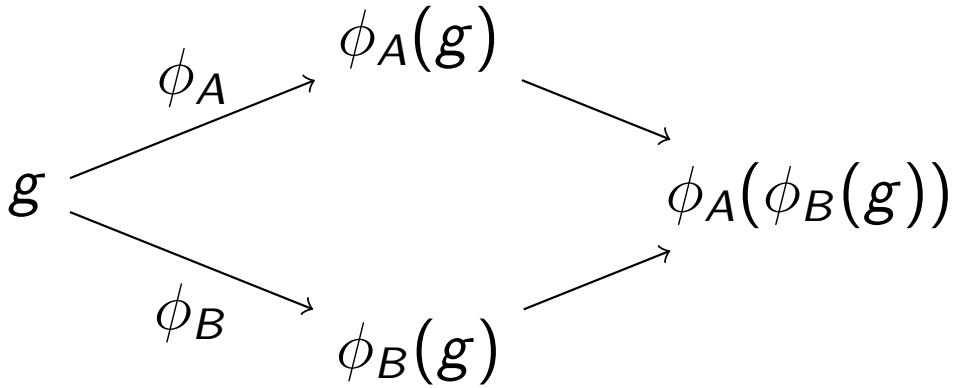
- Let g have order r, and let $0 < N < r$.
  The interval DLP is: Given $h = g^a$ where $0 \leq a < N$, to compute a.
- BSGS in $O(\sqrt{N})$ time and $O(\sqrt{N})$ space.
- Pollard kangaroo method using distinguished points (van Oorschot and Wiener) solves problem in average case expected $(2 + o(1))\sqrt{N}$ group operations.

Pollard kangaroo algorithm

- **Key properties:** one-dimensional space with a sense of direction to the right.
- Let $h = g^a$ with $a \in [0, N]$.
- The tame kangaroo starts in the middle of the interval $[0, N]$.
- The wild kangaroo starts at a.
- Steps in the walk are, on average, distance $m = \sqrt{N}/2$.
- Average case expected running time $(2 + o(1))\sqrt{N}$ group operations.

Generalised Diffie-Hellman 1: Endomorphisms

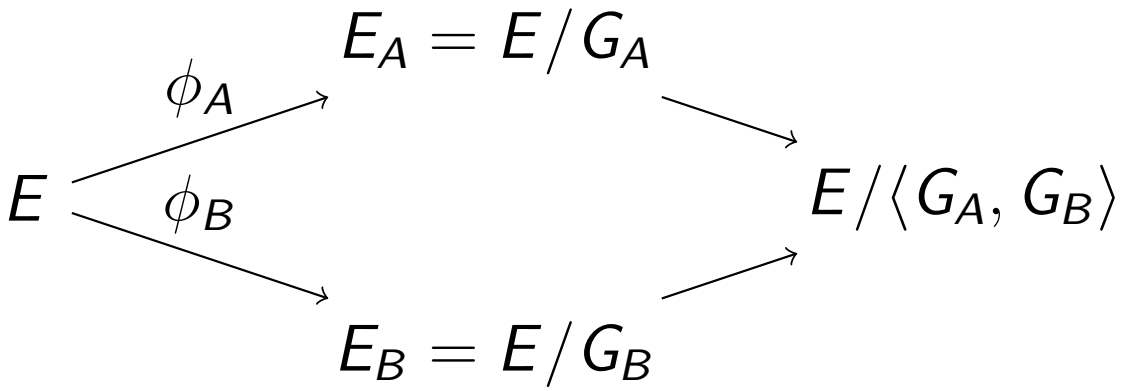
Fix an element g in a group G.
The map $\phi_A(x) = x^a$ is group homomorphism.
DLP: Given g and $\phi_A(g) = g^a$ it is hard to compute ϕ_A.



The maps $\phi_A(x) = x^a$ and $\phi_B(x) = x^b$ are group homomorphisms such that $\phi_A \circ \phi_B = \phi_B \circ \phi_A$.

Generalised Diffie-Hellman 1: Homomorphisms

Let E be an elliptic curve and G_A, G_B finite subgroups such that $G_A \cap G_B = \{0\}$.



Alice and Bob need to give each other enough information that they can compute $\phi_A(G_B)$ and $\phi_B(G_A)$.
Alice computes $E_B/\phi_B(G_A)$ and Bob computes $E_A/\phi_A(G_B)$.

\begin{scary isogeny section}

Example of an isogeny

- Let $E_1 : y^2 = x^3 + x$ and $E_2 : Y^2 = X^3 - 4X$.
- The point $(0, 0)$ on E_1 has order 2.
- There is an isogeny $\phi : E_1 \to E_2$ with kernel generated by $(0, 0)$, given by the rational function

$$\phi(x, y) = \left(\frac{x^2 + 1}{x} \ , \ y \frac{x^2 - 1}{x^2} \right).$$

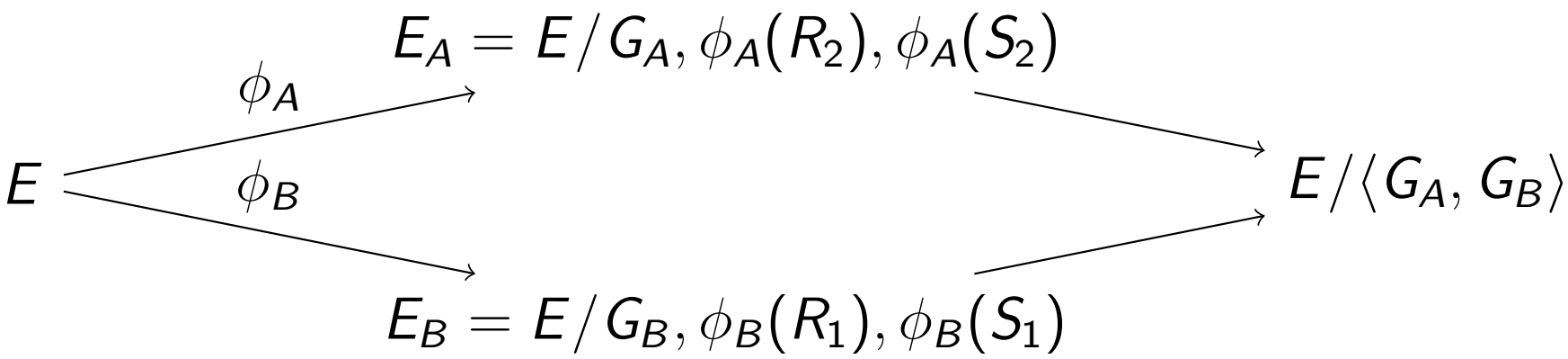
Isogenies

- An **isogeny** $\phi : E_1 \to E_2$ of elliptic curves is a (non-constant) morphism and a group homomorphism.
- An isogeny has finite kernel.
- Given a finite subgroup $G \subseteq E_1(\overline{\mathbb{F}}_q)$ there is a (unique separable) isogeny $\phi_G : E_1 \to E_2$ with kernel G.
- Can compute ϕ_G using Vélu in time **linear** in $\#G$ using operations in $\mathbb{F}_{q^t}$ where $G \subseteq E_1(\mathbb{F}_{q^t})$.
- We will write $E_2 = E_1/G$.
- We focus on separable isogenies, in which case $\deg(\phi) = \#\ker(\phi)$.
- $\mathrm{End}(E) = \{\text{isogenies } \phi : E \to E \text{ over } \overline{\mathbb{F}}_q\} \cup \{0\}$.

Jao and De Feo key exchange (SIDH)

- Let $p = 2^u 3^v f - 1$ be prime, where $2^u \approx 3^v$.
- Let E over $\mathbb{F}_{p^2}$ be a supersingular elliptic curve.
- Then $\#E(\mathbb{F}_{p^2}) = (p+1)^2$ group structure of $E(\mathbb{F}_{p^2})$ is a product of two cyclic groups of order $2^u 3^v f$.
- WTF? E is a supersingular curve with smooth group order?!?!
- They do this because we want to compute isogenies whose kernel is a point of order 2 or 3 defined over $\mathbb{F}_{p^2}$.
- Fix points $R_1, S_1 \in E[2^u]$ such that $\langle R_1, S_1 \rangle = E[2^u]$.
- Fix R_2, S_2 such that $\langle R_2, S_2 \rangle = E[3^v]$.
- The **system parameters** are (E, R_1, S_1, R_2, S_2).

Jao and De Feo key exchange (SIDH)



Jao and De Feo key exchange (SIDH)

- Alice chooses a secret subgroup of $E[2^u]$ by choosing an integer $0 \leq a < 2^u$ and setting $T_1 = R_1 + [a]S_1$.
- Alice computes an isogeny $\phi_A : E \to E_A$ with kernel generated by T_1 and publishes $(E_A, \phi_A(R_2), \phi_A(S_2))$.
- Bob chooses $0 \leq b < 3^v$, computes $\phi_B : E \to E_B$ with kernel generated by $T_2 = R_2 + [b]S_2$ and publishes $(E_B, \phi_B(R_1), \phi_B(S_1))$.
- Alice computes

$$T_1' = \phi_B(R_1) + [a]\phi_B(S_1) = \phi_B(R_1 + [a]S_1) = \phi_B(T_1)$$

  and then computes an isogeny $\phi_A' : E_B \to E_{AB}$ with kernel generated by T_1'.
- Bob computes an isogeny $\phi_B' : E_A \to E_{AB}'$ with kernel $\langle \phi_A(R_2) + [b]\phi_A(S_2) \rangle$.

Jao and De Feo key exchange (SIDH)

- Note: Alice computes E_{AB} and Bob computes E'_{AB}.
- The elliptic curve equations E_{AB} and E'_{AB} computed by Alice and Bob are different, but isomorphic.
- The shared key for Alice and Bob is $j(E_{AB}) = j(E'_{AB})$.
- SIDH gives a convenient plug-and-play substitute for ephemeral Diffie-Hellman key exchange.
- Can use for encryption by making Bob's input static, but need to avoid an active attack due to Galbraith, Petit, Shani, Ti (Asiacrypt 2016).

Performance of SIDH/SIKE

- Alice computes an isogeny $\phi_A : E \to E_A$ of degree 2^u with kernel generated by T_1 and publishes $(E_A, \phi_A(R_2), \phi_A(S_2))$.
- She does this by computing a sequence of u isogenies of degree 2.
- Nice work by De Feo, Jao, Plût on making it efficient.
- **Philosophical question:** Is there a "square-and-multiply" algorithm for isogenies?

SIKE submission to NIST

- SIKE = Supersingular Isogeny Key Exchange.
- Submission to the NIST standardization process on post-quantum cryptography.
- Authors: Jao, Azarderakhsh, Campagna, Costello, De Feo, Hess, Jalali, Koziel, LaMacchia, Longa, Naehrig, Renes, Soukharev and Urbanik.
- Submission contains specification of an IND-CCA KEM.
- `http://sike.org/`
- Strengths: very short ciphertexts. CSIDH is even better.
- Weaknesses: Not as fast as one might hope.

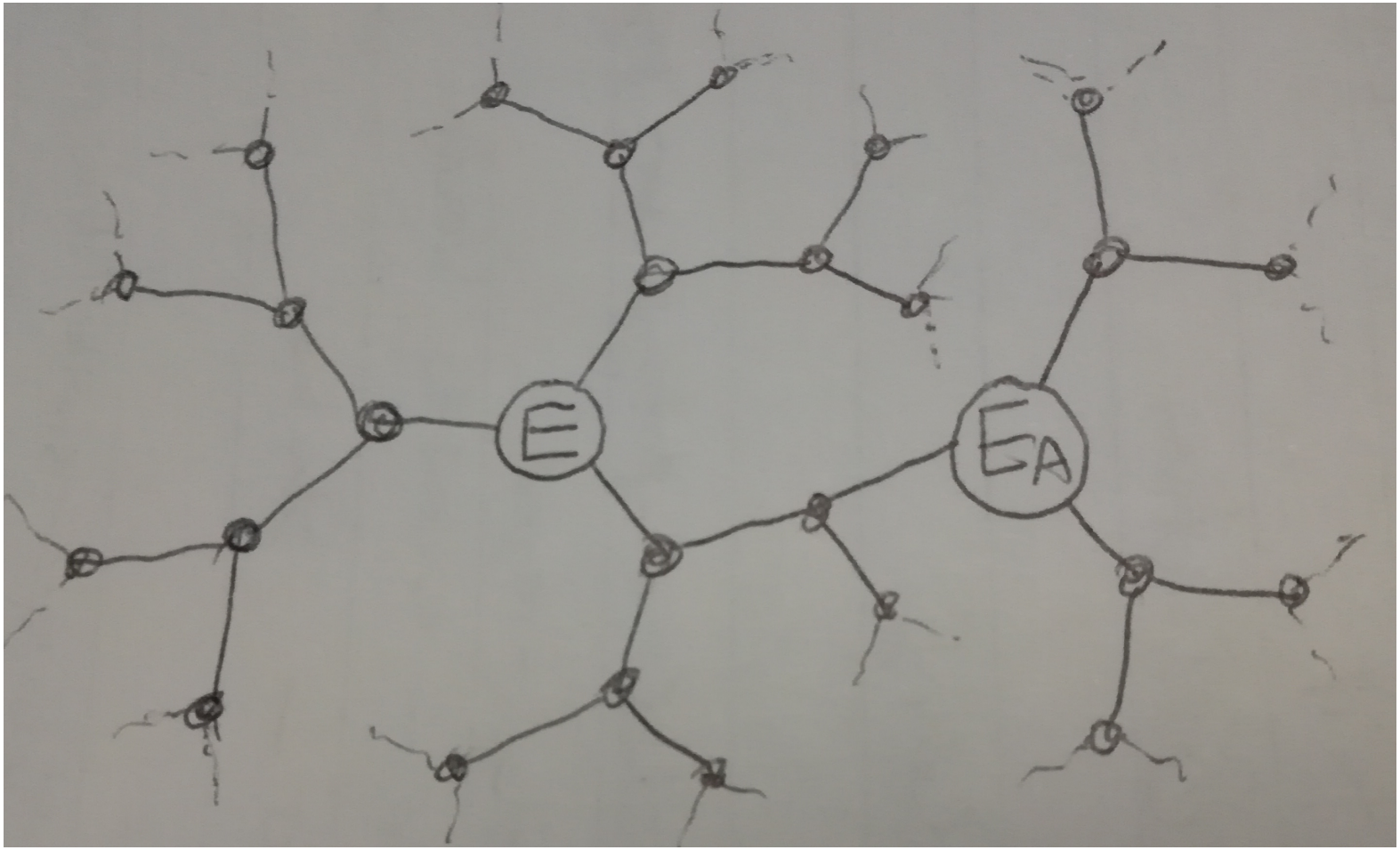
Security of SIDH/SIKE

- Given (E, E_A) an attacker wants to find the isogeny $\phi_A : E \to E_A$ of degree 2^u that maps the points R_2 and S_2 to the given points.
- An attacker needs to have exactly the right isogeny ϕ_A, otherwise they won't compute the correct shared key $E_{AB} = E/\langle G_A, G_B \rangle$.
  (Though, morally, if one can compute any isogeny one ought to be able to compute this special one.)
- There is a baby-step-giant-step algorithm that requires $O(2^{u/2})$ time and space. (Ditto $O(3^{v/2})$.)
  This is $O(p^{1/4})$ isogeny computations in the usual setting.
- Quantum speedup of BSGS using Tani "claw-finding" algorithm, giving $O(2^{u/3}) = O(p^{1/6})$ isogeny computations.

Security of SIDH/SIKE with random walks

- SIDH is analogous to the discrete log in an interval, because E_A is "much closer" to E than a random elliptic curve in the isogeny class would be.
- Can we attack using kangaroos?

Graph of 2-isogenies



Not one-dimensional. No sense of direction.

Security of SIDH/SIKE with random walks

- There is no known low-storage algorithm to break SIDH in $O(2^{u/2})$ isogeny computations.
- Adj, Cervantes-Vázquez, Chi-Domínguez, Menezes and Rodríguez-Henríquez, On the cost of computing isogenies between supersingular elliptic curves, eprint 2018/313 (to appear in proceedings SAC 2018).
- Uses a different idea of van Oorschot and Wiener.
- Gives an algorithm with $O(2^{3u/4})$ complexity.
- See Craig Costello's talk next week.

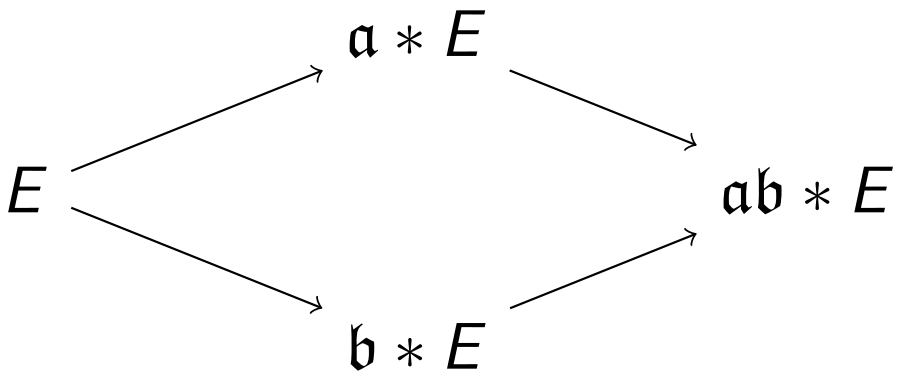
Generalised Discrete Logarithm Problem 2: Homogenous Spaces

(Couveignes 1997)
Let G be a subgroup of $\mathbb{F}_q^*$ or $E(\mathbb{F}_q)$ of prime order r.
For $a \in \mathbb{Z}_r$ and $g \in G$ define $a * g := g^a$.
Given $g \in G$ and $h = a * g$, hard to compute a.
Alternative formulation: an action of $\mathbb{Z}_r^*$ on $G - \{1\}$.

Generalised Diffie-Hellman key exchange:

- Alice chooses $a \in \mathbb{Z}_p$ and sends $t_A = a * g$ to Bob.
- Bob chooses $b \in \mathbb{Z}_p$ and sends $t_B = b * g$ to Alice.
- Alice computes $a * t_B$.
- Bob computes $b * t_A$.

Generalised Diffie-Hellman 2: Group action



Class Group Actions from Isogenies

- J.-M. Couveignes "Hard Homogeneous Spaces", preprint (1997/2006)
- A. Stolbunov, Master thesis (2004)
- A. Rostovtsev, A. Stolbunov, preprint (2006)
- A. Stolbunov "Constructing public-key cryptographic schemes based on class group action on a set of isogenous elliptic curves", Adv. Math. Comm., (2010)
- Couveignes describes a Diffie-Hellman-type key exchange based on group actions.
  Does not mention post-quantum security.
- Rostovtsev and Stolbunov give key exchange and encryption.
  Suggest isogenies could be post-quantum secure.
- Stolbunov's thesis describes also mentions signatures.

Computational problems

- Vectorisation: Given E and $\mathfrak{a} * E$ to compute $\mathfrak{a}$.
- Parallelisation: Given E, $\mathfrak{a} * E$, $\mathfrak{b} * E$ to compute $(\mathfrak{ab}) * E$.
- See Ben Smith paper "Pre- and post-quantum DiffieHellman from groups, actions, and isogenies"
- New result (Galbraith, Panny, Vercauteren): parallelisation + Shor $\rightarrow$ vectorisation.

Pollard rho for vectorisation

- Let $\mathfrak{a} \in G$ act on a set of things called E.
- Problem instance: E_0 and $E_A = \mathfrak{a} * E$.
- Define pseudo-random walk $E \mapsto \mathfrak{b} * E$ for known $\mathfrak{b} \in G$.
- Define walk on pairs $(E, \mathfrak{c}) \mapsto (\mathfrak{b} * E, \mathfrak{b}\mathfrak{c})$.
- Start half the walks on $(\mathfrak{c} * E_0, \mathfrak{c})$ and the other half on $(\mathfrak{c} * E_A, \mathfrak{c})$.
- Run walks until hit a distinguished point.
- With probability $1/2$ a collision gives a solution to $\mathfrak{c}_i * E_0 = \mathfrak{c}_j * E_A$ and so the solution is $\mathfrak{c}_i \mathfrak{c}_j^{-1}$.

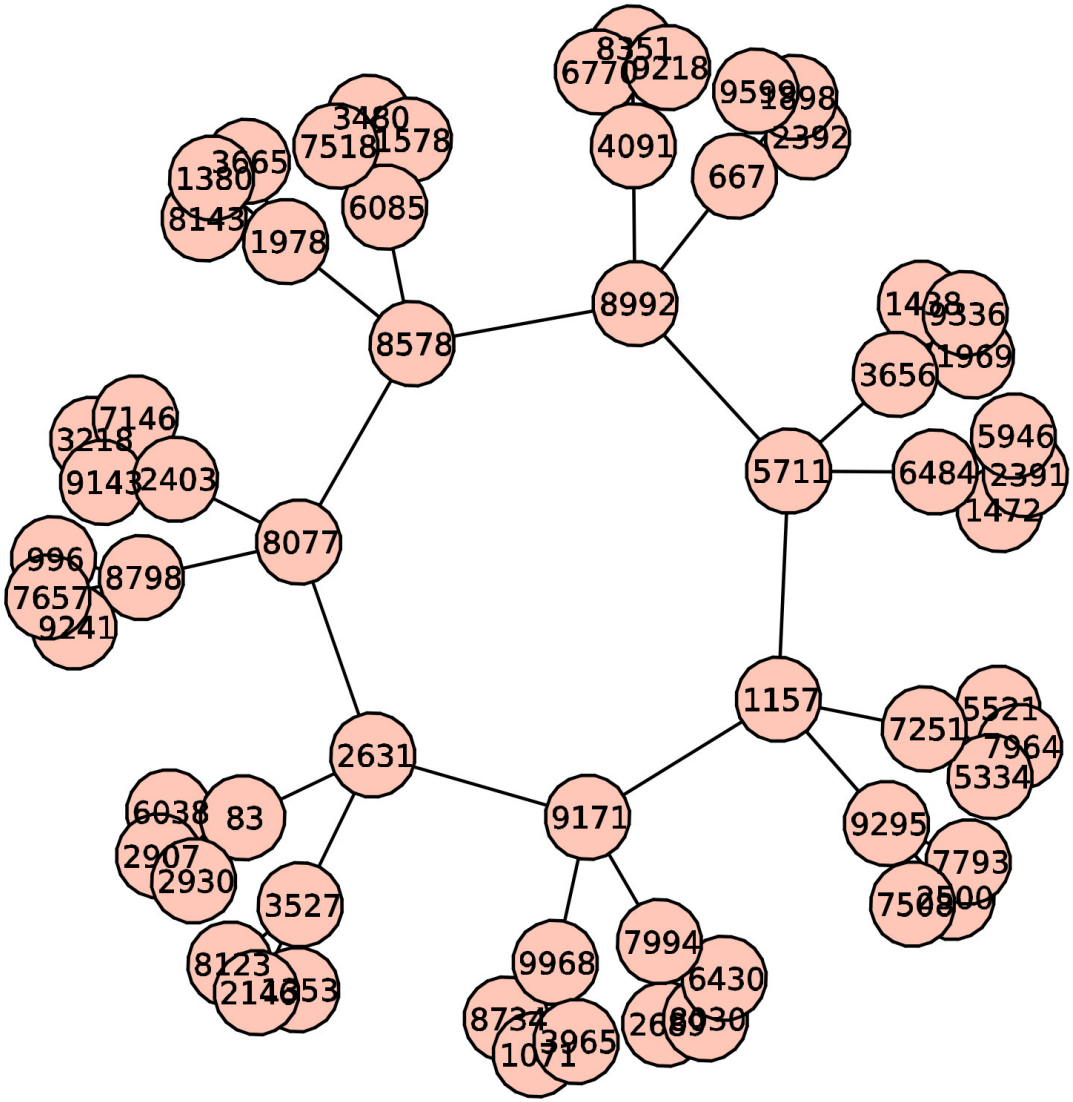
Class Group Action on Elliptic Curves

- Let E be an ordinary elliptic curve over $\mathbb{F}_q$ with $\mathrm{End}(E) \cong \mathcal{O}$ an order in an imaginary quadratic field.
- Let $\mathfrak{a}$ be an invertible $\mathcal{O}$-ideal.
- Can define the subgroup

$$E[\mathfrak{a}] = \{P \in E(\overline{\mathbb{F}}_q) : \phi(P) = 0 \ \forall \phi \in \mathfrak{a}\}.$$

  (Waterhouse 1969)
- There is an isogeny $E \to E'$ with kernel $E[\mathfrak{a}]$. Define $\mathfrak{a} * E$ to be $E' = E/E[\mathfrak{a}]$.
- $\mathfrak{a} * E$ depends only on the ideal class of $\mathfrak{a}$.
- This gives an action of the ideal class group $\mathrm{Cl}(\mathcal{O})$ on the set of E with $\mathrm{End}(E) \cong \mathcal{O}$.

Ordinary Isogeny Graph ($\ell = 3$)



Credit: Dustin Moody

How to compute $\mathfrak{a} * E$?

- Recall we can only compute an isogeny efficiently if its kernel G is small and preferably defined over the ground field.
- For a random ideal class $\mathfrak{a}$ the kernel will not satisfy this requirement.
- Let $\mathfrak{l}_1, \ldots, \mathfrak{l}_n$ be split prime ideals ("Elkies primes") of small norm. Write

$$\mathfrak{a} \equiv \prod_{i=1}^{n} \mathfrak{l}_i^{e_i}.$$

  (Let's assume $\{\mathfrak{l}_i\}$ generates the class group.)
- Couveignes: time required "a few hours".
- Stolbunov: compute $\mathfrak{a} * E$ in 4 minutes or so.
- De Feo, Kieffer and Smith discuss choosing a special curve to make the isogeny computations faster.
- CSIDH improve this (see later)

Efficiency of computation

- In key exchange we turn things around: Choose uniform $e_i \in [-B, B]$ and define $\mathfrak{a} = \prod_i \mathfrak{l}_i^{e_i}$.
- Computing a walk $\prod_{i=1}^{k} \ell_i^{e_i} \pmod{p}$ in a graph requires $\sum_{i=1}^{k} |e_i| \approx kB$ operations.
- To minimise this cost subject to $(2B+1)^k \approx p$ one takes $k \approx \log(p)$ and $B = O(1)$.
- Actually: $\sum_i |e_i| l_i$ operations where l_i is the norm of ℓ_i. Stolbunov uses different intervals $e_i \in [-B_i, B_i]$ to optimise cost.

CSIDH (Castryck, Lange, Martindale, Panny, Renes 2018)

- Let $p = 4\ell_1 \cdots \ell_k - 1$.
- Let X be the set of isomorphism classes of supersingular elliptic curves E with j-invariant in $\mathbb{F}_p$.
  Note that $\#X = O(\sqrt{p}\log(p))$.
- All $E \in X$ have $\mathrm{End}_{\mathbb{F}_p}(E)$ an order in $\mathbb{Q}(\sqrt{-p})$.
  Here $\mathrm{End}_{\mathbb{F}_p}(E) = \{\phi : E \to E \text{ defined over } \mathbb{F}_p\}$.
- C. Delfs and S. D. Galbraith (2016) showed that one can define class group actions on X.
- CSIDH is an instantiation of group action crypto using supersingular curves, which gives **massive** performance improvements.
- Features:
  - No public key validation needed, so can do non-interactive key exchange.
  - Better bandwidth.
  - Only sub-exponentially quantum secure.

Open problems

- How close to uniform is the distribution

$$\mathfrak{a} \equiv \prod_i \mathfrak{l}_i^{e_i}$$

  over uniform $e_i \in [-B, B]$, for fixed small prime ideals $\mathfrak{l}_i$? (Let's assume $\{\mathfrak{l}_i\}$ generates the class group.)
- Can small prime factors of $\#\mathrm{Cl}(\mathcal{O})$ be determined? Can subgroups of ideal class group be exploited?
- (Boneh): Find other homogeneous spaces/torsors for group actions that are efficient and secure for crypto.

Computational problems and algorithms

- Given E and $E' = \mathfrak{a} * E$ to determine the ideal (class) $\mathfrak{a}$.
- Pollard-rho-type algorithm due to Galbraith-Hess-Smart in time $\tilde{O}(\sqrt{\#G})$ (bug fixed by Stolbunov).
  Running time $\tilde{O}(p^{1/4})$ in the CSIDH case.
- Hidden shift problem: G an abelian group and $f, g : G \to S$ such that, for some $s \in G$, $g(x) = f(xs)$ for all $x \in G$. Problem: find s.
- Idea: Given $(E, E' = \mathfrak{a} * E)$ define $f(\mathfrak{b}) = \mathfrak{b} * E$ and $g(\mathfrak{b}) = \mathfrak{b} * E' = f(\mathfrak{b}\mathfrak{a})$.
- Kuperberg's algorithm is a subexponential time and space quantum algorithm for the hidden shift problem.
- A. Childs, D. Jao and V. Soukharev were the first to analyse Kuperberg's algorithm in the isogeny setting.
- See talk by J.-F. Biasse.

Candidate post-quantum pairing

Recent paper by Boneh, Glass, Krashen, Lauter, Sharif, Silverberg, Tibouchi and Zhandry (eprint 2018/665).

- Fix **ordinary** $E/\mathbb{F}_q$
- Fact: $(\mathfrak{a}_1 * E) \times (\mathfrak{a}_2 * E) \cong (\mathfrak{a}_1\mathfrak{a}_2 * E) \times E$ as unpolarized abelian varieties.
  (Result holds more generally for n terms; see Kani 2011.)
- This is essentially a bilinear pairing (resp. multilinear map).
- Not used for key exchange, but other more complex protocols.
- **Open problem:** To find a computable invariant of the isomorphism class.
- **Application:** Algorithm to solve the decisional Diffie-Hellman problem for class group actions in the ordinary case (but not the supersingular case).

Public Key Signatures

- L. De Feo and S. Galbraith "SeaSign: Compact isogeny signatures from class group actions", eprint 2018/824.
- Public key: E and $E_A = \mathfrak{a} * E$ where

$$\mathfrak{a} \equiv \prod_i \mathfrak{l}_i^{e_i}$$

  and $\mathfrak{l}_i$ ideals of small prime norm, $|e_i| \leq B$.
- Signer generates random ideals $\mathfrak{b}_k = \prod_{i=1}^{n} \mathfrak{l}_i^{f_{k,i}}$ for $1 \leq k \leq t$ and computes $\mathcal{E}_k = \mathfrak{b}_k * E$.
- Compute $H(j(\mathcal{E}_1), \ldots, j(\mathcal{E}_t), \text{message})$ where H is a cryptographic hash function with t-bit output $b_1, \ldots, b_t$.
- If $b_k = 0$ signature includes $\mathbf{f}_k = (f_{k,1}, \ldots, f_{k,n})$ and if $b_k = 1$ it includes

$$\mathbf{f}_k - \mathbf{e} = (f_{k,1} - e_1, \ldots, f_{k,n} - e_n).$$

- Use Lyubashevsky's "Fiat-Shamir with aborts".

Thank You