A survey of hyperelliptic pairings

Steven Galbraith

Royal Holloway, University of London

http://www.isg.rhul.ac.uk/∼sdg/

History

- The first papers on pairings in cryptography (Sakai-Ohgishi-Kasahara, Mitsunari-Sakai-Kasahara, Joux, Verheul, Boneh-Franklin) all used pairings on supersingular elliptic curves.

History

- The first papers on pairings in cryptography (Sakai-Ohgishi-Kasahara, Mitsunari-Sakai-Kasahara, Joux, Verheul, Boneh-Franklin) all used pairings on supersingular elliptic curves.
- I suggested using supersingular hyperelliptic curves, to get larger embedding degrees.

History

- The first papers on pairings in cryptography (Sakai-Ohgishi-Kasahara, Mitsunari-Sakai-Kasahara, Joux, Verheul, Boneh-Franklin) all used pairings on supersingular elliptic curves.
- I suggested using supersingular hyperelliptic curves, to get larger embedding degrees.
- We can ask: Was this a good suggestion?

History

- The first papers on pairings in cryptography (Sakai-Ohgishi-Kasahara, Mitsunari-Sakai-Kasahara, Joux, Verheul, Boneh-Franklin) all used pairings on supersingular elliptic curves.
- I suggested using supersingular hyperelliptic curves, to get larger embedding degrees.
- We can ask: Was this a good suggestion?
- After 6 years of research, the answer is:

History

- The first papers on pairings in cryptography (Sakai-Ohgishi-Kasahara, Mitsunari-Sakai-Kasahara, Joux, Verheul, Boneh-Franklin) all used pairings on supersingular elliptic curves.
- I suggested using supersingular hyperelliptic curves, to get larger embedding degrees.
- We can ask: Was this a good suggestion?
- After 6 years of research, the answer is: Yes and No.

History

- The first papers on pairings in cryptography (Sakai-Ohgishi-Kasahara, Mitsunari-Sakai-Kasahara, Joux, Verheul, Boneh-Franklin) all used pairings on supersingular elliptic curves.
- I suggested using supersingular hyperelliptic curves, to get larger embedding degrees.
- We can ask: Was this a good suggestion?
- After 6 years of research, the answer is: Yes and No.
- Yes: Generated lots of interesting research and open problems.
- No: Hyperelliptic curves usually less practical for pairings than elliptic curves.

Executive summary

With current knowledge on pairing implementation, I recommend using elliptic curves for pairing-based cryptography.

Plan of talk

Joint work with Florian Hess and Frederik Vercauteren

- Brief introduction to hyperelliptic curves.
- Ate pairings and pairing implementation.
- Comparison between elliptic and hyperelliptic curves.
- Rubin-Silverberg compression.
- Torsion structure.
- Conclusions and open problems.
- If time: Pairing inversion.

Hyperelliptic curves

A **hyperelliptic curve** over a field $\mathbb{F}_q$ is the curve associated with a non-singular equation of the form

$$C : y^2 + h(x)y = f(x)$$

where $h(x), f(x) \in \mathbb{F}_q[x]$.

Usually: $\deg(f(x)) = 2g + 1$ and $\deg(h(x)) \leq g$ in which case there is a single point ∞ and the curve has genus g.

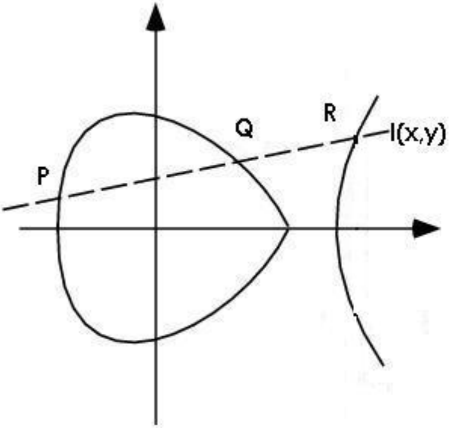
If the genus is 1 then we call the curve **elliptic**.

Example: $y^2 = x^5 + 1$ has genus 2.

Elliptic curve group law

For elliptic curves there is a group law on points given by a geometric procedure.

The rule is that $P + Q + R = 0$ if P, Q and R are the points of intersection (counting multiplicities) of the curve with a line.



Hyperelliptic group law

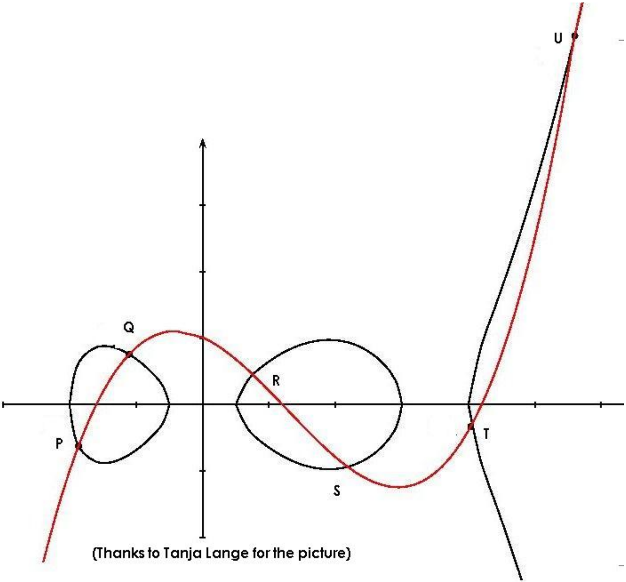
For genus $g \geq 2$ there is not a group law on points, but one can obtain a geometric group law on sets of points.

Let D_1, D_2, D_3 be (multi-)sets of points.

Then $D_1 + D_2 + D_3 = 0$ if there is curve F such that the intersection of C and F (counting multiplicities) is exactly the (multi-)set $D_1 \cup D_2 \cup D_3$.

Hyperelliptic group law

$$\{P, Q\} + \{R, S\} + \{T, U\} = 0$$



(Thanks to Tanja Lange for the picture)

Divisor class group of a curve

The precise definitions use the language of divisors. See the *Handbook of Elliptic and Hyperelliptic Cryptography* for background.

Get divisor class group $\mathrm{Pic}^0_{\mathbb{F}_q}(C)$.

In the case of genus 1, $\mathrm{Pic}^0_{\mathbb{F}_q}(E) = E(\mathbb{F}_q)$.

Mumford representation for divisors

Each divisor class has a **reduced representative**

$$E - d(\infty)$$

where $d \leq g$ and $E = (P_1) + (P_2) + \cdots + (P_d)$ where $P_i = (x_i, y_i) \in C(\overline{\mathbb{F}}_q)$ and $P_i \neq -P_j$ for $i \neq j$.

Such a divisor is represented as a pair $u_E(x), v_E(x) \in \mathbb{F}_q[x]$ where

$$u_E(x) = \prod_{i=1}^{d}(x - x_i),$$

$$v_E(x_i) = y_i$$

and

$$v_E(x)^2 + h(x)v_E(x) - f(x) \equiv 0 \pmod{u_E(x)}.$$

Addition of divisors

Cantor's algorithm gives addition and reduction of divisors in Mumford representation.

See Algorithm 1 in the paper.

It is straightforward to obtain the functions needed for Miller's algorithm.

For fast implementations use optimised explicit formulae (Harley, Lange,...).

Group sizes

Theorem: If C is a curve of genus g then the divisor class group has order

$$(\sqrt{q}-1)^{2g} \leq \#\mathrm{Pic}^0_{\mathbb{F}_q}(C) \leq (\sqrt{q}+1)^{2g}.$$

If q is large compared with g then this is $\approx q^g$.

Advantages of hyperelliptic over elliptic

- Security of the DLP depends on the size of the largest prime divisor of the order of the group.
  So for elliptic curves over $\mathbb{F}_q$ need $q > 2^{160}$.
  For genus 2 can have $q \approx 2^{80}$ and for genus 3 can have $q \approx 2^{54}$.
  In other words, *have a more complicated group law but over a smaller field*.

Advantages of hyperelliptic over elliptic

- Security of the DLP depends on the size of the largest prime divisor of the order of the group.
  So for elliptic curves over $\mathbb{F}_q$ need $q > 2^{160}$.
  For genus 2 can have $q \approx 2^{80}$ and for genus 3 can have $q \approx 2^{54}$.
  In other words, *have a more complicated group law but over a smaller field*.
- Use special curve equations to simplify explicit formulae for the group law (see Lange and Lange-Stevens).

Advantages of hyperelliptic over elliptic

- Security of the DLP depends on the size of the largest prime divisor of the order of the group.
  So for elliptic curves over $\mathbb{F}_q$ need $q > 2^{160}$.
  For genus 2 can have $q \approx 2^{80}$ and for genus 3 can have $q \approx 2^{54}$.
  In other words, *have a more complicated group law but over a smaller field*.
- Use special curve equations to simplify explicit formulae for the group law (see Lange and Lange-Stevens).
- Use **degenerate divisors** $(P) - (\infty)$ rather than $(P_1) + (P_2) + \cdots + (P_g) - g(\infty)$ (Katagi-Akishita-Kitamura-Takagi).

Pairings on hyperelliptic curves

- T. Okamoto and K. Sakurai, CRYPTO 1991.
- G. Frey and H.-G. Rück, Math. Comp. 1994.

Pairings

I assume that everyone here knows something about pairings.

Let C be a curve over $\mathbb{F}_q$.

Let $r \mid \#\mathrm{Pic}^0_{\mathbb{F}_q}(C)$.

Let k be smallest positive integer such that $r \mid (q^k - 1)$.
We call k the embedding degree.

Let $P, Q \in \mathrm{Pic}^0_{\mathbb{F}_{q^k}}(C)$ have order r.
Let $G_1 = \langle P \rangle$ and $G_2 = \langle Q \rangle$.
Let $G_T = \mu_r = \{z \in \mathbb{F}_{q^k} : z^r = 1\}$.

For security need, say, $r > 2^{160}$ and $q^k > 2^{1024}$ (or 2^{2048}?)

Tate-Lichtenbaum pairing

There is a function $f_{r,P}$ which has a zero of multiplicity r at P and pole of multiplicity r at ∞ and which is normalised appropriately at ∞.

The **reduced Tate-Lichtenbaum pairing** is

$$e(P, Q) = f_{r,P}(Q)^{(q^k-1)/r}.$$

Lemma: If $r \mid N \mid (q^k - 1)$ then

$$e(P, Q) = f_{N,P}(Q)^{(q^k-1)/N}.$$

Developments in pairing implementation

- Duursma and Lee were the first to exploit values N which are not multiples of r.
  These ideas were extended by Barreto, Galbraith, O hÉigeartaigh and Scott (eta pairing) and Hess, Smart, Vercauteren (ate pairing).

Developments in pairing implementation

- Duursma and Lee were the first to exploit values N which are not multiples of r.
  These ideas were extended by Barreto, Galbraith, O hÉigeartaigh and Scott (eta pairing) and Hess, Smart, Vercauteren (ate pairing).
- One of the key ideas is to use values N which are smaller than the order of P.
  This is called **loop shortening**.

Developments in pairing implementation

- Duursma and Lee were the first to exploit values N which are not multiples of r.
  These ideas were extended by Barreto, Galbraith, O hÉigeartaigh and Scott (eta pairing) and Hess, Smart, Vercauteren (ate pairing).
- One of the key ideas is to use values N which are smaller than the order of P.
  This is called **loop shortening**.
- One gets fast pairing computation on certain curves.

Elliptic ate pairing

Allows loop shortening depending on the size of the trace of Frobenius t.

Suitable curves maybe generated using the Brezing-Weng method.

Other talks, such as Mike Scott's, will present examples of this.

Hyperelliptic ate pairing

$G_1 = \mathrm{Pic}^0_{F_q}(C)[r]$ (1-eigenspace of $\pi := q$-power Frobenius)
$G_2 = \mathrm{Pic}^0_{\mathbb{F}_{q^k}}(C)[r] \cap \ker(\pi - q)$ (q-eigenspace of π)

Theorem: (Granger, Hess, Oyono, Thériault, Vercauteren)
Let $D_1 = E_1 - d_1(\infty) \in G_1$ and $D_2 = E_2 - d_2(\infty) \in G_2$ be reduced divisors. (Assume supports of E_1 and E_2 are disjoint).
Let $[q]D_2$ be the reduced divisor equivalent to qD_2. Denote by f_{q,D_2} the function with divisor $qD_2 - [q]D_2$ with leading coefficient 1 with respect to an $\mathbb{F}_q$-rational uniformizer at ∞.
Then

$$a(D_2, D_1) = f_{q,D_2}(E_1)$$

defines a non-degenerate bilinear pairing

$$a : G_2 \times G_1 \to \mu_r.$$

Hyperelliptic ate pairing

- **Note:** No final exponentiation!

Hyperelliptic ate pairing

- **Note:** No final exponentiation!
- But only if all denominators computed.
  In practice, it is better to avoid computing denominators and use a final exponentiation.

Hyperelliptic ate pairing

- **Note:** No final exponentiation!
- But only if all denominators computed.
  In practice, it is better to avoid computing denominators and use a final exponentiation.
- Hence, there seems to be no practical advantage to not needing the final exponentiation!
- Worse: lack of final exponentiation may make pairing inversion easier.

Implementation of the ate pairing

- As usual, one uses Miller's algorithm with various standard implementation tricks.
- Have to evaluate functions at divisor $E = (u_E(x), y - v_E(x))$. This can be done using resultants.

Implementation of the ate pairing

- As usual, one uses Miller's algorithm with various standard implementation tricks.
- Have to evaluate functions at divisor $E = (u_E(x), y - v_E(x))$. This can be done using resultants.
- A good trick is to replace y by $v_E(x)$ (so all polynomials depend on x only) and reduce modulo $u_E(x)$ (so all polynomials are of degree $\leq g$.
  Then do a single resultant at the end.

Implementation of the ate pairing

- As usual, one uses Miller's algorithm with various standard implementation tricks.
- Have to evaluate functions at divisor $E = (u_E(x), y - v_E(x))$. This can be done using resultants.
- A good trick is to replace y by $v_E(x)$ (so all polynomials depend on x only) and reduce modulo $u_E(x)$ (so all polynomials are of degree $\leq g$.
  Then do a single resultant at the end.
- There is a connection with norms in function fields.
- See the paper for more details.

Example (Barreto, G., Ó hÉigeartaigh, Scott)

(Using eta pairing)

Consider

$$C : y^2 + y = x^5 + x^3 + d \qquad d \in \{0, 1\}$$

over $\mathbb{F}_{2^m}$ with $\gcd(m, 6) = 1$.

Genus 2, embedding degree $k = 12$.

Pairing degenerate divisors using the eta pairing over $\mathbb{F}_{2^{103}}$ gives the fastest pairing computation time in software.

Example (Barreto, G., Ó hÉigeartaigh, Scott)

But:

- The implementation exploits 128-bit architecture.
- Degenerate divisors not necessarily secure for these parameters.
  (e.g., if using $H(ID) = (P) - (\infty)$ then probability of hash collision only about $1/2^{52}$.)
  Using non-degenerate divisors slower than elliptic case.

Comparing elliptic and hyperelliptic pairings

Consider

$$e : G_1 \times G_2 \to G_T.$$

Many criteria for comparison:

- Computation time for G_1, G_2, G_T and pairing computation.
- Size of representation for G_1, G_2, G_T.
- Flexibility and efficiency of parameter generation.
- Any other special properties.

See paper for full details.

Comparing elliptic and hyperelliptic pairings

Recall that one of the main advantages of hyperelliptic curves is that q can be smaller (i.e., *have a more complicated group law but over a smaller field*).

However, for pairings applications the field $\mathbb{F}_{q^k}$ has to be large independent of the genus.

Hence, at least one of G_1, G_2, *have a more complicated group law but over the same sized field*.

Similarly, in Miller's algorithm, evaluating more complicated functions at more complicated divisors over the same sized field. Hence, intuitively, pairings on curves of genus $g \geq 2$ cannot be faster than elliptic curves.

Comparison of loop shortening methods

In genus g the loop shortening is by a factor $1/g$.

For elliptic curves the loop shortening can be by a factor $\varphi(k)$. (Note: also need k quite large, maximising $\varphi(k)$ alone is not desirable.)

Hence, one can usually match the loop shortening in the hyperelliptic case by using elliptic curves.

Embedding degrees

- The original motivation for hyperelliptic curves was larger k.
- Since group size is q^g and embedding is into (a subfield of) $\mathbb{F}_{q^k}^*$ the right measure of security expansion is k/g.

Embedding degrees

- The original motivation for hyperelliptic curves was larger k.
- Since group size is q^g and embedding is into (a subfield of) $\mathbb{F}_{q^k}^*$ the right measure of security expansion is k/g.
- There are some nice supersingular examples.
  But supersingular curves have $k/g \leq 7.5$ for small genus g.
- Future security needs larger k/g, so use ordinary curves.

Embedding degrees

- The original motivation for hyperelliptic curves was larger k.
- Since group size is q^g and embedding is into (a subfield of) $\mathbb{F}_{q^k}^*$ the right measure of security expansion is k/g.
- There are some nice supersingular examples.
  But supersingular curves have $k/g \leq 7.5$ for small genus g.
- Future security needs larger k/g, so use ordinary curves.
- In the elliptic case there has been great success with ordinary pairing-friendly curves (e.g., Cocks-Pinch, Barreto-Lynn-Scott, Brezing-Weng, Barreto-Naehrig, Freeman-Scott-Teske).
- Much less success for hyperelliptic curves (Freeman).
  Indeed, no good non-supersingular example known, which is a pity.
  Improving these methods is an interesting research problem.

Rubin-Silverberg compression

- Rubin and Silverberg proposed an alternative way to view pairings on abelian varieties.
- They observe that many supersingular abelian varieties can be identified with subvarieties of Weil restrictions of supersingular elliptic curves.
- An alternative way to view their method is as a form of point compression for elliptic curves.
- Their method has the effect of making an elliptic curve look like it has larger embedding degree.
- Their method works for all elliptic curves, not only supersingular curves.
- For details see the paper.

Rubin-Silverberg compression

For example, the supersingular genus 2 curve over $\mathbb{F}_{2^m}$ with $k = 12$ has group order $2^{2m} \pm 2^{(3m+1)/2} + 2^m \pm 2^{(m+1)/2} + 1$.

A subgroup of the same order can be obtained using a supersingular elliptic curve over $\mathbb{F}_{2^{3m}}$ with $k = 4$.

The Rubin-Silverberg compression means that group elements require the same storage.

Hence one gets more-or-less the same functionality using elliptic curves with $k = 4$ as with the genus 2 curve with $k = 12$.

Torsion structures

- Some protocols in pairing-based cryptography (or even non-pairing cryptography, e.g., vector decomposition problem) use fact that elliptic curves give non-cyclic groups.
- Subgroup membership can be tested using the Weil pairing as, for $P, Q \neq \infty$ of prime order r we have
$$e_r(P, Q) = 1 \text{ iff } Q \in \langle P \rangle.$$
- In genus $g \geq 2$ then the torsion structures are even more rich. Are there new applications for this?

Torsion structures

Lemma: Let A be a supersingular abelian surface over $\mathbb{F}_q$ with characteristic polynomial of Frobenius $T^4 + aT^2 + q^2$.
Let $r \| \# A(\mathbb{F}_q)$
Then the eigenvalues of Frobenius on $A[r]$ are $1, -1, q, -q$.
Let D_1, D_2, D_3, D_4 be an ordered eigenbasis for $A[r]$.
Let e be a Galois invariant non-degenerate pairing.
Then

$$e(D_i, D_j) = 1$$

unless $(i, j) = (1, 3), (3, 1), (2, 4), (4, 2)$.

Conclusions

- For non-pairing cryptography there are potential advantages of using hyperelliptic curves of genus $g \geq 2$.
  It is thus natural to consider hyperelliptic curves for pairing-based cryptography.
- Our analysis indicates that, in practice, hyperelliptic curves are not more efficient than elliptic curves for general pairing applications.
- The only potentially significant advantage seems to be the speed of operations in G_1.
  Hence, hyperelliptic curves may be preferable for protocols with few pairing computations but many operations in G_1.
- But, hyperelliptic pairings still a good research area!

Open problems

- Can further loop shortening be performed for the hyperelliptic ate pairing?
- Give methods to construct non-supersingular pairing friendly curves of genus $g \geq 2$ and k in the range, say, $6g \leq k \leq 30g$. Ideally, these curves would have a single point at infinity and would have useful twists.
- Work in progress of my student Dave Mireles gives fast methods to compute pairings on hyperelliptic curves with two points at infinity.
- Can also consider implementation of pairings for non-hyperelliptic curves, although security is less good due to Diem's index calculus method.
- Consider whether efficient and secure pairing-based cryptosystems can be developed for curves of genus $g > 3$, in spite of the index calculus attacks on curves in this case.

Open problems

- Exploit the richer torsion structure available for abelian varieties.
  In particular, find cryptographic applications of pairings on groups which require 3 or more generators.
  A related problem is to give efficient methods to choose divisors in the particular subgroups.
- Improve the efficiency of the Rubin-Silverberg elliptic curve point decompression method.
- Generalise the Rubin-Silverberg method to divisor class groups of curves of genus $g \geq 2$.
- Recall in the Rubin-Silverberg construction one can identify certain abelian varieties with subvarieties of the Weil restriction of supersingular curves.
  In the case where the abelian variety is a Jacobian, is there a way to compute explicit homomorphisms between the elliptic curve representation and the Jacobian representation?

Pairing inversion

Pairing-based cryptography relies on new computational problems.

It is important that these problems are studied, to give assurance that pairing-based cryptography is secure.

See "Aspects of pairing inversion" cryptography eprint 2007/256.

Also see talk by Satoh.

Pairing inversion problems

Let G_1, G_2 and μ_r be cyclic groups of prime order r.

Consider any non-degenerate bilinear pairing

$$e : G_1 \times G_2 \longrightarrow \mu_r.$$

Fixed Argument Pairing Inversion 1 (FAPI-1): Given $P \in G_1$ and $z \in \mu_r$, compute $Q \in G_2$ such that $e(P, Q) = z$.

Fixed Argument Pairing Inversion 2 (FAPI-2): Given $Q \in G_2$ and $z \in \mu_r$, compute $P \in G_1$ such that $e(P, Q) = z$.

Evidence that these are hard problems

- Verheul considered the problem of computing a group homomorphism from μ_r to $E(\mathbb{F}_q)[r]$ and showed a number of striking consequences.
  We generalise his result.

Evidence that these are hard problems

- Verheul considered the problem of computing a group homomorphism from μ_r to $E(\mathbb{F}_q)[r]$ and showed a number of striking consequences.
  We generalise his result.
- **Theorem:** Let E be an elliptic curve with a pairing as above. Suppose one can solve FAPI-1 and FAPI-2 in polynomial time. Then the Diffie-Hellman problem in μ_r and the Diffie-Hellman problem in $E(\mathbb{F}_q)[r]$ may be solved in polynomial time.

Connection to bilinear Diffie-Hellman

Bilinear-Diffie-Hellman problem (BDH-1) is: given $P, aP, bP \in G_1$ and $Q \in G_2$ to compute $e(P,Q)^{ab}$.

Theorem: Suppose one can solve FAPI-1 in polynomial time, then one can solve BDH-1 in polynomial time.

Note: It is sufficient to be able to invert **one** pairing on $G_1 \times G_2$ to break BDH-1 for **all** pairings on $G_1 \times G_2$.

Example

The curve $E : y^2 = x^3 + 4$ over $\mathbb{F}_p$ with $p = 4176171311231184 5269$ has $t = -1$, $r = 715827883$, $k = 31$ and $D = -3$.

Now $T = -2$ and the ate pairing is

$$e(Q, P) = \left(y_P - (3x_Q^2)/(2y_Q)x_P - (-x_Q^3 + 8)/(2y_Q)\right)^{(q^k-1)/(3r)}$$

Pairing inversion for fixed (Q, z) is to find $x, y \in \mathbb{F}_p$ such that

$$(y - \lambda x - \nu)^d = z$$

for $d = (q^k - 1)/(3r)$ and $\lambda, \nu \in \mathbb{F}_{p^k}$ as above.

Example

Problem is that there are too many d-th roots of z to try.

Is it possible to efficiently determine a d-th root of z of the correct form?

Seems that the final exponentiation gives the security in this case.

Duursma-Lee example

Consider the Duursma-Lee curves

$$C : y^2 = x^p - x + b \qquad \text{where} \qquad b = \pm 1$$

over $\mathbb{F}_p$ with $p \equiv 3 \pmod 4$.

Have genus $g = (p-1)/2$ and embedding degree $k = 2p$

Take $p = 83$.

Then $\#\mathrm{Pic}^0_{\mathbb{F}_{83}}(C) \approx 2^{262}$ and $k = 2p = 166$ so $\mathbb{F}_{83^k} \approx 2^{1058}$.

Duursma-Lee example

Let $P, Q \in C(\mathbb{F}_{83})$.

The Duursma-Lee/eta pairing of P with $\psi(Q)$ for usual distortion map ψ is

$$z = (g_P(\psi(Q))/(x_{\psi(Q)} - x_P^{p^2} - 2b))^2.$$

To solve FAPI-1 for this pairing: For each square root $z^{1/2}$:
Compute $f(x, y) = g_P(x, y) - z^{1/2}(x - x_P^{p^2} - 2b)$.
Take resultant with $y^2 = x^p - x + b$ to get a polynomial in x of degree $p + 1$.
Find roots in $\mathbb{F}_p$, recover y and check solution.

Duursma-Lee example

So pairing inversion to a single point is easy.

Problem is that expect the pre-image to be a general divisor.

Solving for a general divisor involves solving a large system of polynomial equations and seems to be hard.

Conclusions

So far we have not found a single example of a pairing which can be efficiently inverted.

Hence our research supports the claim that pairing-based cryptography is secure.