

# COUNTING RICHELOT ISOGENIES BETWEEN SUPERSPECIAL ABELIAN SURFACES

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ABSTRACT. Castryck, Decru, and Smith used superspecial genus-2 curves and their Richelot isogeny graph for basing genus-2 isogeny cryptography, and recently, Costello and Smith devised an improved isogeny path-finding algorithm in the genus-2 setting. In order to establish a firm ground for the cryptographic construction and analysis, we give a new characterization of *decomposed Richelot isogenies* in terms of *involutive reduced automorphisms* of genus-2 curves over a finite field, and explicitly count such decomposed (and non-decomposed) Richelot isogenies between *superspecial* principally polarized abelian surfaces. As a corollary, we give another algebraic geometric proof of Theorem 3 in the paper of Castryck et al.

## 1. INTRODUCTION

Isogenies of supersingular elliptic curves give a computationally intractable problem even against quantum computers, and based on it, isogeny-based cryptosystems are now widely studied as one candidate for post-quantum cryptography, e.g., [3, 5, 10, 2]. Recently, by several authors, the cryptosystems are extended to higher genus isogenies, especially the genus-2 case [15, 6, 1, 4].

In particular, Castryck, Decru, and Smith [1] showed that *superspecial* genus-2 curves and their isogeny graphs give a correct foundation for constructing genus-2 isogeny cryptography. In addition, the subgraph whose vertices consist of decomposed principally polarized abelian varieties is important in cryptography since it was employed in the recent cryptanalysis by Costello and Smith [4].

Castryck et al. also presented concrete algebraic formulas for computing  $(2, 2)$ -isogenies by using the Richelot construction. In the genus-2 case, the isogenies may have decomposed principally polarized abelian surfaces as codomain, and we call them decomposed isogenies. In [1], the authors gave explicit formulas for the decomposed isogenies and a theorem stating that the number of decomposed Richelot isogenies outgoing from the Jacobian  $J(C)$  of a superspecial curve  $C$  of genus 2 is *at most six* (Theorem 3 in [1]), but they *do not precisely determine* this number. Moreover, their proof is *computer-aided*, that is, using the Gröbner basis computation.

Therefore, we revisit the isogeny counting based on an intrinsic algebraic geometric characterization. In 1960, Igusa [9] classified the curves of genus 2 with given reduced group of automorphisms, and in 1986, Ibukiyama, Katsura, and Oort [7] explicitly counted such superspecial curves according to the classification. Based on the classical results, we first count the number of Richelot isogenies from a superspecial Jacobian to decomposed surfaces (Cases (0)–(6) in Section 5) in terms of *involutive (i.e., of order 2) reduced automorphisms* which are

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called long elements. As a corollary, we give an algebraic geometric proof of Theorem 3 in [1] together with a *precise count of decomposed Richelot isogenies* (Remark 5.1). Moreover, by extending the method, we also count the total number of (decomposed) Richelot isogenies up to isomorphism outgoing from irreducible superspecial curves of genus 2 (resp. decomposed principally polarized superspecial abelian surfaces) in Theorem 6.2 (resp. Theorem 6.4).

Our paper is organized as follows: Section 2 gives mathematical preliminaries including the Igusa classification and the Ibukiyama–Katsura–Oort curve counting. Section 3 presents an abstract description of Richelot isogenies and Section 4 gives the main characterization of decomposed Richelot isogenies in terms of reduced groups of automorphisms. Section 5 counts the number of long elements of order 2 in reduced groups of automorphisms based on the results in Section 4. Section 6 gives the total numbers of (decomposed) Richelot isogenies outgoing from the irreducible superspecial curves of genus 2 and products of two elliptic curves, respectively. Section 7 gives some examples in small characteristic. Finally, Section 8 gives a concluding remark.

For an abelian surface  $A$  or a nonsingular projective variety  $X$ , we use the following notation:

- $A[n]$  : the group of  $n$ -torsion points of  $A$ ,
- $A^t$  : the dual of  $A$ ,
- $\text{NS}(A)$  : the Néron-Severi group of  $A$ ,
- $T_v$  : the translation by an element  $v$  of  $A$ ,
- $D \sim D'$  : linear equivalence for divisors  $D$  and  $D'$  on  $X$ ,
- $D \approx D'$  : numerical equivalence for divisors  $D$  and  $D'$  on  $X$ ,
- $id_X$  : the identity morphism of a variety  $X$ .

## 2. PRELIMINARIES

Let  $k$  be an algebraically closed field of characteristic  $p > 5$ . An abelian surface  $A$  defined over  $k$  is said to be superspecial if  $A$  is isomorphic to  $E_1 \times E_2$  with  $E_i$  supersingular elliptic curves ( $i = 1, 2$ ). Since for any supersingular elliptic curves  $E_i$  ( $i = 1, 2, 3, 4$ ) we have an isomorphism  $E_1 \times E_2 \cong E_3 \times E_4$  (cf. Shioda [14, Theorem 3.5], for instance), this notion does not depend on the choice of supersingular elliptic curves. For a nonsingular projective curve  $C$  of genus 2, we denote by  $J(C)$  the (canonically polarized) Jacobian variety of  $C$ . The curve  $C$  is said to be superspecial if  $J(C)$  is superspecial as an abelian surface. We denote by  $\text{Aut}(C)$  the group of automorphisms of  $C$ . Since  $C$  is hyperelliptic,  $C$  has the hyperelliptic involution  $\iota$  such that the quotient curve  $C/\langle \iota \rangle$  is isomorphic to the projective line  $\mathbf{P}^1$ :

$$\psi : C \longrightarrow \mathbf{P}^1.$$

There exist 6 ramification points on  $C$ . We denote them by  $P_i$  ( $1 \leq i \leq 6$ ). Then,  $Q_i = \psi(P_i)$ 's are the branch points of  $\psi$  on  $\mathbf{P}^1$ . The group  $\langle \iota \rangle$  is a normal subgroup of  $\text{Aut}(C)$ . We put  $\text{RA}(C) \cong \text{Aut}(C)/\langle \iota \rangle$  and we call it the reduced group of automorphisms of  $C$ . We call an element of  $\text{RA}(C)$  a reduced automorphism of  $C$ . For  $\sigma \in \text{RA}(C)$ ,  $\tilde{\sigma}$  is an element of  $\text{Aut}(C)$  such that  $\tilde{\sigma} \bmod \langle \iota \rangle = \sigma$ .

**Definition 2.1.** An element  $\sigma \in \text{RA}(C)$  of order 2 is said to be long if  $\tilde{\sigma}$  is of order 2. Otherwise, an element  $\sigma \in \text{RA}(C)$  of order 2 is said to be short (cf. Katsura–Oort [11, Definition 7.15]).

This definition does not depend on the choice of  $\tilde{\sigma}$ .

**Lemma 2.2.** *If an element  $\sigma \in \text{RA}(C)$  of order 2 acts freely on 6 branch points, then  $\sigma$  is long.*

*Proof.* By a suitable choice of coordinate  $x$  of  $\mathbf{A}^1 \subset \mathbf{P}^1$ , taking 0 as a fixed point of  $\sigma$ , we may assume  $\sigma(x) = -x$ , and  $Q_1 = 1, Q_2 = -1, Q_3 = a, Q_4 = -a, Q_5 = b, Q_6 = -b$  ( $a \neq 0, \pm 1; b \neq 0, \pm 1; a \neq \pm b$ ). Then, the curve is defined by

$$y^2 = (x^2 - 1)(x^2 - a^2)(x^2 - b^2),$$

and  $\tilde{\sigma}$  is given by  $x \mapsto -x, y \mapsto \pm y$ . Therefore,  $\tilde{\sigma}$  is of order 2.  $\square$

**Lemma 2.3.** *If  $\text{RA}(C)$  has an element  $\sigma$  of order 2, then there exists a long element  $\tau \in \text{RA}(C)$  of order 2.*

*Proof.* If  $\sigma$  acts freely on 6 branch points, then by Lemma 2.2,  $\sigma$  itself is a long element of order 2. We assume that the branch point  $Q_1 = \psi(P_1)$  is a fixed point of  $\sigma$ . Since  $\sigma$  is of order 2, it must have one more fixed point among the branch points, say  $Q_2 = \psi(P_2)$ . By a suitable choice of coordinate  $x$  of  $\mathbf{A}^1 \subset \mathbf{P}^1$ , we may assume  $Q_1 = 0$  and  $Q_2 = \infty$ . We may also assume  $Q_3 = 1$ . Then,  $\sigma$  is given by  $x \mapsto -x$  and the six branch points are  $0, 1, -1, a, -a, \infty$  ( $a \neq \pm 1$ ). The curve  $C$  is given by

$$y^2 = x(x^2 - 1)(x^2 - a^2) \quad (a \neq 0, \pm 1).$$

We consider an element  $\tau \in \text{Aut}(\mathbf{P}^1)$  defined by  $x \mapsto \frac{a}{x}$ . Then, we have an automorphism  $\tilde{\tau}$  of  $C$  defined by  $x \mapsto \frac{a}{x}, y \mapsto \frac{a\sqrt{ay}}{x^3}$ . Therefore, we see  $\tau \in \text{RA}(C)$ . Since  $\tilde{\tau}$  is of order 2,  $\tau$  is long.  $\square$

$\text{RA}(C)$  acts on the projective line  $\mathbf{P}^1$  as a subgroup of  $\text{PGL}_2(k)$ . The structure of  $\text{RA}(C)$  is classified as follows (cf. Igusa [9, p. 644], and Ibukiyama–Katsura–Oort [7, p. 130]):

$$(0) 0, (1) \mathbf{Z}/2\mathbf{Z}, (2) S_3, (3) \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, (4) D_{12}, (5) S_4, (6) \mathbf{Z}/5\mathbf{Z}.$$

We denote by  $n_i$  the number of superspecial curves of genus 2 whose reduced group of automorphisms is isomorphic to the group  $(i)$ . Then,  $n_i$ 's are given as follows (cf. Ibukiyama–Katsura–Oort [7, Theorem 3.3]):

$$\begin{aligned} (0) \quad n_0 &= (p-1)(p^2-35p+346)/2880 - \{1 - (\frac{-1}{p})\}/32 - \{1 - (\frac{-2}{p})\}/8 - \{1 - (\frac{-3}{p})\}/9 \\ &\quad + \begin{cases} 0 & \text{if } p \equiv 1, 2 \text{ or } 3 \pmod{5}, \\ -1/5 & \text{if } p \equiv 4 \pmod{5}, \end{cases} \\ (1) \quad n_1 &= (p-1)(p-17)/48 + \{1 - (\frac{-1}{p})\}/8 + \{1 - (\frac{-2}{p})\}/2 + \{1 - (\frac{-3}{p})\}/2, \\ (2) \quad n_2 &= (p-1)/6 - \{1 - (\frac{-2}{p})\}/2 - \{1 - (\frac{-3}{p})\}/3, \\ (3) \quad n_3 &= (p-1)/8 - \{1 - (\frac{-1}{p})\}/8 - \{1 - (\frac{-2}{p})\}/4 - \{1 - (\frac{-3}{p})\}/2, \\ (4) \quad n_4 &= \{1 - (\frac{-3}{p})\}/2, \\ (5) \quad n_5 &= \{1 - (\frac{-2}{p})\}/2, \\ (6) \quad n_6 &= \begin{cases} 0 & \text{if } p \equiv 1, 2 \text{ or } 3 \pmod{5}, \\ 1 & \text{if } p \equiv 4 \pmod{5}. \end{cases} \end{aligned}$$

Here, for a prime number  $q$  and an integer  $a$ ,  $(\frac{a}{q})$  is the Legendre symbol. The total number  $n$  of superspecial curves of genus 2 is given by

$$\begin{aligned} n &= n_0 + n_1 + n_2 + n_3 + n_4 + n_5 + n_6 \\ &= (p-1)(p^2 + 25p + 166)/2880 - \{1 - (\frac{-1}{p})\}/32 + \{1 - (\frac{-2}{p})\}/8 \\ &\quad + \{1 - (\frac{-3}{p})\}/18 + \begin{cases} 0 & \text{if } p \equiv 1, 2 \text{ or } 3 \pmod{5}, \\ 4/5 & \text{if } p \equiv 4 \pmod{5}. \end{cases} \end{aligned}$$

For an abelian surface  $A$ , we have  $A^t = \text{Pic}^0(A)$  (Picard variety of  $A$ ), and for a divisor  $D$  on  $A$ , there exists a homomorphism

$$\begin{aligned} \varphi_D : A &\longrightarrow A^t \\ v &\longmapsto T_v^* D - D. \end{aligned}$$

If  $D$  is ample, then  $\varphi_D$  is surjective, i.e., an isogeny. We know  $(D \cdot D)^2 = 4 \deg \varphi_D$ . We set  $K(D) = \text{Ker } \varphi_D$ . If  $D$  is ample, then  $K(D)$  is finite and it has a non-degenerate alternating bilinear form  $e^D(v, w)$  on  $K(D)$  (cf. Mumford [13, Section 23]). Let  $G$  be an isotropic subgroup scheme of  $K(D)$  with respect to  $e^D(v, w)$ . In case  $D$  is ample,  $G$  is finite and we have an isogeny

$$\pi : A \longrightarrow A/G.$$

The following theorem is due to Mumford [13, Section 23, Theorem 2, Corollary]:

**Theorem 2.4.** *Let  $G$  be an isotropic subgroup scheme of  $K(D)$ . Then, there exists a divisor  $D'$  on  $A/G$  such that  $\pi^* D' \sim D$ .*

Let  $n$  be a positive integer which is prime to  $p$ . Then, we have the Weil pairing  $e_n : A[n] \times A^t[n] \longrightarrow \mu_n$ . Here,  $\mu_n$  is the multiplicative group of order  $n$ . By Mumford [13, Section 23 “Functional Properties of  $e^L(5)$ ”], we have the following.

**Lemma 2.5.** *For  $v \in A[n]$  and  $w \in \varphi_D^{-1}(A^t[n])$ , we have*

$$e_n(v, \varphi_D(w)) = e^{nD}(v, w).$$

If  $D$  is a principal polarization, the homomorphism  $\varphi_D : A \longrightarrow A^t$  is an isomorphism. Therefore, by this identification we can identify the pairing  $e^{nD}$  with the Weil pairing  $e_n$ .

### 3. RICHELLOT ISOGENIES

We recall the abstract description of Richelot isogenies. (For the concrete construction of Richelot isogenies, see Castryck–Decru–Smith [1, Section 3], for instance.)

Let  $A$  be an abelian surface with a principal polarization  $C$ . Then, we may assume that  $C$  is effective, and we have the self-intersection number  $C^2 = 2$ . It is easy to show (or as was shown by A. Weil) that there are two cases for effective divisors with self-intersection 2 on an abelian surface  $A$ :

(1) There exists a nonsingular curve  $C$  of genus 2 such that  $A$  is isomorphic to the Jacobian variety  $J(C)$  of  $C$  and that  $C$  is the divisor with self-intersection 2. In this case,  $(J(C), C)$  is said to be non-decomposed.

(2) There exist two elliptic curves  $E_1, E_2$  with  $(E_1 \cdot E_2) = 1$  such that  $E_1 \times \{0\} + \{0\} \times E_2$  is a divisor with self-intersection 2 and that  $A \cong E_1 \times E_2$ . In this case,  $(A, E_1 \times \{0\} + \{0\} \times E_2)$  is said to be decomposed.

Since  $\varphi_C$  is an isomorphism by the fact that  $C$  is a principal polarization, we have  $K(2C) = \text{Ker } \varphi_{2C} = \text{Ker } 2\varphi_C = A[2]$ . Let  $G$  be a maximal isotropic subgroup of  $K(2C) = A[2]$  with respect to the pairing  $e^{2C}$ . Since we have  $|G|^2 = |A[2]| = 2^4$  (cf. Mumford [13, Section 23, Theorem 4]), we have  $|G| = 4$  and  $G \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ . We have a quotient homomorphism

$$\pi : A \longrightarrow A/G.$$

By Theorem 2.4, there exists a divisor  $C'$  on  $A/G$  such that  $2C \sim \pi^* C'$ . Since  $\pi$  is a finite morphism and  $2C$  is ample, we see that  $C'$  is also ample. We have the self-intersection number  $(2C \cdot 2C) = 8$ , and we have

$$8 = (2C \cdot 2C) = (\pi^* C' \cdot \pi^* C') = \deg \pi (C' \cdot C') = 4(C' \cdot C').$$

Therefore, we have  $(C' \cdot C') = 2$ , that is,  $C'$  is a principal polarization on  $A/G$ . By the Riemann–Roch theorem of an abelian surface for ample divisors, we have

$$\dim H^0(A/G, \mathcal{O}_{A/G}(C')) = (C' \cdot C')/2 = 1.$$

Therefore, we may assume  $C'$  is an effective divisor.

Using these facts, we see that  $C'$  is either a nonsingular curve of genus 2 or  $E_1 \cup E_2$  with elliptic curves  $E_i$  ( $i = 1, 2$ ) which intersect each other transversely. In this situation, the correspondence from  $(A, C)$  to  $(A/G, C')$  is called a Richelot isogeny. It is easy to show that if there exists a Richelot isogeny from  $(A, C)$  to  $(A/G, C')$ , then there exists a Richelot isogeny from  $(A/G, C')$  to  $(A, C)$ .

Now, we consider the case where  $A$  is a superspecial abelian surface. Then, since  $\pi$  is separable,  $A/G$  is also a superspecial abelian surface. We will use this fact freely.

From here on, for abelian surface  $E_1 \times E_2$  with elliptic curves  $E_i$  ( $i = 1, 2$ ) we denote by  $E_1 + E_2$  the divisor  $E_1 \times \{0\} + \{0\} \times E_2$ , if no confusion occurs. We sometimes call  $E_1 \times E_2$  a principally polarized abelian surface. In this case, the principal polarization on  $E_1 \times E_2$  is given by  $E_1 + E_2$ .

**Definition 3.1.** Let  $(A, C)$ ,  $(A', C')$  and  $(A'', C'')$  be principally polarized abelian surfaces with principal polarizations  $C, C', C''$ , respectively. The Richelot isogeny  $\pi : A \rightarrow A'$  is said to be isomorphic to the Richelot isogeny  $\varpi : A \rightarrow A''$  if there exist an automorphism  $\sigma \in A$  with  $\sigma^*C \approx C$  and an isomorphism  $g : A' \rightarrow A''$  with  $g^*C'' \approx C'$  such that the following diagram commutes:

$$\begin{array}{ccc} \sigma : & A & \longrightarrow & A \\ & \pi \downarrow & & \downarrow \varpi \\ g : & A' & \longrightarrow & A'' \end{array}$$

#### 4. DECOMPOSED RICHELOT ISOGENIES

In this section, we use the same notation as in Section 3.

**Definition 4.1.** Let  $A$  and  $A'$  be abelian surfaces with principal polarizations  $C, C'$ , respectively. A Richelot isogeny  $A \rightarrow A'$  is said to be decomposed if  $C'$  consists of two elliptic curves. Otherwise, the Richelot isogeny is said to be non-decomposed.

**Example 4.2.** Let  $C_{a,b}$  be a nonsingular projective model of the curve of genus 2 defined by the equation

$$y^2 = (x^2 - 1)(x^2 - a)(x^2 - b) \quad (a \neq 0, 1; b \neq 0, 1; a \neq b).$$

Let  $\iota$  be the hyperelliptic involution defined by  $x \mapsto x, y \mapsto -y$ .  $\text{RA}(C_{a,b})$  has an element of order 2 defined by

$$\sigma : x \mapsto -x, y \mapsto y.$$

We put  $\tau = \iota \circ \sigma$ . We have two elliptic curves  $E_\sigma = C_{a,b}/\langle \sigma \rangle$  and  $E_\tau = C_{a,b}/\langle \tau \rangle$ . The elliptic curve  $E_\sigma$  is isomorphic to an elliptic curve  $E_\lambda : y^2 = x(x-1)(x-\lambda)$  with

$$(4.1) \quad \lambda = (b-a)/(1-a)$$

and the elliptic curve  $E_\tau$  is isomorphic to an elliptic curve  $E_\mu : y^2 = x(x-1)(x-\mu)$  with

$$(4.2) \quad \mu = (b-a)/b(1-a).$$

The map given by (4.1) and (4.2) yields a bijection

$$\begin{aligned} & \{(a, b) \mid a, b \in k; a \neq 0, 1; b \neq 0, 1; a \neq b, \text{ and } J(C_{a,b}) \text{ is superspecial}\} \\ & \longrightarrow \{(\lambda, \mu) \mid \lambda, \mu \in k; \lambda \neq \mu; E_\lambda, E_\mu \text{ are supersingular}\} \end{aligned}$$

(for the details, see Katsura–Oort [12, p. 259]). We have a natural morphism  $C_{a,b} \longrightarrow E_\sigma \times E_\tau$  and this morphism induces an isogeny

$$\pi : J(C_{a,b}) \longrightarrow E_\sigma \times E_\tau.$$

By Igusa [9, p. 648], we know  $\text{Ker } \pi \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$  and  $\text{Ker } \pi$  consists of  $P_1 - \sigma(P_1)$ ,  $P_3 - \sigma(P_3)$ ,  $P_5 - \sigma(P_5)$  and the zero point. Here,  $P_1 = (1, 0)$ ,  $P_3 = (a, 0)$ ,  $P_5 = (b, 0)$ . Since  $P_i - \sigma(P_i)$  is a divisor of order 2, we have  $P_i - \sigma(P_i) \sim \sigma(P_i) - P_i$ .

Comparing the calculation in Castryck–Decru–Smith [1, Proposition 1 (2)] with the one in Katsura–Oort [12, Lemma 2.4], we see that  $\pi : J(C_{a,b}) \longrightarrow E_\sigma \times E_\tau$  is a decomposed Richelot isogeny with  $C'_{a,b} = E_\sigma + E_\tau$  (also see Katsura–Oort [11, Proof of Proposition 7.18 (iii)]). We will use the bijection above to calculate decomposed Richelot isogenies.

**Proposition 4.3.** *Let  $C$  be a nonsingular projective curve of genus 2. Then, the following three conditions are equivalent.*

- (i)  $C$  has a decomposed Richelot isogeny outgoing from  $J(C)$ .
- (ii)  $\text{RA}(C)$  has an element of order 2.
- (iii)  $\text{RA}(C)$  has a long element of order 2.

*Proof.* (i)  $\Rightarrow$  (ii). By assumption, we have a Richelot isogeny

$$(4.3) \quad \pi : J(C) \longrightarrow J(C)/G$$

such that  $G$  is an isotropic subgroup of  $J(C)[2]$  with respect to  $2C$ , and that  $C'$  is a principal polarization consisting of two elliptic curves  $E_i$  ( $i = 1, 2$ ) on  $J(C)/G$  with  $2C \sim \pi^*(E_1 + E_2)$ . Since  $C$  is a principal polarization, we have an isomorphism  $\varphi_C : J(C) \cong J(C)^t$ . In a similar way, we have  $J(C)/G \cong (J(C)/G)^t$ . Dualizing (4.3), we have

$$\eta = \pi^t : J(C)/G \longrightarrow J(C)$$

with  $J(C)/G \cong E_1 \times E_2$ ,  $C' = E_1 + E_2$  and  $\eta^*(C) \sim 2(E_1 + E_2)$ . The kernel  $\text{Ker } \eta$  is an isotropic subgroup of  $(E_1 \times E_2)[2]$  with respect to the divisor  $2(E_1 + E_2)$ .

Denoting by  $\iota_{E_1}$  the inversion of  $E_1$ , we set

$$\bar{\tau} = \iota_{E_1} \times id_{E_2}.$$

Then,  $\bar{\tau}$  is an automorphism of order 2 which is not the inversion of  $E_1 \times E_2$ . By the definition, we have

$$\bar{\tau}^*(E_1 + E_2) = E_1 + E_2.$$

Moreover, since  $\text{Ker } \eta$  consists of elements of order 2 and  $\bar{\tau}$  fixes the elements of order 2,  $\bar{\tau}$  preserves  $\text{Ker } \eta$ . Therefore,  $\bar{\tau}$  induces an automorphism  $\tau : J(C) \cong (J(C)/G)/\text{Ker } \eta \cong (E_1 \times E_2)/\text{Ker } \eta$ . Therefore, we have the following diagram:

$$\begin{array}{ccc} E_1 \times E_2 & \xrightarrow{\bar{\tau}} & E_1 \times E_2 \\ \eta \downarrow & & \downarrow \eta \\ J(C) & \xrightarrow{\tau} & J(C). \end{array}$$

We have

$$\eta^* \tau^* C = \bar{\tau}^* \eta^* C = \bar{\tau}^*(2(E_1 + E_2)) = 2(E_1 + E_2).$$

On the other hand, we have

$$\eta^* C = 2(E_1 + E_2).$$

Since  $\eta^*$  is an injective homomorphism from  $\text{NS}(J(C))$  to  $\text{NS}(E_1 \times E_2)$ , we have  $C \approx \tau^* C$ . Therefore,  $\tau^* C - C$  is an element of  $\text{Pic}^0(J(C)) = J(C)^t$ . Since  $C$  is ample, the

homomorphism

$$\begin{aligned} \varphi_C : J(C) &\longrightarrow J(C)^t \\ v &\longmapsto T_v^*C - C \end{aligned}$$

is surjective. Therefore, there exists an element  $v \in J(C)$  such that

$$T_v^*C - C \sim \tau^*C - C,$$

that is,  $T_v^*C \sim \tau^*C$ . Since  $T_v^*C$  is a principal polarization, we see

$$\dim H^0(J(C), \mathcal{O}_{J(C)}(T_v^*C)) = 1.$$

Therefore, we have  $T_v^*C = \tau^*C$ , that is,  $T_{-v}^*\tau^*C = C$ . Since  $\tau$  is of order 2, we have  $(\tau \circ T_{-v})^2 = T_{-v-\tau(v)}$ , a translation. Therefore, we have  $T_{-v-\tau(v)}^*C = C$ . However, since  $C$  is a principal polarization, we have  $\text{Ker } \varphi_C = \{0\}$ . Therefore, we have  $T_{-v-\tau(v)} = id$ . This means  $\tau \circ T_{-v}$  is an automorphism of order 2 of  $C$ . By definition, this is not the inversion  $\iota$ . Hence, this gives an element of order 2 in  $\text{RA}(C)$ .

(ii)  $\Rightarrow$  (iii) This follows from Lemma 2.3.

(iii)  $\Rightarrow$  (i) This follows from Lemma 2.2 and Example 4.2.  $\square$

*Remark 4.4.* In the proof of the proposition, the automorphism  $\tau \circ T_{-v}$  really gives a long element of order 2 in  $\text{RA}(C)$ .

By Castryck–Decru–Smith [1, Subsection 3.3], if the curve  $C$  of genus 2 is obtained from a decomposed principally polarized abelian surface by a Richelot isogeny, then the curve  $C$  has a long reduced automorphism of order 2. As is well-known, for a curve  $C$  of genus 2, the Jacobian variety  $J(C)$  has 15 Richelot isogenies (cf. Castryck–Decru–Smith [1, Subsection 3.2], for instance). If we have a Richelot isogeny  $(A, C) \longrightarrow (A', C')$ , then we also have a Richelot isogeny  $(A', C') \longrightarrow (A, C)$ . Therefore, we have the following proposition.

**Proposition 4.5.** *Let  $C$  be a nonsingular projective curve of genus 2. Among 15 Richelot isogenies outgoing from  $J(C)$ , the number of decomposed Richelot isogenies is equal to the number of long elements of order 2 in  $\text{RA}(C)$ .*

In this proposition, we consider that a different isotropic subgroup gives a different Richelot isogeny. However, two different Richelot isogenies may be isomorphic to each other by a suitable automorphism (see Definition 3.1). From the next section, we will compute the number of Richelot isogenies up to isomorphism.

## 5. THE NUMBER OF LONG ELEMENTS OF ORDER 2

In this section, we count the number of long elements of order 2 in  $\text{RA}(C)$ . For an element  $f \in \text{RA}(C)$ , we express the reduced automorphism by

$$f : x \mapsto f(x)$$

with a suitable coordinate  $x$  of  $\mathbf{A}^1 \subset \mathbf{P}^1$ . We will give the list of  $f(x)$  corresponding to elements of order 2. Here, we denote by  $\omega$  a primitive cube root of unity, by  $i$  a primitive fourth root of unity, and by  $\zeta$  a primitive sixth root of unity.

Case (0)  $\text{RA}(C) \cong \{0\}$ .

There exist no long elements of order 2.

Case (1)  $\text{RA}(C) \cong \mathbf{Z}/2\mathbf{Z}$ .

The curve  $C$  is given by  $y^2 = (x^2 - 1)(x^2 - a^2)(x^2 - b^2)$ .

There exists only one long element of order 2 given by  $f(x) = -x$ .

Case (2)  $\text{RA}(C) \cong S_3$ .

The curve  $C$  is given by  $y^2 = (x^3 - 1)(x^3 - a^3)$ .

There exist three long elements of order 2 given by  $f(x) = \frac{a}{x}, \frac{\omega a}{x}, \frac{\omega^2 a}{x}$ .

Case (3)  $\text{RA}(C) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ .

The curve  $C$  is given by  $y^2 = x(x^2 - 1)(x^2 - a^2)$ .

There exist two long elements of order 2 given by  $f(x) = \frac{a}{x}, \frac{-a}{x}$ ,  
and there exists one short element of order 2 given by  $f(x) = -x$ .

Case (4)  $\text{RA}(C) \cong D_{12}$ .

The curve is given by  $y^2 = x^6 - 1$ .

There exist four long elements of order 2 given by  $f(x) = -x, \frac{\zeta}{x}, \frac{\zeta^3}{x}, \frac{\zeta^5}{x}$ ,  
and there exist three short elements of order 2 given by  $f(x) = \frac{1}{x}, \frac{\zeta^2}{x}, \frac{\zeta^4}{x}$ .

Case (5)  $\text{RA}(C) \cong S_4$ .

The curve  $C$  is given by  $y^2 = x(x^4 - 1)$ .

There exist six long elements of order 2 given by  $f(x) = \frac{x+1}{x-1}, -\frac{x-1}{x+1}, \frac{i(x+i)}{x-i}, \frac{i}{x}, -\frac{i}{x}, -\frac{i(x-i)}{x+i}$ ,  
and there exist three short elements of order 2 given by  $f(x) = -x, \frac{1}{x}, -\frac{1}{x}$ .

Case (6)  $\text{RA}(C) \cong \mathbf{Z}/5\mathbf{Z}$ .

The curve is given by  $y^2 = x^5 - 1$ .

There exist no long elements of order 2.

*Remark 5.1.* By Proposition 4.5 and the calculation above, we see that for a curve  $C$  of genus 2, the number of outgoing decomposed Richelot isogenies from  $J(C)$  is at most six. This result coincides with the one given by Castryck–Decru–Smith [1, Theorem 3].

## 6. COUNTING RICHELLOT ISOGENIES

**6.1. Richelot isogenies from Jacobians of irreducible genus-2 curves.** Let  $C$  be a non-singular projective curve of genus 2, and let  $J(C)$  be the Jacobian variety of  $C$ . For a fixed  $C$ , we consider the set  $\{(J(C), G)\}$  of pairs of  $J(C)$  and an isotropic subgroup  $G$  for the polarization  $2C$ . The group  $\text{Aut}(C)$  acts on the ramification points of  $C \rightarrow \mathbf{P}^1$ . Using this action,  $\text{Aut}(C)$  induces the action on the set  $\{(J(C), G)\}$ . Since the inversion  $\iota$  of  $C$  acts on  $J(C)[2]$  trivially, the reduced group  $\text{RA}(C)$  of automorphisms acts on the set  $\{(J(C), G)\}$  which consists of 15 elements.

Let  $P_i$  ( $i = 1, 2, \dots, 6$ ) be the ramification points of  $\psi : C \rightarrow \mathbf{P}^1$ . A division into the sets of 3 pairs of these 6 points gives an isotropic subgroup  $G$ , that is,

$$\{P_{i_1} - P_{i_2}, P_{i_3} - P_{i_4}, P_{i_5} - P_{i_6}, \text{the identity}\}$$

gives an isotropic subgroup of  $J(C)[2]$ . The action of  $\text{RA}(C)$  on the set  $\{(J(C), G)\}$  is given by the action of  $\text{RA}(C)$  on the set

$$\{((P_{i_1}, P_{i_2}), (P_{i_3}, P_{i_4}), (P_{i_5}, P_{i_6}))\},$$

which contains 15 sets. Here, the pair  $(P_i, P_j)$  is unordered. In this section, we count the number of orbits of this action for each case.

Let  $C$  be a curve of genus 2 with  $\text{RA}(C) \cong \mathbf{Z}/2\mathbf{Z}$ . Such a curve is given by the equation

$$y^2 = (x^2 - 1)(x^2 - a)(x^2 - b)$$

with suitable conditions for  $a$  and  $b$ . The branch points  $Q_i = \psi(P_i)$  are given by

$$Q_1 = 1, Q_2 = -1, Q_3 = \sqrt{a}, Q_4 = -\sqrt{a}, Q_5 = \sqrt{b}, Q_6 = -\sqrt{b}.$$

The generator of the reduced group  $\text{RA}(C)$  of automorphisms is given by

$$\sigma : x \mapsto -x.$$



Since the inversion  $\iota$  acts trivially on the ramification points,  $\text{RA}(C)$  acts on the set of the ramification points  $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ , and the action of  $\sigma$  on the ramification points is given by

$$P_{2i-1} \mapsto P_{2i}, P_{2i} \mapsto P_{2i-1} \quad (i = 1, 2, 3).$$

The isotropic subgroup which corresponds to  $\langle (P_1, P_2), (P_3, P_4), (P_5, P_6) \rangle$  gives a decomposed Richelot isogeny and the other isotropic subgroups give non-decomposed isogenies. Moreover,  $\langle (\sigma(P_{i_1}), \sigma(P_{i_2})), (\sigma(P_{i_3}), \sigma(P_{i_4})), (\sigma(P_{i_5}), \sigma(P_{i_6})) \rangle$  gives the Richelot isogeny isomorphic to the one given by  $\langle (P_{i_1}, P_{i_2}), (P_{i_3}, P_{i_4}), (P_{i_5}, P_{i_6}) \rangle$ . We denote  $P_i$  by  $i$  for the sake of simplicity. Then, the action  $\sigma$  is given by the permutation  $(1, 2)(3, 4)(5, 6)$ , and by the action of  $\text{RA}(C)$ , the set  $\{ \langle (P_{i_1}, P_{i_2}), (P_{i_3}, P_{i_4}), (P_{i_5}, P_{i_6}) \rangle \}$  of 15 elements is divided into the following 11 loci:

$$\begin{aligned} & \{[(1, 2), (3, 4), (5, 6)]\}, \{[(1, 2), (3, 5), (4, 6)]\}, \{[(1, 2), (3, 6), (4, 5)]\}, \\ & \{[(1, 3), (2, 4), (5, 6)]\}, \{[(1, 3), (2, 5), (4, 6)], [(1, 6), (2, 4), (3, 5)]\}, \\ & \{[(1, 3), (2, 6), (4, 5)], [(1, 5), (2, 4), (3, 6)]\}, \{[(1, 4), (2, 3), (5, 6)]\}, \\ & \{[(1, 4), (2, 5), (3, 6)], [(1, 6), (2, 3), (4, 5)]\}, \{[(1, 4), (2, 6), (3, 5)], [(1, 5), (2, 3), (4, 6)]\}, \\ & \{[(1, 5), (2, 6), (3, 4)]\}, \{[(1, 6), (2, 5), (3, 4)]\}. \end{aligned}$$

The reduced automorphism  $\sigma$  is a long one of order 2 and the element  $[(1, 2), (3, 4), (5, 6)]$  is pairwise fixed by  $\sigma$ . Therefore, the element  $[(1, 2), (3, 4), (5, 6)]$  gives a decomposed isogeny. The other 10 loci give non-decomposed isogenies. In the same way, we have the following proposition.

**Proposition 6.1.** *Under the notation above, the number of Richelot isogenies up to isomorphism in each case and the number of elements in each orbit are listed as follows. Here, in the list, for example,  $(1 \times 6, 2 \times 4)(1 \times 1)$  means that there exist 6 orbits which contain one element and 4 orbits which contain 2 elements for non-decomposed Richelot isogenies, and there exists one orbit which contains one element for decomposed Richelot isogenies.*

- (0)  $\text{RA}(C) \cong \{0\}$ : 15 Richelot isogenies. No decomposed one.  
(1 × 15)(0).
- (1)  $\text{RA}(C) \cong \mathbf{Z}/2\mathbf{Z}$ : 11 Richelot isogenies. 1 decomposed one.  
(1 × 6, 2 × 4)(1 × 1).
- (2)  $\text{RA}(C) \cong S_3$ : 7 Richelot isogenies. 1 decomposed one.  
(1 × 3, 3 × 3)(3 × 1).
- (3)  $\text{RA}(C) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ : 8 Richelot isogenies. 2 decomposed ones.  
(1 × 1, 2 × 4, 4 × 1)(1 × 2).
- (4)  $\text{RA}(C) \cong D_{12}$ : 5 Richelot isogenies. 2 decomposed ones.  
(2 × 1, 3 × 1, 6 × 1)(1 × 1, 3 × 1).
- (5)  $\text{RA}(C) \cong S_4$ : 4 Richelot isogenies. 1 decomposed one.  
(1 × 1, 4 × 2)(6 × 1).
- (6)  $\text{RA}(C) \cong \mathbf{Z}/5\mathbf{Z}$ : 3 Richelot isogenies. No decomposed one.  
(5 × 3)(0).

**Theorem 6.2.** *The total number of Richelot isogenies up to isomorphism outgoing from the irreducible superspecial curves of genus 2 is equal to*

$$\frac{(p-1)(p+2)(p+7)}{192} - 3\left\{1 - \left(\frac{-1}{p}\right)\right\}/32 + \left\{1 - \left(\frac{-2}{p}\right)\right\}/8.$$

The total number of decomposed Richelot isogenies up to isomorphism outgoing from the irreducible superspecial curves of genus 2 is equal to

$$(6.1) \quad \frac{(p-1)(p+3)}{48} - \left\{1 - \left(\frac{-1}{p}\right)\right\}/8 + \left\{1 - \left(\frac{-3}{p}\right)\right\}/6.$$

*Proof.* The total number of Richelot isogenies up to isomorphism outgoing from the irreducible superspecial curves of genus 2 is equal to

$$15n_0 + 11n_1 + 7n_2 + 8n_3 + 5n_4 + 4n_5 + 3n_6$$

and the total number of decomposed Richelot isogenies up to isomorphism outgoing from the irreducible superspecial curves of genus 2 is equal to

$$n_1 + n_2 + 2n_3 + 2n_4 + n_5.$$

The results follow from these facts.  $\square$

**6.2. Richelot isogenies from elliptic curve products.** Let  $E, E'$  be supersingular elliptic curves, and we consider a decomposed principal polarization  $E + E'$  and a Richelot isogeny  $(E \times E', E + E') \rightarrow (J(C), C)$ . For the principally polarized abelian surface  $(E \times E', E + E')$ , we put  $\text{RA}(E \times E') = \text{Aut}(E)/\langle \iota_E \rangle \times \text{Aut}(E')/\langle \iota_{E'} \rangle$ , and call it the reduced group of automorphisms of  $E \times E'$ . Let  $\{P_1, P_2, P_3\}$  (resp.  $\{P_4, P_5, P_6\}$ ) be the 2-torsion points of  $E$  (resp.  $E'$ ). Then, the six points  $P_i$  ( $1 \leq i \leq 6$ ) on  $E \times E'$  play the role of ramification points of irreducible curves of genus 2, and  $\text{RA}(E \times E')$  acts on the set  $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ . In this section, let  $E_2$  be the elliptic curve defined by  $y^2 = x^3 - x$  and  $E_3$  the elliptic curve defined by  $y^2 = x^3 - 1$ . We know  $\text{Aut}E_2 \cong \mathbf{Z}/4\mathbf{Z}$  and  $\text{Aut}E_3 \cong \mathbf{Z}/6\mathbf{Z}$ . The elliptic curve  $E_2$  is supersingular if and only if  $p \equiv 3 \pmod{4}$  and  $E_3$  is supersingular if and only if  $p \equiv 2 \pmod{3}$ . In this section, the abelian surface  $E \times E'$  means an abelian surface  $E \times E'$  with principal polarization  $E + E'$ .

Now, let  $E, E'$  be supersingular elliptic curves which are neither isomorphic to  $E_2$  nor to  $E_3$ . We also assume  $E$  is not isomorphic to  $E'$ . Using these notations, we have the following list of the orders of reduced groups of automorphisms.

$$\begin{aligned} |\text{RA}(E \times E')| &= 1, & |\text{RA}(E \times E_2)| &= 2, & |\text{RA}(E \times E_3)| &= 3, \\ |\text{RA}(E_2 \times E_2)| &= 8, & |\text{RA}(E_3 \times E_3)| &= 18, & |\text{RA}(E_2 \times E_3)| &= 6. \end{aligned}$$

The isotropic subgroups for the polarization  $2(E + E')$  are determined in Castryck–Decru–Smith [1, Subsection 3.3]. Using their results and the same method as in Subsection 6.1, we have the following proposition.

**Proposition 6.3.** *Let  $E, E'$  be supersingular elliptic curves which are neither isomorphic to  $E_2$  nor to  $E_3$  with  $E_2$  and  $E_3$  defined as above. We also assume that  $E$  is not isomorphic to  $E'$ . The number of Richelot isogenies up to isomorphism outgoing from a decomposed principally polarized superspecial abelian surface in each case and the number of elements in each orbit are listed as follows. Here, in the list, for example,  $(1 \times 3, 2 \times 1)(1 \times 4, 2 \times 3)$  means that there exist 3 orbits which contain one element and one orbit which contains 2 elements for non-decomposed Richelot isogenies, and there exist 4 orbits which contain one element and 3 orbits which contain 2 elements for decomposed Richelot isogenies.*

- (i)  $E \times E'$  : 15 Richelot isogenies, 6 non-decomposed ones.  
(1 × 6)(1 × 9).
- (ii)  $E \times E$  : 11 Richelot isogenies, 4 non-decomposed ones.  
(1 × 3, 2 × 1)(1 × 4, 2 × 3).
- (iii)  $E \times E_2$  : 9 Richelot isogenies, 3 non-decomposed ones ( $p \equiv 3 \pmod{4}$ ).  
(2 × 3)(1 × 3, 2 × 3).

- (iv)  $E \times E_3$  : 5 Richelot isogenies, 2 non-decomposed ones ( $p \equiv 2 \pmod{3}$ ).  
( $3 \times 2$ )( $3 \times 3$ ).
- (v)  $E_2 \times E_2$  : 5 Richelot isogenies, 1 non-decomposed one ( $p \equiv 3 \pmod{4}$ ).  
( $4 \times 1$ )( $1 \times 1, 2 \times 1, 4 \times 2$ ).
- (vi)  $E_3 \times E_3$  : 3 Richelot isogenies, 1 non-decomposed one ( $p \equiv 2 \pmod{3}$ ).  
( $3 \times 1$ )( $3 \times 1, 9 \times 1$ ).
- (vii)  $E_2 \times E_3$  : 3 Richelot isogenies, 1 non-decomposed one ( $p \equiv 11 \pmod{12}$ ).  
( $6 \times 1$ )( $3 \times 1, 6 \times 1$ ).

*Proof.* We give a proof for the case (iv) (cf. Castryck–Decru–Smith [1, Subsection 3.3]). For the other cases, the arguments are quite similar. Since the elliptic curve  $E_3$  is defined by  $y^2 = x^3 - 1$ , the 2-torsion points  $(x, y)$  of  $E_3$  are given by  $P_1 = (1, 0)$ ,  $P_2 = (\omega, 0)$  and  $P_3 = (\omega^2, 0)$ . Here,  $\omega$  is a primitive cube root of unity. We denote by  $P_4, P_5$  and  $P_6$  the 2-torsion points of  $E$ . The automorphism of  $E_3$

$$\sigma : x \mapsto \omega x, y \mapsto y$$

gives an element of the reduced group  $\text{RA}(E_3) \cong \text{Aut}(E_3)/\langle \iota \rangle$  of automorphisms of  $E_3$ . As in the case of Subsection 6.1, we describe the isotropic subgroups  $G$ . We know that a division into the sets of 3 pairs of these 6 points  $P_i$  ( $1 \leq i \leq 6$ ) on  $E \times E_3$  gives an isotropic subgroup  $G$ , that is,

$$\{P_{i_1} - P_{i_2}, P_{i_3} - P_{i_4}, P_{i_5} - P_{i_6}, \text{the identity}\}$$

gives an isotropic subgroup of  $(E \times E_3)[2]$ . Here, we consider  $P_i$  ( $1 \leq i \leq 3$ ) as the point  $(0, P_i)$  on  $E \times E_3$ , and  $P_i$  ( $4 \leq i \leq 6$ ) as the point  $(P_i, 0)$  on  $E \times E_3$ . This set contains 15 elements. In the case (iv), we have  $E \not\cong E_3$ . Therefore, by Castryck–Decru–Smith [1, Subsection 3.3], among the 15 isotropic subgroups the 9 cases such that  $P_{i_1}, P_{i_2}, P_{i_3} \in E$  and  $P_{i_4}, P_{i_5}, P_{i_6} \in E_3$  give the decomposed Richelot isogenies and the rest gives the non-decomposed Richelot isogenies. For the abbreviation, we denote by  $P_i$  by  $i$ . Then, on the set  $\{1, 2, 3, 4, 5, 6\}$ ,  $id_E \times \sigma$  acts as the cyclic permutation  $(1, 2, 3)$ . The isotropic subgroup  $G$  is determined by the set of 3 pairs of 2-torsion points:

$$\{(i_1, i_2), (i_3, i_4), (i_5, i_6)\},$$

and the reduced group  $\langle id_E \times \sigma \rangle$  of automorphisms induces the action on the set of the 15 isotropic subgroups. By this action, the set of the 15 isotropic subgroups is divided into the following 5 orbits:

$$\begin{aligned} & \{(1, 2), (3, 4), (5, 6), [(2, 3), (1, 4), (5, 6)], [(1, 3), (2, 4), (5, 6)]\}, \\ & \{(1, 2), (3, 5), (4, 6), [(2, 3), (1, 5), (4, 6)], [(1, 3), (2, 5), (4, 6)]\}, \\ & \{(1, 2), (3, 6), (4, 5), [(2, 3), (1, 6), (4, 5)], [(1, 3), (2, 6), (4, 5)]\}, \\ & \{(1, 4), (2, 5), (3, 6), [(1, 6), (2, 4), (3, 5)], [(1, 5), (2, 6), (3, 5)]\}, \\ & \{(1, 4), (2, 6), (3, 5), [(1, 5), (2, 4), (3, 6)], [(1, 6), (2, 5), (3, 4)]\}. \end{aligned}$$

By the criterion above, the first 3 sets correspond with the decomposed Richelot isogenies, and the last 2 sets correspond with the non-decomposed Richelot isogenies.  $\square$

We denote by  $h$  the number of supersingular elliptic curves defined over  $k$ . Then, we know

$$h = \frac{p-1}{12} + \{1 - (\frac{-3}{p})\}/3 + \{1 - (\frac{-1}{p})\}/4$$

(cf. Igusa [8], for instance). We denote by  $h_1$  the number of supersingular elliptic curves with trivial reduced group of automorphisms,  $h_2$  the number of supersingular elliptic curves with  $\text{Aut}(E_2) \cong \mathbf{Z}/4\mathbf{Z}$ ,  $h_3$  the number of supersingular elliptic curves with  $\text{Aut}(E_3) \cong \mathbf{Z}/6\mathbf{Z}$ . We have  $h = h_1 + h_2 + h_3$  and  $h_2 = \{1 - (\frac{-1}{p})\}/2$  and  $h_3 = \{1 - (\frac{-3}{p})\}/2$ .

**Theorem 6.4.** *The total number of non-decomposed Richelot isogenies up to isomorphism outgoing from decomposed principally polarized superspecial abelian surfaces is equal to*

$$(6.2) \quad \frac{(p-1)(p+3)}{48} - \left\{1 - \left(\frac{-1}{p}\right)\right\}/8 + \left\{1 - \left(\frac{-3}{p}\right)\right\}/6.$$

*The total number of decomposed Richelot isogenies up to isomorphism outgoing from decomposed principally polarized superspecial abelian surfaces is equal to*

$$\frac{(p-1)(3p+17)}{96} + (p+6)\left\{1 - \left(\frac{-1}{p}\right)\right\}/16 + \left\{1 - \left(\frac{-3}{p}\right)\right\}/3.$$

*Proof.* The total number of non-decomposed Richelot isogenies up to isomorphism outgoing from decomposed principally polarized superspecial abelian surfaces is equal to

$$6\left\{\frac{h_1(h_1-1)}{2}\right\} + 4h_1 + 3h_2h_1 + 2h_3h_1 + h_2 + h_3 + h_2h_3.$$

The total number of decomposed Richelot isogenies up to isomorphism outgoing from decomposed principally polarized superspecial abelian surfaces is equal to

$$9\left\{\frac{h_1(h_1-1)}{2}\right\} + 7h_1 + 6h_2h_1 + 3h_3h_1 + 4h_2 + 2h_3 + 2h_2h_3.$$

Since  $\left\{1 - \left(\frac{-1}{p}\right)\right\}^2 = 2\left\{1 - \left(\frac{-1}{p}\right)\right\}$  and  $\left\{1 - \left(\frac{-3}{p}\right)\right\}^2 = 2\left\{1 - \left(\frac{-3}{p}\right)\right\}$ , the result follows from these facts.  $\square$

*Remark 6.5.* Since the total number of *decomposed* Richelot isogenies up to isomorphism outgoing from the *irreducible* superspecial curves of genus 2 is equal to the total number of *non-decomposed* Richelot isogenies up to isomorphism outgoing from *decomposed* principally polarized superspecial abelian surfaces, (6.1) and (6.2) give the same number.

## 7. EXAMPLES

By Ibukiyama–Katsura–Oort [7, Subsection 1.3], we have the following normal forms of curves  $C$  of genus 2 with given reduced group  $\text{RA}(C)$  of automorphisms:

(1) For  $S_3 \subset \text{RA}(C)$ , the normal form is  $y^2 = (x^3 - 1)(x^3 - \alpha)$ .

This curve is superspecial if and only if  $\alpha$  is a zero of the polynomial

$$g(z) = \sum_{l=0}^{\lfloor p/3 \rfloor} \binom{(p-1)/2}{((p+1)/6) + l} \binom{(p-1)/2}{l} z^l.$$

(2) For  $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \subset \text{RA}(C)$ , the normal form is  $y^2 = x(x^2 - 1)(x^2 - \beta)$ .

This curve is superspecial if and only if  $\beta$  is a zero of the polynomial

$$h(z) = \sum_{l=0}^{\lfloor p/4 \rfloor} \binom{(p-1)/2}{((p+1)/4) + l} \binom{(p-1)/2}{l} z^l.$$

(3) For  $\text{RA}(C) \cong D_{12}$ , the normal form is  $y^2 = x^6 - 1$ .

This curve is superspecial if and only if  $p \equiv 5 \pmod{6}$  (cf. Ibukiyama–Katsura–Oort [7, Proposition 1.11]).

(4) For  $\text{RA}(C) \cong S_4$ , the normal form is  $y^2 = x(x^4 - 1)$ .

This curve is superspecial if and only if  $p \equiv 5$  or  $7 \pmod{8}$  (cf. Ibukiyama–Katsura–Oort [7, Proposition 1.12]).

Finally, the elliptic curve  $E$  defined by  $y^2 = x(x-1)(x-\lambda)$  is supersingular if and only if  $\lambda$  is a zero of the Legendre polynomial

$$\Phi(z) = \sum_{l=0}^{(p-1)/2} \binom{(p-1)/2}{l}^2 z^l.$$

Using these results, we construct some examples.

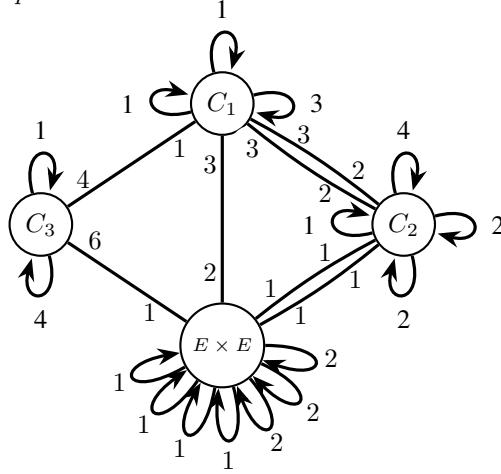
**7.1. Examples in characteristic 13.** Assume the characteristic  $p = 13$ . Over  $k$  we have only one supersingular elliptic curve  $E$ , and three superspecial curves  $C_1$ ,  $C_2$  and  $C_3$  of genus 2 with  $\text{RA}(C_1) \cong S_3$ ,  $\text{RA}(C_2) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$  and  $\text{RA}(C_3) = S_4$ , respectively (cf. Ibukiyama–Katsura–Oort [7, Remark 3.4]). In characteristic 13, we know  $h(z) = 7z^3 + 12z^2 + 12z + 7$ , and the zeros are  $-1$  and  $-5 \pm \sqrt{6}$ . We also know  $g(z) = 2z^4 + 3z^3 + 4z^2 + 3z + 2$ , and one of zeros is  $-4 + \sqrt{2}$ . The Legendre polynomial is given by  $\Phi(z) = z^6 + 10z^5 + 4z^4 + 10z^3 + 4z^2 + 10z + 1$ , and one of zeros is  $3 - 2\sqrt{2}$ . Using these facts, we know that the curves above are given by the following equations:

- (1)  $E: y^2 = x(x-1)(x-3+2\sqrt{2})$  ( $\text{RA}(E) \cong \{0\}$ ),
- (2)  $C_1: y^2 = (x^3-1)(x^3+4-\sqrt{2})$  ( $\text{RA}(C_1) \cong S_3$ ),
- (3)  $C_2: y^2 = x(x^2-1)(x^2+5+2\sqrt{6})$  ( $\text{RA}(C_2) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ ),
- (4)  $C_3: y^2 = x(x^4-1)$  ( $\text{RA}(C_3) \cong S_4$ ).

Therefore, outgoing from superspecial curves of genus 2, we have, in total,  $1 + 2 + 1 = 4$  decomposed Richelot isogenies up to isomorphism by Proposition 6.1. On the other hand, outgoing from the unique decomposed principally polarized abelian surface  $(E \times E, E + E)$ , we have 5 non-decomposed Richelot isogenies (not up to isomorphism) (cf. Igusa [8] and Castryck–Decru–Smith [1, Figure 1]). Using the method in Castryck–Decru–Smith [1, Subsection 3.3], as the images of 5 non-decomposed Richelot isogenies, we have the following superspecial curves of genus 2:

- (a)  $C_a: y^2 = (x^2-1)(x^2-4+7\sqrt{2})(x^2-6+6\sqrt{2})$  ( $\text{RA}(C_a) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ ),
- (b)  $C_b: y^2 = (x^2-1)(x^2+3-2\sqrt{2})(x^2-4-\sqrt{2})$  ( $\text{RA}(C_b) \cong S_4$ ),
- (c)  $C_c: y^2 = (x^2-1)(x^2+3-4\sqrt{2})(x^2+1+3\sqrt{2})$  ( $\text{RA}(C_c) \cong S_3$ ),
- (d)  $C_d: y^2 = (x^2-1)(x^2-3)(x^2+3-4\sqrt{2})$  ( $\text{RA}(C_d) \cong S_3$ ),
- (e)  $C_e: y^2 = (x^2-1)(x^2-6-6\sqrt{2})(x^2-2+2\sqrt{2})$  ( $\text{RA}(C_e) \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ ).

We see that  $C_a \cong C_e \cong C_2$ ,  $C_c \cong C_d \cong C_1$  and  $C_b \cong C_3$ . As Richelot isogenies,  $(E \times E, E + E) \rightarrow (J(C_c), C_c)$  is isomorphic to  $(E \times E, E + E) \rightarrow (J(C_d), C_d)$ , but  $(E \times E, E + E) \rightarrow (J(C_a), C_a)$  is not isomorphic to  $(E \times E, E + E) \rightarrow (J(C_e), C_e)$ . Compare our graph with Castryck–Decru–Smith [1, Figure 1]. In the graph the numbers along the edges are the multiplicities of Richelot isogenies outgoing from the nodes.



### 7.2. Examples in characteristic 11.

Assume the characteristic  $p = 11$ . Over  $k$  we have two supersingular elliptic curves  $E_2, E_3$  and two superspecial curves  $C_1, C_2$  of genus 2 with  $\text{RA}(C_1) \cong S_3$ ,  $\text{RA}(C_2) \cong D_{12}$ , respectively (cf. Ibukiyama–Katsura–Oort [7, Remark 3.4]). In characteristic 11, we know

$$g(z) = 10(z^3 + 5z^2 + 5z + 1),$$

and the roots are  $-1, 3$  and  $4$ . Using this fact, we know that the curves above are given by the following equations:

- (1)  $E_2: y^2 = x^3 - x$  ( $\text{RA}(E_2) \cong \mathbf{Z}/2\mathbf{Z}$ ),
- (2)  $E_3: y^2 = x^3 - 1$  ( $\text{RA}(E_3) \cong \mathbf{Z}/3\mathbf{Z}$ ),
- (3)  $C_1: y^2 = (x^3 - 1)(x^3 - 3)$  ( $\text{RA}(C_1) \cong S_3$ ),
- (4)  $C_2: y^2 = x^6 - 1$  ( $\text{RA}(C_2) \cong D_{12}$ ).

We have three decomposed principally polarized abelian surfaces:

$$E_2 \times E_2, E_3 \times E_3, E_2 \times E_3.$$

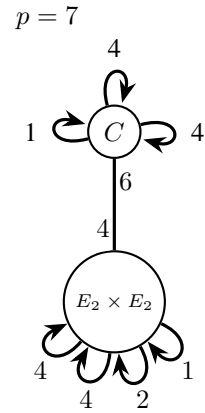
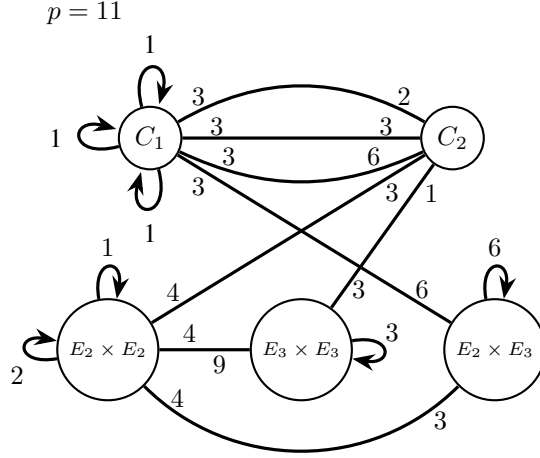
Therefore, from the superspecial curves of genus 2 we have, in total,  $1 + 2 = 3$  decomposed Richelot isogenies up to isomorphism by Proposition 6.1. On the other hand, from the decomposed principally polarized abelian surfaces, we have  $1 + 1 + 1 = 3$  non-decomposed Richelot isogenies up to isomorphism by Proposition 6.3 (cf. Castryck–Decru–Smith [1, Subsections 3.2 and 3.3]). For the decomposed principally polarized abelian surface  $E_2 \times E_2$  the image of the only one non-decomposed Richelot isogeny is given by  $C_2$ . For the decomposed principally polarized abelian surface  $E_3 \times E_3$  the image of the only one non-decomposed Richelot isogeny is also given by  $C_2$ . For the decomposed principally polarized abelian surface  $E_2 \times E_3$  the image of the only one non-decomposed Richelot isogeny is given by  $C_1$ .

**7.3. Examples in characteristic 7.** Assume the characteristic  $p = 7$ . Over  $k$  we have only one supersingular elliptic curve  $E_2$  and only one superspecial curves  $C$  of genus 2, which has the reduced group  $\text{RA}(C) \cong S_4$  of automorphisms (cf. Ibukiyama–Katsura–Oort [7, Remark 3.4]).

They are given by the following equations:

- (1)  $E_2: y^2 = x^3 - x$  ( $\text{RA}(E_2) \cong \mathbf{Z}/2\mathbf{Z}$ ),
- (2)  $C: y^2 = x(x^4 - 1)$  ( $\text{RA}(C) \cong S_4$ ).

We have only one decomposed principally polarized abelian surface  $E_2 \times E_2$ . Therefore, outgoing from the superspecial curves of genus 2 we have only one decomposed Richelot isogeny up to isomorphism. From the decomposed principally polarized abelian surface, we also have only one non-decomposed Richelot isogeny up to isomorphism (cf. Castryck–Decru–Smith [1, Subsections 3.2 and 3.3]). For the decomposed principally polarized abelian surface  $E_2 \times E_2$  the image of the only one non-decomposed Richelot isogeny is given by  $C$ .



## 8. CONCLUDING REMARK

Section 5 clarified a concrete situation on decomposed Richelot isogenies, and it gave a firm understanding of the isogeny graph for genus-2 isogeny cryptography. Further applications (or implications) of our results to cryptography are left as an interesting open problem.

For example, a very recent cryptanalytic algorithm by Costello and Smith [4] is considered as a promising target. They proposed a new isogeny path-finding algorithm in the genus-2 superspecial Richelot isogeny graph. They reduced the original problem to the elliptic curve path-finding problem and improved the time complexity of the original genus-2 path-finding problem. The key ingredient of the reduction is a sub-algorithm for finding a path connecting a given irreducible genus-2 curve and the (connected) subset consisting of elliptic curve products.

In Proposition 4.3, we showed the equivalence of existence of a decomposed Richelot isogeny outgoing from  $J(C)$  and existence of a (long) element of order 2 in the reduced group of automorphisms of  $C$ . It implies that the subset of elliptic curve products are adjacent to genus-2 curves having involutive reduced automorphisms in the superspecial graph. We expect this new characterization can be applied to improving the Costello–Smith attack.

## REFERENCES

- [1] W. Castryck, T. Decru, and B. Smith. Hash functions from superspecial genus-2 curves using Richelot isogenies. In *NutMiC 2019: Number-Theoretic Methods in Cryptology*, 2019. To appear in *J. of Math. Crypt.*
- [2] W. Castryck, T. Lange, C. Martindale, L. Panny, and J. Renes. CSIDH: An efficient post-quantum commutative group action. In *ASIACRYPT 2018, Part III*, pages 395–427, 2018.
- [3] D. Charles, K. Lauter, and E. Goren. Cryptographic hash functions from expander graphs. *J. Crypt.*, 22(1):93–113, 2009.
- [4] C. Costello and B. Smith. The supersingular isogeny problem in genus 2 and beyond. In *PQCrypto 2020*, pages 151–168, 2020.
- [5] L. D. Feo, D. Jao, and J. Plût. Towards quantum-resistant cryptosystems from supersingular elliptic curve isogenies. *J. Math. Crypt.*, 8(3):209–247, 2014.
- [6] E. V. Flynn and Y. B. Ti. Genus two isogeny cryptography. In *PQCrypto 2019*, pages 286–306, 2019.
- [7] T. Ibukiyama, T. Katsura, and F. Oort. Supersingular curves of genus two and class numbers. *Compositio Math.*, 57:127–152, 1986.
- [8] J.-I. Igusa. Class number of a definite quaternion with prime discriminant. *Proc. Nat. Acad. Sci. U.S.A.*, 44:312–314, 1958.
- [9] J.-I. Igusa. Arithmetic variety of moduli for genus two. *Ann of Math.*, 72:612–649, 1960.
- [10] D. Jao, R. Azarderakhsh, M. Campagna, C. Costello, L. D. Feo, B. Hess, A. Jalali, B. Koziel, B. LaMacchia, P. Longa, M. Naehrig, G. Pereira, J. Renes, V. Soukharev, and D. Urbanik. SIKE: Supersingular isogeny key encapsulation. *submission to the NIST's PQC standardization, round 2, updated version*, April 2020.
- [11] T. Katsura and F. Oort. Families of supersingular abelian surfaces. *Compositio Math.*, 62:107–167, 1987.
- [12] T. Katsura and F. Oort. Supersingular abelian varieties of dimension two or three and class numbers. *Advanced Studies in Pure Math.*, 10:253–281, 1987.
- [13] D. Mumford. *Abelian Varieties*. Oxford Univ. Press, 1970.
- [14] T. Shioda. Supersingular K3 surfaces. In *Algebraic Geometry, Proc. Copenhagen 1978 (K. Lønsted, ed.)*, Lecture Notes in Math. 732, pages 563–591. Springer-Verlag, Berlin-Heidelberg-New York, 1979.
- [15] K. Takashima. Efficient algorithms for isogeny sequences and their cryptographic applications. In *Mathematical Modelling for Next-Generation Cryptography: CREST Crypto-Math Project*, pages 97–114. Springer Verlag, 2017.

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