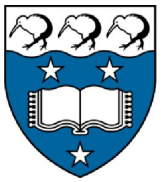
Kangaroo Methods for Solving the Interval Discrete Logarithm Problem



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Abstract

The interval discrete logarithm problem is defined as follows: Given some g, h in a group G, and some $N \in \mathbb{N}$ such that $g^z = h$ for some z where $0 \leq z < N$, find z.
At the moment, kangaroo methods are the best low memory algorithm to solve the interval discrete logarithm problem. The fastest non parallelised kangaroo methods to solve this problem are the three kangaroo method, and the four kangaroo method. These respectively have expected average running times of $(1.818 + o(1))\sqrt{N}$, and $(1.714 + o(1))\sqrt{N}$ group operations.
It is currently an open question as to whether it is possible to improve kangaroo methods by using more than four kangaroos. Before this dissertation, the fastest kangaroo method that used more than four kangaroos required at least $2\sqrt{N}$ group operations to solve the interval discrete logarithm problem. In this thesis, I improve the running time of methods that use more than four kangaroos significantly, and almost beat the fastest kangaroo algorithm, by presenting a seven kangaroo method with an expected average running time of $(1.7195 + o(1))\sqrt{N} \pm O(1)$ group operations. The question, 'Are five kangaroos worse than three?' is also answered in this thesis, as I propose a five kangaroo algorithm that requires on average $(1.737 + o(1))\sqrt{N}$ group operations to solve the interval discrete logarithm problem.

Contents

Chapter 1

Introduction

1.1 Introduction

The interval discrete logarithm problem (IDLP) is defined in the following manner: Given a group G, some $g, h \in G$, and some $N \in \mathbb{N}$ such that $g^z = h$ for some $0 \leq z < N$, find z. In practice, N will normally be much smaller than $|G|$, and g will be a generator of G. The probability that z takes any integer value between 0 and $N - 1$ is deemed to be uniform.

To our current knowledge, the IDLP is hard over some groups. Examples of such groups include Elliptic curves of large prime order, and $\mathbb{Z}_p^*$, where p is a large prime. Hence, cryptosystems such as the Boneh-Goh-Nissim homomorphic encryption scheme [1] derive their security from the hardness of the IDLP. The IDLP also arises in a wide range of other contexts. Examples include, counting points on elliptic curves [4], small subgroup and side channel attacks [5,6], and the discrete logarithm problem with c-bit exponents [4]. The IDLP is therefore regarded as a very important problem in contemporary cryptography.

Kangaroo methods are the best generic low storage algorithm to solve the IDLP. In this thesis, I examine serial kangaroo algorithms, although all serial kangaroo methods can be parallelised in a standard way, giving a speed up in running time in the process. Currently, the fastest kangaroo algorithm is the 4 kangaroo method of Galbraith, Pollard and Ruprai [3]. On an interval of size N, this algorithm has an estimated average running time of $(1.714 + o(1))\sqrt{N}$ group operations and requires $O(\log(N))$ memory. It should be noted that over some specific groups, there are better algorithms to solve

the IDLP. For example, over groups where inversion is fast, such as elliptic curves, an algorithm proposed by Galbraith and Ruprai in [2] has an expected average case running time of $(1.36 + o(1))\sqrt{N}$ group operations, while requiring a constant amount of memory. This method is much slower than the four kangaroo method over groups where inversion is slow however. Baby-step giant-step algorithms, which were first proposed by Shanks in [10], are also faster than kangaroo methods. The fastest Baby-step Giant-step algorithm was illustrated by Pollard in [8], and requires on average $4/3\sqrt{N}$ group operations to solve the IDLP. However, these algorithms are unusable over large intervals, since they have $O(\sqrt{N})$ memory requirements. If one is solving the IDLP in an arbitrary group, on an interval of size N, one would typically use baby-step giant-step algorithms if $N < 2^{30}$, and would use the four kangaroo method if $N > 2^{30}$.
The first kangaroo method was proposed by Pollard in 1978 in [9]. This had an estimated average running time of $3.3\sqrt{N}$ group operations [11]. The next improvement came from van Oorshot and Wiener in [12,13], with the introduction of an algorithm with an estimated average running time of $(2 + o(1))\sqrt{N}$ group operations. This was the fastest kangaroo method for over 15 years, until Galbraith, Pollard and Ruprai published their three and four kangaroo methods in [3]. These respectively require on average $(1.818 + o(1))\sqrt{N}$, and $(1.714 + o(1))\sqrt{N}$ group operations to solve the IDLP. The fastest kangaroo method that uses more than four kangaroos is a five kangaroo method, proposed in [3]. This requires at least $2\sqrt{N}$ group operations to solve the IDLP.
It is currently believed, but not proven, that the four kangaroo method is the optimal kangaroo method. This is a major gap in our knowledge of kangaroo methods, and so the main question that this dissertation attempts to answer is the following.

- **Question 1:** Can we improve kangaroo methods by using more than four kangaroos?

This dissertation also attempts to answer the following lesser, but also interesting problem.

- **Question 2:** Are five kangaroos worse than three?

In this report, I attempt to answer these questions, by investigating five kangaroo methods in detail. I then state how a five kangaroo method can be adapted to give a seven kangaroo method, giving an improvement in running time in the process.

Chapter 2

Current State of Knowledge of Kangaroo Methods

In this section I will give a brief overview of how kangaroo methods work, and of our current state of knowledge of kangaroo algorithms.

2.1 General intuition behind kangaroo methods

The key idea behind all kangaroo methods known today, is that if we can express any $x \in G$ in two of the forms out of $g^p h$,g^q or $g^r h^{-1}$, where $p, q, r \in \mathbb{N}$, then one can find z, and hence solve the IDLP. In all known kangaroo methods, we have a herd of kangaroos who randomly 'hop' around various elements of G. Eventually, 2 different kangaroos can be expected to land on the same group element. If we define the kangaroo's walks such that all elements of a kangaroo's walk are in one of the forms out of $g^p h$,g^q or $g^r h^{-1}$, then when 2 kangaroos land on the same group element, the IDLP may be able to be solved.

2.2 van Oorshot and Weiner Method

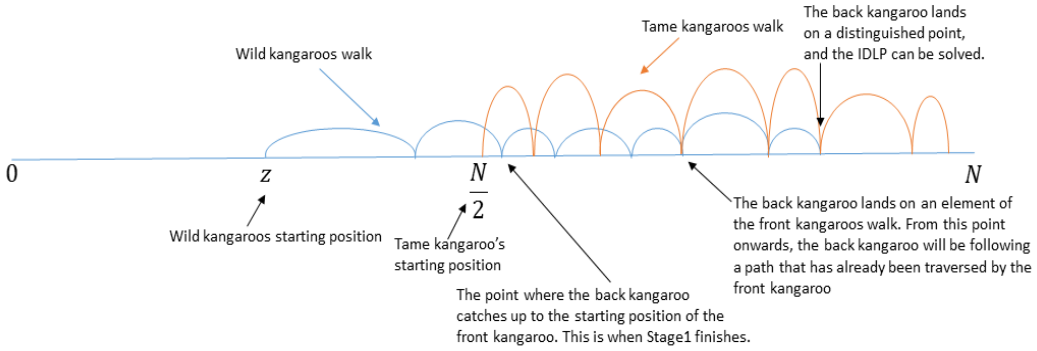
The van Oorshot and Weiner method of [12,13] was the fastest kangaroo method for over 15 years. In this method, there is one 'tame' kangaroo (labelled T), and one 'wild' kangaroo (labelled W). Letting t_i and w_i respectively denote the group elements T and W are at after i 'jumps' of their walk, T and W's walks are defined recursively in the following manner.

- How T's walk is defined
  - T starts his walk at $t_0 = g^{\frac{N}{2}}$. [1]
  - $t_{i+1} = t_i g^n$, for some $n \in \mathbb{N}$.
- How W's walk is defined
  - W starts his walk at $w_0 = h$.
  - $w_{i+1} = w_i g^m$, for some $m \in \mathbb{N}$.

One should note also that the algorithm is arranged so that T and W jump alternately. Clearly, all elements of T's walk are expressed in the form g^q, while all elements of W's walk are expressed in the form $g^p h$, where $p, q \in \mathbb{N}$. Hence when T and W both visit the same group element, we will have $w_i = t_j$, for some $i, j \in \mathbb{N}$, from which we can obtain an equation of the form $g^q = g^p h$, for some $p, q \in \mathbb{N}$, which implies $z = q - p$. Now if we structure the algorithm so that the amount each kangaroo jumps by at each step is dependent only on its current group element, then from the point where both kangaroos have visited a common group element onwards, both kangaroos walks will follow the same path. As a consequence of this, we can detect when T and W have visited the same group element, while only storing a small number of the elements of each kangaroo's walk. All kangaroo methods employ this same idea, and this is why kangaroo methods only require $O(\log(N))$ memory.
To arrange the algorithm so that the amount each kangaroo jumps by at each step is dependent only on its current group element, we create a hash function H, which randomly assigns a 'step size' to each element of G. When a kangaroo lands on some $x \in G$, the amount it jumps forward by at that step is $H(x)$ (so its current group element is multiplied by the precomputed value of $g^{H(x)}$). To use this property so that the algorithm requires only $O(\log(N))$ memory, we create a set of 'distinguished

[1] Note that the method assumes N is even, so $N/2 \in \mathbb{N}$

points', D, where $D \subset G$. D is defined such that $\forall x \in G$, the probability that $x \in D$ is $c\log(N)/\sqrt{N}$, for some $c > 0$. If a kangaroo lands on some $x \in D$, we first check to see if the other kangaroo has landed on x also. If it has, we can solve the IDLP. If the other kangaroo hasn't landed on x, then if T is the kangaroo that landed on x, we store x, the q such that $x = g^q$, and a flag indicating that T landed on x. On the other hand, if W landed on x, we store x, the p such that $g^p h = x$, and a flag indicating that W landed on x. Hence we can only detect a collision after the kangaroos have visited the same distinguished point. At this stage, the IDLP can be solved. A diagram of the process the algorithm undertakes in solving the IDLP is shown below.



To analyse the running time of the van Oorshot and Wiener method, we break the algorithm into the three disjoint stages shown in Stage 1, Stage 2, and Stage 3 below. In this analysis (and for the remainder of this thesis), I will use the following definitions.

Step. *A period where each kangaroo makes exactly one jump.*

Position (of a kangaroo). *A kangaroo is at position p if and only if it's current group element is g^p. For instance, T starts his walk at position $N/2$.*

Distance (between kangaroos). *The difference between two kangaroos positions.*

We analyse the running time of the algorithm (and of all kangaroo methods) by considering the expected average number of group operations it requires to solve the IDLP. The reason for analysing the running time using this metric is explained in section 2.3.

- **Stage 1.** The period between when the kangaroos start their walks, and when the back kangaroo B catches up to the front kangaroo F's starting position. Now

since the probability that z takes any integer value $[0, N)$ is uniform, the average distance between B and F before they start their walks is $N/4$. Therefore, the expected number of steps required in stage 1 is $N/4m$, where m is the average step size of the kangaroos walks

- **Stage 2.** This is the period between when stage 1 finishes, and B lands on a group element that has been visited by F. John Pollard showed experimentally in [8] that the number of steps required in this stage is m. One can see this intuitively in the following way. Once B has caught up to F's starting position, he will be jumping over a region in which F has visited on average $1/m$ group elements. Hence the probability that F lands on a element of F's path at each step in Stage 2 is $1/m$. Therefore, we can expect B's walk to join with F's after m steps.
- **Stage 3.** This is the period between when stage 2 finishes, and B lands on a distinguished point. Since the probability that any $x \in G$ is distinguished is $c\log(N)/\sqrt{N}$, for some $c > 0$, we can expect B to make $\frac{1}{c\log(N)/\sqrt{N}} = \sqrt{N}/c\log(N)$ steps in this stage.

Hence we can expect the algorithm to require $N/4m + m + \sqrt{N}/c\log(N)$ steps to solve the IDLP. Now since each of the two kangaroos make one jump at each step, and in each jump a kangaroo makes we multiply two already known group elements together, each step requires two group operations. Hence the algorithm requires $2\big(N/4m + m + \sqrt{N}/c\log(N)\big)$ group operations to solve the IDLP. The optimal choice of m in this expression is $m = \sqrt{N}/2$, which gives an expected average running time of $\big(2 + 1/c\log(N)\big)\sqrt{N} = \big(2 + o(1)\big)\sqrt{N}$ group operations.
Note that this analysis (and the analysis of the three and four kangaroo methods) ignores the number of group operations required to initialise the algorithm (in the initialisation phase we find the starting positions of the kangaroos, assign a step size to each $x \in G$, and precompute the group elements $g^{H(x)}$ for each $x \in G$). In section 3.3, I show however that the number of group operations required in this stage is constant.

2.3 How we analyse the running time of Kangaroo Methods

I can now explain why we analyse running time of kangaroo methods in terms of the expected average number of group operations they require to solve the IDLP.

Why only consider group operations? Generally speaking, in kangaroo methods, the computational operations that need to be carried out are group operations (e.g. multiplying a kangaroo's current group element to move it around the group), hashing (e.g. computing the step size assigned to a group element), and memory access comparisons (e.g. when a kangaroo lands on a distinguished point, checking to see if that distinguished point has already been visited by another kangaroo). The groups which one is required to solve the IDLP over are normally elliptic curve groups of large prime order, or $\mathbb{Z}_p^*$, for a large prime p [11]. Group operations over such groups require far more computational operations than hashing, or memory access comparisons do. Hence when analysing the running time, we get an accurate approximation to the number of computational operations required by only counting the number of group operations. Counting the number of hashing operations, and memory access comparisons increases the difficulty of analysing the running time substantially, so it is desirable to only count group operations.
Why consider the *expected average* number of group operations? In the van Oorshot and Weiner method (and in all other kangaroo methods), the hash function which assigns a step size to each element of G is chosen randomly. Now for fixed z and varied hash functions, there are many possibilities for how the walks of the kangaroos will pan out. Hence in practise, the number of group operations until the IDLP is solved can take many possible values for each z. Hence we consider the *expected* running time for each z, as being the average running time across all possible walks the kangaroos can make for each z.
Now z can take any integer value between 0 and $N-1$ with equal probability. Hence we refer to the expected *average* running time, as being the average of the expected running times across all $z \in \mathbb{N}$ in the interval $[0, N)$.

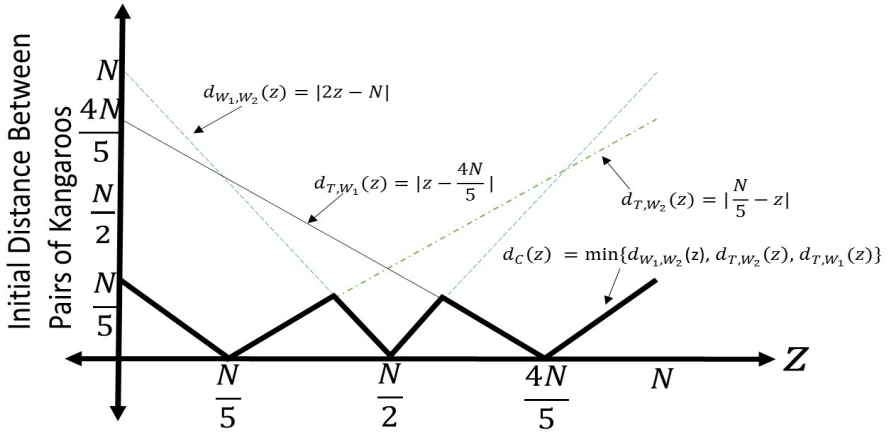
2.4 Three Kangaroo Method

The next major breakthrough in kangaroo methods came from Galbraith, Pollard and Ruprai in [3] with the introduction of their three kangaroo method. The algorithm assumes that g has odd order, and that $10|N$. The method uses three different types of kangaroos, labelled W_1, W_2 and T. All of the elements of W_1,W_2 and T's walks are respectively expressed in the forms $g^p h$, $g^r h^{-1}$, and g^q, where $p, q, r \in \mathbb{N}$. If any pair of kangaroos collides, then we can express some $x \in G$ in 2 of the forms out of $g^p h$,g^q and $g^r h^{-1}$. If we have $x = g^p h = g^q$, then $z = p - q$, while if we have $x = g^q = g^r h^{-1}$, then $z = r - q$. On the other hand, if we have $x = g^p h = g^q h^{-1}$, then since g has odd order, we can solve $z = 2^{-1}(q - p) \mod (|g|)$. Hence a collision between any pair of

kangaroos can solve the IDLP.
To enable us to express the elements of the kangaroos walks in these ways, as in the van Oorshot and Wiener method, we create a hash function H which assigns a step size to each element of G. Defining $W_{1,i}$,$W_{2,i}$ and T_i to respectively denote the group element W_1, W_2 and T are at after i jumps in their walk, we then define the walks of the kangaroos in the following way.

- How W_1's walk is defined
  - W_1 starts at the group element $W_{1,0} = g^{-N/2}h$
  - To move W_1 to the next step, $W_{1,i+1} = W_{1,i}g^{H(W_{1,i+1})}$
- How W_2's walk is defined
  - We start W_2 at the group element $W_{2,0} = g^{N/2}h^{-1}$
  - To move W_2 to the next step, $W_{2,i+1} = W_{2,i}g^{H(W_{2,i})}$
- How T's walk is defined
  - We start T at $T_0 = g^{3N/10}$.
  - To move T to the next step, $T_{i+1} = T_i g^{H(T_1)}$

As in the van Oorshot and Wiener method, the algorithm is arranged so that the kangaroos jump alternately. One can see that W_1,W_2, and T start their walks respectively at the positions $z - N/2$,$N/2 - z$, and $3N/10$. Hence if we let d_{T,W_1},d_{T,W_2}, and d_{W_1,W_2} be the functions for the initial distances between T and W_1, T and W_2, and W_1 and W_2 over all $0 \leq z < N$, then $d_{T,W_1}(z) = \left|(z - N/2) - (3N/10)\right| = \left|z - 4N/5\right|$, $d_{T,W_2}(z) = \left|N/5 - z\right|$, and $d_{W_1,W_2}(z) = \left|2z - N\right|$. We also define d_C to be the function which denotes the initial distance between the closest pair of kangaroos over all $0 \leq z < N$. Hence $d_C(z) = \min\left\{d_{T,W_1}(z), d_{T,W_2}(z), d_{W_1,W_2}(z)\right\}$. The starting positions of the kangaroos in this method are chosen so that the average distance between the closest pair of kangaroos (the average of $d_C(z)$ for $0 \leq z < N$) is minimised. A diagram of the distance between all pairs of kangaroos, and of d_C is shown below.



The following table shows the the formula for d_C, what C (the closest pair of kangaroos) is, what B (the back kangaroo in C) is, and what F (the front kangaroo in C) is, across all $0 \leq z < N$.

	$d_C(z)$	C	B	F
$0 \leq z \leq N/5$	$d_{T,W_2}(z) = N/5 - z$	T and W_2	T	W_2
$N/5 \leq z \leq 2N/5$	$d_{T,W_2}(z) = z - N/5$	T and W_2	W_2	T
$2N/5 \leq z \leq N/2$	$d_{W_1,W_2} = N - 2z$	W_1 and W_2	W_1	W_2
$N/2 \leq z \leq 3N/5$	$d_{W_1,W_2} = 2z - N$	W_1 and W_2	W_2	W_1
$3N/5 \leq z \leq 4N/5$	$d_{T,W_1}(z) = 4N/5 - z$	T and W_1	W_1	T
$4N/5 \leq z \leq N/2$	$d_{T,W_1}(z) = z - 4N/5$	T and W_1	T	W_1

The expected number of steps until the IDLP is solved from a collision between the closest pair can be analysed in the same way as the running time was analysed in the van Oorshot and Weiner method.

- **Stage 1.** The period between when the kangaroos start their walks, and when B catches up with F's starting position. The average of d_C can easily be seen to be $N/10$. Hence if we let m be the average step size used, the expected number of steps for B to catch up to F's starting position is $N/10m$.
- **Stage 2.** The period between when stage 1 finishes, and when B lands on an elements of F's walk. The same analysis as was applied in the van Oorshot and Weiner method shows that the expected number of steps required in this stage is m.

- **Stage 3.** The period between when B lands on an element of F's path, and B lands on a distinguished point. In the three kangaroo method, the probability of a group element being distinguished is the same as it is in the van Oorshot and Wiener method, so the expected number of steps required in this stage is $\sqrt{N}/c\log(N)$.

If we make the pessimistic assumption that the IDLP will always be solved from a collision between the closest pair of kangaroos, we can expect the algorithm to require $N/10m+m+\sqrt{N}/c\log(N)$ steps to solve the IDLP. This expression is minimised when m is taken to be $\sqrt{N/10}$. In this case, the algorithm requires $(2\sqrt{1/10}+o(1)))\sqrt{N}$ steps to solve the IDLP. Since there are three kangaroos jumping at each step, the expected number of group operations until the IDLP is solved is $(1.897+o(1))\sqrt{N}$ group operations.
When Galbraith, Pollard and Ruprai considered the expected number of group operations until the IDLP was solved from a collision between any pair of kangaroos (so not just from a collision between the closest pair), they found through a complex analysis, that the three kangaroo method has an expected average case running time of $(1.818+o(1))\sqrt{N}$ group operations. This is a huge improvement on the running time of the van Oorshot and Weiner method.

2.5 Four Kangaroo Method

The Four kangaroo method is a very simple, but clever extension of the 3 kangaroo method. As in the three kangaroo method, we start three kangaroos, T_1, W_1 and W_2 at the positions $^{3N}/_{10}$, $z-{}^{N}/_{2}$, and ${}^{N}/_{2}-z$ respectively. Here however, we add in one extra tame kangaroo (T_2), who starts his walk at $^{3N}/_{10}+1$. One can see that the starting positions of W_1 and W_2 have the same parity, while exactly one of T_1 and T_2's starting positions will have the same parity as W_1 and W_2's starting positions. Therefore, if the step sizes are defined to be even, then in any walk, both of the wild, and one of the tame kangaroos will be able to collide, while one of the tame kangaroos will be unable to collide with any other kangaroo. Therefore, the three kangaroos that can collide are effectively simulating the three kangaroo method, except over an interval of half the size. Hence, from the analysis of the three kangaroo method, the three kangaroos that can collide in this method require $(1.818+o(1))\sqrt{N/2}$ group operations to solve the IDLP. However, since there is one 'useless' kangaroo that requires just as many group operations as the three other useful kangaroos, the expected number of group operations required to solve the IDLP by the four kangaroo method is ${}^{(3+1)}/_{3}(1.818+o(1))\sqrt{N/2}=$

$(1.714 + o(1))\sqrt{N}$ group operations.

2.6 Can we do better by using more kangaroos?

I now return to the main question this dissertation seeks to address. The answer to this question is not clear through intuition, since there are arguments both for and against using more kangaroos.

On the one hand, if one uses more kangaroos, we both increase the number of pairs of kangaroos that can collide, and we can make the kangaroos closer together. Therefore, by using more kangaroos, the number of steps until the first collision occurs will decrease. However, by increasing the number of kangaroos, there will more kangaroos jumping at each step, so the number of group operations required at each step increases.

Chapter 3

Five Kangaroo Methods

This section will attempt to answer the two main questions of this dissertation (see Question 1 and Question 2 in the introduction), by investigating kangaroo methods which use five kangaroos.

In this section, I will answer Question 2, and partially answer Question 1, by presenting a five kangaroo algorithm which requires on average $(1.737 + o(1))\sqrt{N} \pm O(1)$ group operations to solve the IDLP. To find a five kangaroo algorithm with this running time, I answered the following questions, in the order stated below.

- How should the walks of the kangaroos be defined in a 5 kangaroo algorithm?
- How many kangaroos of each type should be used?
- Where abouts should the kangaroos start their walks?
- What average step size should be used?

3.1 How the Walks of the Kangaroos are Defined

I will first investigate 5 kangaroo methods where a kangaroo's walk can be defined in one of the same three ways as they were in the three and four kangaroo methods. This

means, that a kangaroo can either be of type WILD1,WILD2, or TAME, where the types of kangaroos are defined in the following way.

- **Wild1 Kangaroo** - A kangaroo for which we express all elements of its walk in the form $g^p h$, where $p \in \mathbb{N}$.
- **Wild2 Kangaroo** - A kangaroo for which we express all elements of its walk in the form $g^r h^{-1}$, where $r \in \mathbb{N}$.
- **Tame Kangaroo** - A kangaroo for which we express all elements of its walk in the form g^q, where $q \in \mathbb{N}$.

As in the three and four kangaroo methods, there will be a hash function H which assigns a step size to each $x \in G$. To walk the kangaroos around the group, if x is a kangaroos current group element, the group element it will jump to next will be $xg^{H(x)}$. As in all previous kangaroo methods, the algorithm will be arranged so that the kangaroos each take one jump during each step of the algorithm.

3.2 How many kangaroos of each type should be used?

If we let N_{W1}, N_{W2} and N_T respectively denote the number of WILD1, WILD2, and TAME kangaroos used in any 5 kangaroo method. Then $N_{W1} + N_{W2} + N_T = 5$. The following theorem will prove to be very useful in working out how many kangaroos of each type should be used, given this constraint.

Theorem 3.2. *Let A be any 5 kangaroo algorithm, where the kangaroos can be of type* TAME*,* WILD1*, or* WILD2*. Then the expected number of group operations until the closest 'useful' pair of kangaroos in A collides is no less than $10\sqrt{\frac{N}{2N_{W1}N_{W2}+4N_T(N_{W1}+N_{W2})}}$ (A pair is called useful if the IDLP can be solved when the pair collides).*

My proof of this requires Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.3, Lemma 3.2.4, and Lemma 3.2.5. Before stating and proving these lemmas, I will make the following two important remarks.

- **Remark 1.** I showed in my description of the three kangaroo method how a collision between kangaroos of different types could solve the IDLP. On the other

hand, we can gain no information about z from a collision between kangaroos of the same type. Hence a pair of kangaroos is 'useful' if and only if it features two kangaroos of different types.

- **Remark 2.** In this section, for any choice of starting positions, I will let d (where d is a function of z) be the function that denotes the initial distance between the closest useful pair of kangaroos across all $0 \leq z < N$. d is analogous to the function d_C in the three kangaroo method, where for any z with $0 \leq z < N$, $d(z)$ will be defined to be the smallest distance between pairs of kangaroos of different types, at that particular z. Note that $d(z)$ is completely determined by the starting positions of the kangaroos.

Lemma 3.2.1. *For any choice of starting positions in a 5 kangaroo algorithm, the minimal expected number of group operations until the closest useful pair collides is* $10\sqrt{Ave(d(z))}$*, where* $Ave(d(z))$ *is the average starting distance between the closest useful pair of kangaroos over all instances of the IDLP (i.e. over all* $z \in \mathbb{N}$ *where* $z \in [0, N) = [0, N-1]$*).*

Proof of Lemma 3.2.1. Let A be a 5 kangaroo algorithm that starts all kangaroos at some specified choice of starting positions, and let m be the average step size used in A. Also let $d(z)$ be defined as in remark 2. For each z with $0 \leq z < N$, using an argument very similar to that provided in section 2.2, the expected number of steps until the closest useful pair collides for this specific z is $d(z)/m + m$. Since 5 kangaroos jump at each step, the expected number of group operations until the closest useful pair collides for this z is $5(d(z)/m + m)$. Therefore, the expected average number of group operations until the closest useful pair collides across all instances of the IDLP (i.e. across all $z \in \mathbb{N}$ with $0 \leq z < N$) is

$$\frac{1}{N}\sum_{z=0}^{N-1} 5\left(\frac{d(z)}{m} + m\right)$$

$$\approx \frac{1}{N}\int_0^{N-1} 5\left(\frac{d(z)}{m} + m\right) dz$$

$$= 5\left(\frac{\frac{1}{N}\int_0^{N-1} d(z)dz}{m} + m\right)$$

$$\approx 5\left(\frac{\frac{1}{N}\sum_{z=0}^{N-1} d(z)}{m} + m\right) = 5\left(\frac{Ave(d(z))}{m} + m\right).$$

Now simple differentiation shows that the m that minimises this is $m = \sqrt{Ave(d(z))}$. Substituting in $m = \sqrt{Ave(d(z))}$ gives the required result. □

Lemma 3.2.2. *For each z, $d(z)$ is either of the form $|C_1 \pm z|$ or $|C_2 \pm 2z|$, where C_1 and C_2 are constants independent of z. Furthermore, letting p_{g_1} and p_{g_2} respectively denote the number of pairs with initial distance function of the form $|C \pm z|$, and $|C \pm 2z|$, $p_{g_1} = N_T(N_{W1} + N_{W2})$, and $p_{g_2} = N_{W1}N_{W2}$.*

Proof of Lemma 3.2.2. Since a pair of kangaroos is useful if and only if it features two kangaroos of different types, a pair is useful if and only if it is a TAME/WILD1, a TAME/WILD2, or a WILD1/WILD2 pair (here a TYPE1/TYPE2 pair means a pair of kangaroos featuring one kangaroo of TYPE1, and another kangaroo of TYPE2). Since all WILD1,WILD2 and TAME kangaroos start their walks respectively at group elements of the form $g^p h$,$g^r h^{-1}$ and g^q, for some $p, q, r \in \mathbb{N}$, the starting positions of all WILD1,WILD2 and TAME kangaroos will be of the forms $p + z$, $r - z$, and q respectively. Hence the initial distance functions between TAME/WILD1, TAME/WILD2, and WILD1/WILD2 pairs are respectively $|p + z - q|$, $|r - z + q|$, and $|p + z - (r - z)|$. Therefore, the distance function between all TAME/WILD1 and TAME/WILD2 pairs can be expressed in the form $|C_1 \pm z|$, where C_1 is independent of z. The number of such pairs is $N_T N_{W1} + N_T N_{W2}$. On the other hand, the initial distance function between a WILD1/WILD2 pair can be expressed in the form $|C_2 \pm 2z|$, where C_2 is also independent of z. The number of such pairs is $N_{W1}N_{W2}$. □

From the graph of d_C in section 2.4, we can see that the function for the distance between the closest pair of kangaroos in the three kangaroo method is a sequence of triangles. The same is generally true in five kangaroo methods. In lemma 3.2.3, I will show that the area under the function d is minimised in five kangaroo methods when d is a sequence of triangles.

Lemma 3.2.3. *Suppose $d_{1,1}, d_{1,2}, d_{1,3}, ..., d_{p_1}$ and $d_{2,1}, d_{2,2}, ..., d_{2,p_2}$ are functions of z, defined on the interval $[0, N)$, where for all i and j, $d_{1,i}(z) = |C_i \pm z|$, and $d_{2,j}(z) = |C_2 \pm 2z|$, where C_i and C_j are constants. Let $d(z)$ be defined such that $\forall\ 0 \leq z < N$, $d(z) = \min\{d_{1,1}(z), d_{1,2}(z), ..., d_{1,p_1}(z), d_{2,1}(z), ..., d_{2,p_2}(z)\}$. Then assuming that we have full control over the constants C_i and C_j, in every case where d is not a sequence of triangles, the area under d can be decreased by making d into a sequence of triangles. This can be done by changing some of the constants C_i and C_j.*

Proof. By d being a sequence of triangles, I mean that if we shade in the region beneath d, then the shaded figure is a sequence of triangles (see figure 3.2(a)). Now d is a

sequence of triangles if and only if S_1,S_2 and S_3 hold, where S_1 is the statement '$d(0) = 0$, or $d(0) > 0$ and $d'(0) < 0$', S_2 is the statement '$d(N-1) = 0$, or $d(N-1) > 0$ and $d'(N-1) > 0$', and S_3 is the statement 'for every z where the gradient of d changes, the sign of the gradient of d changes'. This is equivalent to the statement, 'for every z where the function which is smallest changes (from say d_{a_1,b_1} to d_{a_2,b_2}), the gradients of both d_{a_1,b_1} and d_{a_2,b_2} have opposite sign at z'.
Hence if d is not a sequence of triangles, d must not satisfy at least one of the properties out of S_1 (see figure 3.2(b)),S_2 (see figure 3.2(c)), or S_3 (see figures 3.2 (d) and (e)).

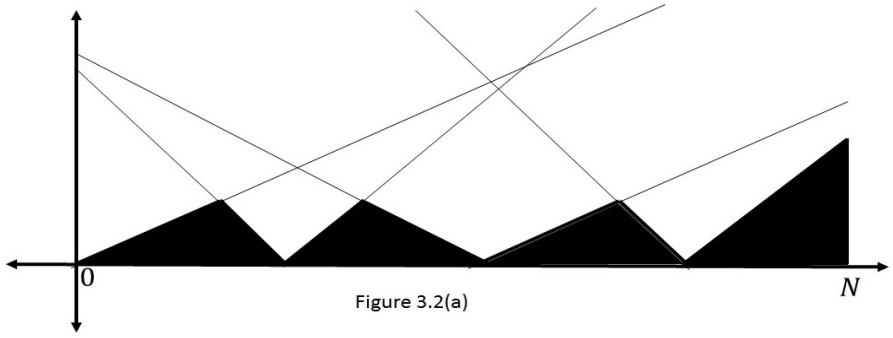


Figure 3.2(a)

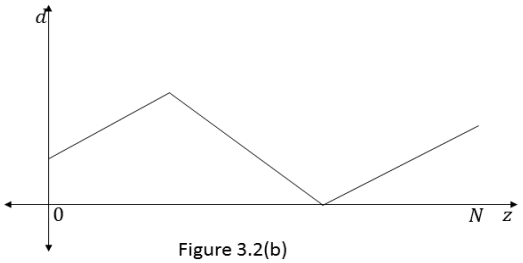


Figure 3.2(b)

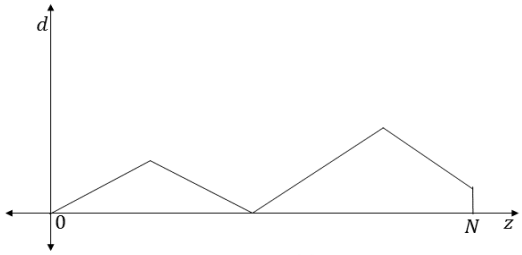


Figure 3.2(c)

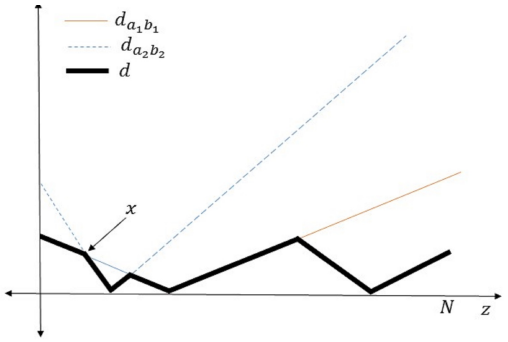


Figure 3.2(d): Case where there exists a x with $0 \leq x \leq$ N where the gradient of d changes from a negative, to a different negative value.

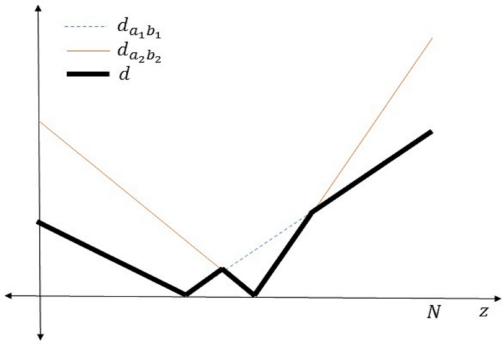


Figure 3.2(e): Case where there exists a x with $0 \leq x \leq$ N where the gradient of d changes from a negative, to a different negative value.

First consider the case where S_1 doesn't hold. Then $d(0) > 0$, and $d'(0) > 0$. Let d_{C0} be the function which is smallest out out of all $\{d_{1,1}(z), d_{1,2}(z), ..., d_{1,p_1}(z), d_{2,1}(z), ..., d_{2,p_2}(z)\}$ at $z = 0$. Hence $d_{C0} = |C + gz|$, where $C > 0$, and g is 1 or 2, and $d(z) = d_C(z)$ for all z between 0 and p, for some $p < N$. If we change d_{C0} so that $d_{C0} = |gz|$, and define all other functions apart from d_{C0} in the same way, then the area under d decreases by Cp over the region $[0, p]$ (see figure 3.2(f)), while the area under d over $[p, N)$ will be no larger than it was before $d_{C0}(z)$ was redefined to be $|gz|$. Hence the area under d decreases by at least Cp over $[0, N)$. Therefore, in every case where S_1 doesn't hold, we can decrease the area under d by redefining d so that S_1 holds (PROP1).

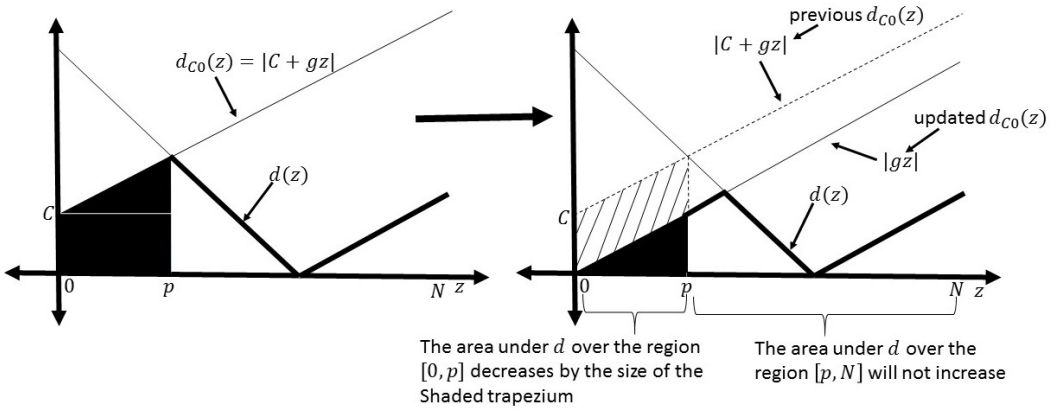


Figure 3.2(f)

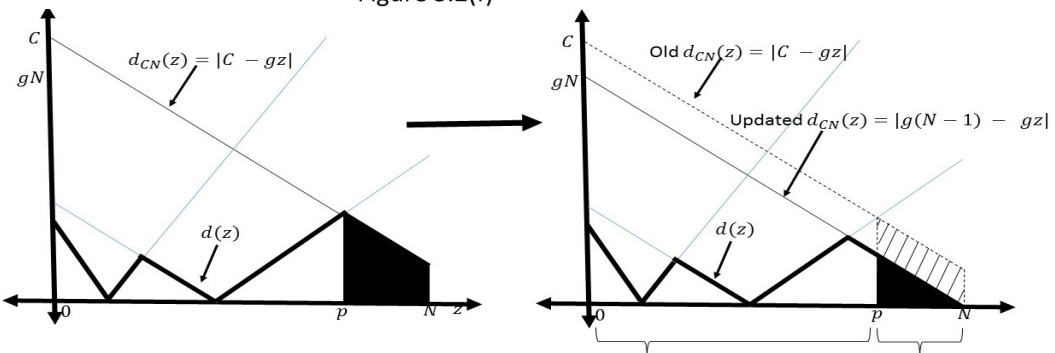


Figure 3.2(g)

Similarly, in the case where S_2 doesn't hold, $d(N-1) = C > 0$ and $d'(N-1) < 0$. If d_{CN} is the function such that $d_{CN}(N-1) = d(N-1)$, then $d_{CN}(z) = |C - gz|$, where $C > g(N-1)$ (since $d_{CN}(N-1) > 0$). Hence if we redefine d_{CN} such that $d_{CN}(z) = |g(N-1) - gz|$, if p is defined such that $d_{CN}(z) = d(z)\ \forall\ \ p \leq z < N$ (before $d_{CN}(z)$ was defined to be $|g(N-1) - gz|$), then the area under d decreases by at least

$(N-1-p)(C-gN)$ (see figure 3.2(g)). Therefore, in every case where S_2 doesn't hold, we can decrease the area under d by redefining d so that S_2 holds (PROP2).

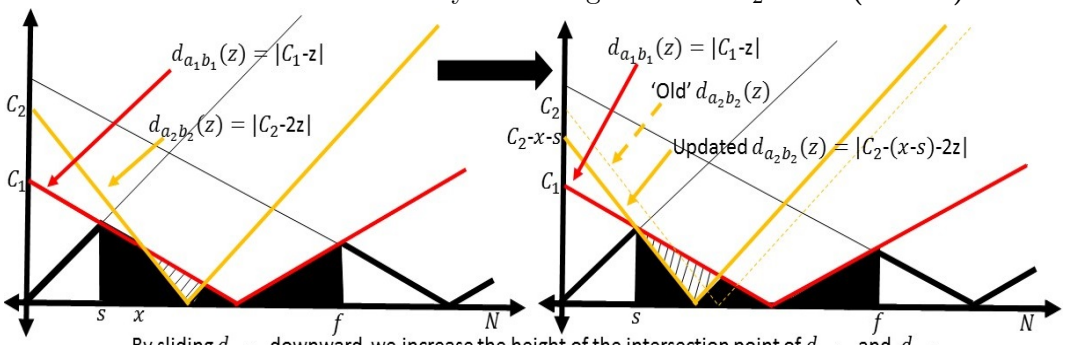


Figure 3.2(h)

Now consider the case where S_3 doesn't hold. Then there exists an x with $0 \leq x < N$, and functions d_{a_1,b_1} and d_{a_2,b_2} such that the function which is smallest (so the function which d equals) changes from d_{a_1,b_1} to d_{a_2,b_2} at x, and $d'_{a_1,b_1}(x)$ and $d'_{a_2,b_2}(x)$ have the same sign.

In the case where $d'_{a_1,b_1}(x) < 0$ and $d'_{a_2,b_2}(x) < 0$, (see figure 3.2(h)) $d'_{a_2,b_2}(x) < d'_{a_1,b_1}(x)$ (since $d_{a_2,b_2}(z) > d_{a_1,b_1}(z)$ for $z < x$ with $z \approx x$, and $d_{a_2,b_2}(z) < d_{a_1,b_1}(z)$ for $z > x$ with $z \approx x$. Hence $d_{a_2,b_2}(x)' = -2$ and $d_{a_1,b_1}(x)' = -1$. Therefore $d_{a_2,b_2}(z) = |C_2 - 2z|$, and $d_{a_1,b_1}(z) = |C_1 - z|$ for some $C_1, C_2 > 0$. Now I will let $[s, f]$ be the region such that for all z where $s \leq z \leq f$, $d(z) = d_{a_1,b_1}(z)$ or $d(z) = d_{a_2,b_2}(z)$, and h be such that $h = d_{a_1,b_1}(x) = d_{a_2,b_2}(x)$. Now if we fix d_{a_1,b_1} (and all other functions in $\{d_{1,1}(z), d_{1,2}(z), ..., d_{1,p_1}(z), d_{2,1}(z), ..., d_{2,p_2}(z)\}$), and allow d_{a_2,b_2} to vary (by varying the constant C_2), then assuming d_{a_1,b_1} and d_{a_2,b_2} intersect somewhere on the region $[s, f]$ when both $d'_{a_1,b_1} < 0$ and $d'_{a_2,b_2} < 0$, then the area under d over the region $[s, f]$ is determined purely by the height of this intersection point (Mathematically, it can be easily shown that if we define H to be the height of the intersection point of d_{a_1,b_1} and d_{a_2,b_2} when d_{a_1,b_1} and d_{a_2,b_2} are decreasing, and $A1_{s,f}$ to be the area under d_{a_1,b_1} over $[s, f]$, then the area under d over $[s, f]$ is $A1_{s,f} - 2H^2/9$). Now suppose we redefine d_{a_2,b_2} so that $d_{a_2,b_2}(z) = |C_2 - (x - s) - 2z|$ (as in figure 3.2(h)). Then d_{a_1,b_1} and d_{a_2,b_2} intersect with negative gradient when $z = s$, and the height of this intersection point is $h + (x - s)$. The following basic lemmas will be very useful for the remainder this proof.

Lemma 3.2.3.1. *If $s > 0$, $d'(z) > 0$ for $z < s$, with $z \approx s$.*

Proof. At any z, $d'(z)$ is either $-2, -1, 1$ or 2. If $d'(z) < 0$ for $z < s$ with $z \approx s$, then since $d_{a_1,b_1}(z) = d(z)$ for $z > s$ with $z \approx s$, d_{a_1,b_1} would still be the smallest function

for $z < s$, with $z \approx s$. But this is a contradiction to $[s, f]$ being the region which either d_{a_1,b_1} or d_{a_1,b_1} are the smallest functions over. □

Lemma 3.2.3.2. *In the case where the intersection point of d_{a_1,b_1} and d_{a_2,b_2} when $d'_{a_1,b_1}, d'_{a_2,b_2} < 0$ is when $z = s$, d satisfies S_3 over $[s, f]$.*

Proof. After d_{a_2,b_2} is redefined, d_{a_1,b_1} and d_{a_2,b_2} are clearly still the smallest functions over $[s, f]$. Hence we only need to consider the intersection points of d_{a_1,b_1} and d_{a_2,b_2} to prove the lemma. By the nature of the functions d_{a_1,b_1} and d_{a_2,b_2}, d_{a_1,b_1} and d_{a_2,b_2} can have at most one intersection point when both d'_{a_1,b_1} and d'_{a_2,b_2} have the same sign. Now d_{a_1,b_1} and d_{a_2,b_2} intersect when both have negative gradient when $z = s$. Hence the only z where d's gradient changes, but the sign of d's gradient remains the same over $[s, f]$, is when $z = s$. But Lemma 3.2.3.1 implies that the gradient of d changes from a positive to a negative value when $z = s$. Hence S_3 is satisfied over $[s, f]$. □

As a result, we can conclude $x > s$, since otherwise S_3 would hold over $[s, f]$ when $d_{a_2,b_2}(z)$ was $|C_2 - 2z|$. Hence the height of the intersection point of d_{a_1,b_1} and d_{a_2,b_2} when both $d'_{a_1,b_1}, d'_{a_2,b_2} < 0$ is increased when d_{a_2,b_2} is redefined, so the area under d over $[s, f]$ is decreased by redefining d_{a_2,b_2} in such a way that d satisfies S_3 over $[s, f]$. Since the value of $d(z)$ doesn't increase when d_{a_2,b_2} is redefined for $0 \leq z < s$ and $f < z < N$, the area under d over all other regions apart from $[s, f]$ does not increase when d_{a_2,b_2} is redefined. Hence the area under d over $[0, N)$ is decreased by redefining d so that S_3 is satisfied over $[s, f]$.
A very similar argument can show that in the case where there exists a z such that the function which is smallest changes from d_{A_1,B_1} to d_{A_2,B_2} at z, and $d'_{A_1,B_1}, d'_{A_2,B_2} > 0$, then we can redefine d_{A_2,B_2} so that the area under d over $[0, N)$ is decreased and S_3 is satisfied over $[S, F]$ (where S and F are defined such that d equals either d_{A_1,B_1} or d_{A_2,B_2} for all z with $S \leq z \leq F$).
Therefore, in every case where there exists z such that the gradient of d changes but keeps the same sign at z, we can decrease the area under d over $[0, N)$ by redefining one of the functions in $\{d_{1,1}(z), d_{1,2}(z), ..., d_{1,p_1}(z), d_{2,1}(z), ..., d_{2,p_2}(z)\}$, so that d satisfies S_3 over each of the intervals $[s, f]$ and $[S, F]$. Hence, in such a case we can decrease the area under d while making d satisfy S_3 over $[0, N)$ (PROP3).
By combining PROP1,PROP2, and PROP3, we can conclude that in every case where d is not a sequence of triangles (so at least one of S_1,S_2 or S_3 doesn't hold for d), we can decrease the area under d by redefining d so that d is a sequence of triangles (so S_1,S_2 and S_3 all hold for d on $[0, N)$). □

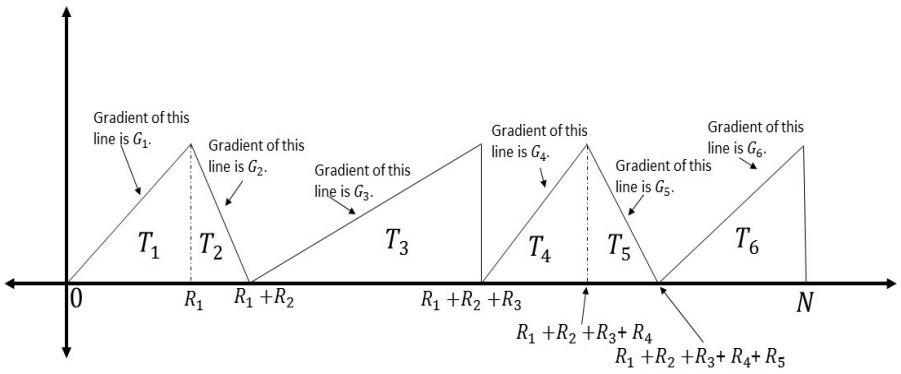


Figure 3.1: Diagram of the kind of situation being considered in lemma 3.2.4

In any five kangaroo method, we are given a set of functions of the same form as those in lemma 3.2.3. To minimise the expected number of group operations until the closest useful pair collides, we need to minimise the area under d, by choosing the C_i and C_js appropriately in functions of the form of lemma 3.2.3. Hence we may assume that when the area under d is as small as possible, d will be a sequence of triangles. Therefore, since this theorem is stating a lower bound on the expected number of group operations until the closest useful pair collides, for the purposes of the proof of this theorem, d can be assumed to be a sequence of triangles.
The following lemma will be useful in finding how small the area under d can be, given that d can be assumed to be a sequence of triangles.

Lemma 3.2.4. *Fix $n, N \in \mathbb{N}$, and let the gradients G_1,G_2,...,$G_n \in \mathbb{R} \setminus \{0\}$ be fixed. Let R_1,R_2,...,R_n be such that $R_i \geq 0$, $\sum_{i=1}^{n} R_i = N$, and such that the sum of areas of triangles of base R_i and gradient G_i is minimised. Then all triangles have the same height.*

Proof of Lemma 3.2.4. I will label the triangles such that the i^{th} triangle (T_i) is the triangle which has $i-1$ triangles to the left of it. G_i can be considered to be the gradient of the slope of T_i, and R_i can be considered to be the size of the region which T_i occupies. Also let A_T be the sum of area under these triangles. A diagram of the situation is shown in Figure 3.1.

Then we have

$$\sum_{i=1}^{n} R_i = N$$

and

$$A_T = \sum_{i=1}^{n} \frac{|G_i|.R_i^2}{2}$$

In the arrangement where the heights of all the triangles are the same, for each $1 \leq i < j < n$, $R_i/R_j = |G_j|/|G_i|$. Suppose we change the ratio of R_i to R_j, so that the new R_i is $R_i + \epsilon$, and R_j becomes $R_j - \epsilon$ (note that ϵ can be greater than or less than 0). Then the area of T_i becomes $(G_i(R_i + \epsilon)^2)/2 = (G_i(R_i^2 + 2R_i\epsilon + \epsilon^2))/2$ while the area of T_j becomes $G_j(R_j^2 - 2R_j\epsilon + \epsilon^2)/2$. A_T therefore changes by

$$\frac{|G_i|(R_i+\epsilon)^2}{2} + \frac{|G_j|(R_j-\epsilon)^2}{2} - \frac{|G_i|R_i^2}{2} - \frac{|G_j|R_j^2}{2}$$

$$= \frac{|G_i|R_i^2}{2} + \frac{|G_j|R_j^2}{2} + |G_i|R_i\epsilon - |G_j|R_j\epsilon - \frac{|G_i|R_i^2}{2} - \frac{|G_j|R_j^2}{2} + \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}$$

$$= |G_i|R_i\epsilon - |G_j|R_j\epsilon + \epsilon^2 = \epsilon^2 > 0$$

Therefore, adjusting the ratio of the size of any two regions away from that which ensures the height of the triangles is the same, always strictly increases the sum of the area of the triangles. It follows that when the heights of all the triangles are equal, the sum of the area beneath these triangles is minimised. □

Lemma 3.2.5. *In the case where d is a sequence of triangles, each useful pair is either never the closest useful pair, or it is the closest useful pair over a single region.*

Proof. By a useful pair (P) being the closest useful pair only over a single region, I mean that there is a single interval $[s, f]$, with $0 \leq s < f < N$, such that P is the closest useful pair for all z where $s \leq z \leq f$. This is in contrast to there being intervals $[s_1, f_1]$, and $[s_2, f_2]$, where $0 \leq s_1 < f_1 < N$, and $f_1 < s_2 < f_2 < N$, such that P is the closest useful pair for all z with $s_1 \leq z \leq f_1$, and $s_2 \leq z \leq f_2$, but P is not the closest useful pair for all z where $f_1 < z < s_2$.
The result of this lemma follows easily from the fact that if d is a sequence of triangles, then for every z where the closest useful pair changes (say from P_1 to P_2), the gradients of the initial distance functions between P_1, and P_2 have opposite sign. □

Proof of Theorem 3.2. The lemmas can now be used to prove the theorem. Firstly, label all useful pairs in any way from 1 to n, where n is the number of useful pairs of

kangaroos in A. Now since d can be assumed to be a a sequence of triangles, lemma 3.2.5 implies that a useful pair is either never the closest useful pair, or there exists a single region for which a useful pair is closest over. For each i, in the case where pair i is the closest useful pair over some region, let R_i denote the single region which pair i is the closest useful pair over, and R_{i_s} denote the size of R_i. In the case where pair i is never the closest useful pair, define R_{i_s} to be 0. Then $\sum_{i=1}^{n} R_{i_s} = N$. Letting d_i denote the distance function between pair i for each i, by Lemma 3.2.2, $d_i(z)$ is of the form $|C \pm g_i z|$, where g_i is 1 or 2, and $C \in \mathbb{R}$. It is clear from the diagram in section 2.4 that such functions feature at most two triangles, both of which have slopes of gradient with an absolute value of g_i. Hence d_i must feature at most two such triangles over R_i. Now for every i, pair i is the closest useful pair over R_i, so $d(z) = d_i(z)\ \forall z \in R_i$. Therefore, d features at most two triangles over R_i, both of which have slopes that have a gradient with an absolute value of g_i. Therefore, if we let T_{g_1} and T_{g_2} denote the number of triangles in d where the absolute values of their gradients are 1 and 2 respectively, $T_{g_1} \leq 2p_{g_1} = 2N_T(N_{W1} + N_{W2})$ and $T_{g_2} \leq 2p_{g_2} = 2N_{W1}N_{W2}$. Now by Lemma 3.2.4, for the area under d to be minimised, all triangles must have the same height. This implies that all triangles with the same gradient in d must cover a region of the same size. Defining R_{T_1} and R_{T_2} to denote the size of the regions covered by each triangle of gradients 1 and 2 respectively, Lemma 3.2.4 also implies that $R_{T_1} = 2R_{T_2}$. Then since all triangles cover the domain of d, $N = T_{g_1}R_{T_1} + T_{g_2}R_{T_2}$ $\leq 2N_T(N_{W1} + N_{W2})R_{T_1} + 2N_{W1}N_{W2}R_{T_2} = 4N_T(N_{W1} + N_{W2})R_{T_2} + 2N_{W1}N_{W2}R_{T_2}$. Hence $R_{T_2} \geq N/(4N_T(N_{W1}+N_{W2})+2N_{W1}N_{W2})$. Now since all triangles in d have the same height, the average of d over all z with $0 \leq z < N$ is half the height of all triangles. Therefore, since the height of a triangle of gradient 2 is $2R_{T_2}$, R_{T_2} gives the average of d, which is the average distance between the closest useful pair. Then by Lemma 3.2.1, the expected number of group operations until the closest useful pair collides in the case where the area under d is minimised (and hence the average distance between the closest useful pair is minimised) is $10\sqrt{R_{T_2}} \geq 10\sqrt{N/(4N_T(N_{W1} + N_{W2}) + 2N_{W1}N_{W2})}$. □

By plugging in various values of N_T, N_{W1} and N_{W2} into this formula under the constraint that $N_T + N_{W1} + N_{W2} = 5$, we can gather a lower bound on the expected number of group operations until the closest useful pair collides, when different numbers of each type of walk are used. The following table shows the methods which achieved the three best lower bounds. In the table, a tuple of the form (x, y, z) denotes an algorithm that uses x TAME kangaroos, y WILD1 kangaroos, and z WILD2 kangaroos.

$10\sqrt{\frac{N}{(4N_T(N_{W1}+N_{W2})+2N_{W1}N_{W2})}}$	Number of walks of each type used
$1.8898\sqrt{N}$	$(2,1,2),(2,2,1)$
$1.9611\sqrt{N}$	$(3,1,1)$
$2.0412\sqrt{N}$	$(1,2,2),(2,0,3),(2,3,0),$ $(3,0,2),(3,2,0)$

From this, it can be seen that a $(2,2,1)$ 5 kangaroo method has the minimal lower bound on the expected number of group operations until the closest useful pair collides, of $1.8898\sqrt{N}$ group operations. A $(2,1,2)$ method doesn't need to be considered as a separate case, since this is clearly equivalent to a $(2,2,1)$ method. If one starts two WILD1 kangaroos at h and $g^{0.7124N}h$, a WILD2 kangaroo at $g^{1.3562N}h^{-1}$, and two TAME kangaroos at $g^{0.9274N}$ and $g^{0.785N}$, the lower bound $(2,2,1)$ method of $1.8898\sqrt{N}$ group operations is realised. The table shows that in any other 5 kangaroo method, the closest useful pair can't collide in less than $1.9611\sqrt{N}$ group operations on average.

Since the closest useful pair of kangaroos is by far the most significant in determining the running time of any kangaroo algorithm (for instance, in the 3 kangaroo method the expected number of group operations until the closest useful pair collides could be as low as $1.8972\sqrt{N}$ group operations, while the expected number of group operations until any pair collides can only be as low as $1.818\sqrt{N}$ group operations), a method that uses 2 TAME, 2 WILD1, and 1 WILD2 kangaroos is most likely to be the optimal 5 kangaroo method, out of all methods that only use TAME,WILD1, and WILD2 kangaroos.

3.3 Where the Kangaroos should start their walks, and what average step size should be used?

Given that, in any 5 kangaroo method that uses only TAME, WILD1, and WILD2 kangaroos one should use 2 WILD1, 1 WILD2, and 2 TAME kangaroos, the next question to consider is where abouts the kangaroos should start their walks. In the analysis that answers this question, the question of what average step size to use will be answered also. I will now state some definitions and remarks that will be used throughout the remainder of this thesis.

- **Remark 1:** Firstly, I will redefine the IDLP to be to find z, given $h = g^{zN}$, when we're given g and h, and that $0 \leq z < 1$.

- **Remark 2:** At this stage, I will place one constraint on the starting positions of all kangaroos. This being, that all WILD1,WILD2, and TAME kangaroos will respectively start their walks at positions of the form $aN + zN$, $bN - zN$, and cN, for universal constants a,b, and c, that are independent of the interval size N. This is the only constraint I will place on the starting positions at this stage.

- **Remark 3:** I will let $D_{i,z}$ denote the initial distance between the i^{th} closest useful pair of kangaroos for some specified z. It follows from Remark 2 that $D_{i,z} = d_{i,z}N$, for some $d_{i,z}$ independent of N.

- **Remark 4:** The average step size m will be defined to be $c_m\sqrt{N}$, for some c_m independent of N.

- **Remark 5:** $S_{i,z}$ will be defined such that $S_{i,z}$ denotes the expected number of steps the back kangaroo requires to catch up to the front kangaroo's starting position in the i^{th} closest useful pair, for some specified z. It is clear that $S_{i,z} = \lceil \frac{D_{i,z}}{m} \rceil$. Since $D_{i,z} = d_{i,z}N$, and $m = c_m\sqrt{N}$ for some $d_{i,z}$, and c_m which are independent of N, we can say $S_{i,z} = \lceil s_{i,z}\sqrt{N} \rceil$ for some $s_{i,z}$ independent of N. For the typical interval sizes over which one uses kangaroo methods to solve the IDLP ($N > 2^{30}$), this can be considered to be $s_{i,z}\sqrt{N}$.

- **Remark 6:** $C_{i,z}$ and $c_{i,z}$ will be defined such that $C_{i,z} = \sum_{j=1}^{i} S_{i,z}$, and $c_{i,z}\sqrt{N} = C_{i,z}$. One can see from the definition of $S_{i,z}$ that $c_{i,z}$ is independent of the interval size N.

I will answer the question that titles this section by first presenting a formula that can compute the running time of any (2,2,1) 5-kangaroo algorithm (see section 3.3.1), and then by showing how this formula can be used to find the best starting positions and average step size to use (see section 3.3.2).

3.3.1 Formula for computing the running time of a (2,2,1)-5 kangaroo algorithm

Theorem 3.3.1. *In any 5 kangaroo method which uses 2* TAME*, 1* WILD2*, and 2* WILD1 *kangaroos, the expected number of group operations required to solve the IDLP is approximately* $5\left(\left(\int_0^1 c_z dz\right) + o(1)\right)\sqrt{N} + O(\log(N))$*, where*

$$c_z = \sum_{i=1}^{8}\left(e^{\frac{(-is_{i,z}+c_{i,z})}{c_m}}(\frac{c_m}{i}+s_{i,z}) - e^{\frac{-is_{i+1,z}+c_{i,z}}{c_m}}(\frac{c_m}{i}+s_{i+1,z})\right)$$

.

Proof. Any 5 kangaroo method can be broken into the following 4 disjoint stages;

- **Stage 1-** The stage where the algorithm is initialised. This involves computing the starting positions of the kangaroos, assigning a step size to each group element, and computing and storing the group elements which kangaroos are multiplied by at each step (so computing $g^{H(x)}$ for each x in G).
- **Stage 2-** The period between when the kangaroos start their walks, and the first collision between a useful pair occurs.
- **Stage 3-** The period between when the first collision occurs, and when both kangaroos have visited the same distinguished point.
- **Stage 4-** The stage where z is computed, using the information gained from a useful collision.

3.3.1.1 Number of group operations required in Stage 1

To find the starting positions of the kangaroos, we require one inversion (to find the starting position of the kangaroo of type WILD2), 3 multiplications (in finding the starting positions of all wild kangaroos), and 5 exponentiations (one to find the starting position of each kangaroo). All of these operations are $O(\log(N))$ in any group. As explained in [8], the step sizes can be assigned in $O(\log(N))$ time also using a hash function. To pre-compute the group elements which kangaroos can be multiplied by at each step, we need to compute g^s, for each step size s that can be assigned to a group

element (this is the number of values which the function H can take). The number of step sizes used in kangaroo methods is generally between 20 and 100. This was suggested by Pollard in [8]. Applying less than a constant number (100) of exponentiation operations requires a constant number of group operations (so independent of the interval size).
Summing together the number of group operations required by each part of Stage 1, we see that the number of group operations required in Stage 1 is $O(\log(N))$.

3.3.1.2 Number of group operations required in Stage 2

To analyse the number of group operations required in stage 2, I will define $Z(z)$ to be a random variable on $\mathbb{N}$ such that $\Pr\big(Z(z) = k\big)$ denotes the probability that for some specified z, the first collision occurs after k steps. From this, one can see that $\mathrm{E}\big(Z(z)\big) = \sum_{k=0}^{\infty} k \Pr\big(Z(z) = k\big)$, gives the expected number of steps until the first collision occurs, at our specified z.

Important Remark

To compute $\mathrm{E}(Z(z))$, I will compute the expected number of steps until the first useful collision occurs in the case where in every useful pair of kangaroos, the back kangaroo takes the expected number of steps to catch up to the front kangaroos starting position, and make the assumption that this is proportional to the expected number of steps until the first useful collision occurs across all possible random walks. This assumption was used implicitly in computing the running time of the three kangaroo method in [3], and is a necessary assumption to make, since calculating the expected number of steps until the first useful collision occurs across all possible walks is extremely difficult.
The following lemma will be useful in computing $\mathrm{E}(Z(z))$.

Lemma 3.3.1. *In the case where in every useful pair of kangaroos, the back kangaroo takes the expected number of steps to catch up to the starting position of the front kangaroo, for every k,* $\Pr(Z(z) = k) = \sum_{j=1}^{i} \binom{i}{j} \frac{1}{m^j} e^{\frac{-ik-i+C_{i,z}+j}{m}}$ *where i is the number of pairs of kangaroos for which the back kangaroo has caught up to the front after k steps.*

Proof. Let $k \in \mathbb{N}$, and i be such that $S_i(z) \leq k < S_{i+1}(z)$. In the case where in every pair of kangaroos, the back kangaroo takes the expected number of steps to catch up

to the starting position of the front kangaroo, when $S_i(z) \leq k < S_{i+1}(z)$, there will be exactly i pairs where the back kangaroo will be walking over a region that has been traversed by the front kangaroo (these will be the i closest useful pairs). Therefore, exactly i pairs can collide after k steps, for all $S_i(z) \leq k < S_{i+1}(z)$. In order for the first collision to occur after exactly k steps, we require that the i pairs that can collide avoid each other for the first $k-1$ steps, and then on the k^{th} step, j pairs collide for some j between 1 and i. I will define $E_{k,j}$ to be the event that there are no collisions in the first $k-1$ steps, and then on the k^{th} step, exactly j pairs collide. Since $E_{k,j}$ and $E_{k,l}$ are disjoint events for $j \neq l$, we can conclude that $\Pr(Z(z) = k) = \sum_{j=1}^{i} P(E_{k,j})$ **(1)**. Now $\Pr(E_{k,j})$ can be computed as follows; In any instance where $E_{k,j}$ occurs, we can define sets X and Y such that X is the set of all x where the x^{th} closest useful pair of kangaroos doesn't collide in the first $k-1$) steps, but does collide on the k^{th} step, and Y is the set of all y where the y^{th} closest useful pair doesn't collide in the first k steps. Now in Stage 2 of the van Oorshot and Weiner method, I explained how at any step, the probability that a pair collides once the back kangaroo has caught up to the path of the front is $1/m$. Hence for any $x \in X$, the probability that the back kangaroo in the x^{th} closest useful pair avoids the path of the front kangaroo for the first $k-1$ steps, but lands on an element in the front kangaroos walk on the k^{th} step is $\frac{1}{m}(1-\frac{1}{m})^{k-S_{x,z}} \approx \frac{1}{m}e^{\frac{-k+S_{x,z}}{m}}$, while for any $y \in Y$, the probability that the y^{th} closest useful pair doesn't collide in the first k steps is $(1-\frac{1}{m})^{k+1-S_{y,z}} \approx e^{\frac{-k-1+S_{y,z}}{m}}$. Now before any collisions have taken place, the walks of any 2 pairs of kangaroos are independent of each other. Therefore, the probability that the pairs in X all first collide on the k^{th} step, while all the pairs in Y don't collide in the first k steps is $\prod_{x\in X}\frac{1}{m}e^{\frac{-k+S_{x,z}}{m}}\prod_{y\in Y}e^{\frac{-k-1+S_{y,z}}{m}}$ $= \frac{1}{m^j}e^{\frac{-jk+\sum_{x\in X}S_{x,z}}{m}}e^{\frac{-(i-j)k-(i-j)+\sum_{y\in Y}S_{y,z}}{m}} = \frac{1}{m^j}e^{\frac{-ik-i+j+C_{i,z}}{m}}$. Now since there are $\binom{i}{j}$ ways for j out of the i possible pairs to collide on the k^{th} step, we obtain the formula $P(E_{k,j}) = \binom{i}{j}\frac{1}{m^j}e^{\frac{-ik-i+j+C_{i,z}}{m}}$. When substituting this result back into **(1)**, we obtain the required result of $\Pr(Z(z)=k) = \sum_{j=1}^{i}\binom{i}{j}\frac{1}{m^j}e^{\frac{-ik-i+j+C_{i,z}}{m}}$ □

I will now define p_z to be the function such that $p_z(k) = k\Pr(Z(z) = k)$, for all $k \in N$. Hence $\mathrm{E}(Z(z)) = \sum_{k=0}^{\infty} p_z(k)$. Now in a 5 kangaroo method that uses 2 TAME, 2 WILD1, and 1 WILD2 walks, there there are 8 pairs that can collide to yield a useful collision (4 TAME/WILD1 pairs, 2 TAME/WILD2 pairs, and 2 WILD1/WILD2 pairs). Hence for any k, the number of pairs where the back kangaroo has caught up to the

front kangaroos starting position can be anywhere between 0 and 8. Therefore,

$$p_z(k) = \begin{cases} 0 & 1 \le k < S_{1,z} \\ \frac{k}{m} e^{\frac{-k+C_{1,z}}{m}} & S_{1,z} \le k < S_{2,z} \\ k(\frac{1}{m} e^{\frac{-2k-1+C_{2,z}}{m}} + \frac{1}{m^2} e^{\frac{-2k+C_{2,z}}{m}}) & S_{2,z} \le k < S_{3,z} \\ \sum_{j=1}^{3} k\binom{3}{j} \frac{1}{m^j} e^{\frac{-3k-3+j+C_{3,z}}{m}} & S_{3,z} \le k < S_{4,z} \\ \sum_{j=1}^{4} k\binom{4}{j} \frac{1}{m^j} e^{\frac{-4k-4+j+C_{4,z}}{m}} & S_{4,z} \le k < S_{5,z} \\ \sum_{j=1}^{5} k\binom{5}{j} \frac{1}{m^j} e^{\frac{-5k-5+j+C_{5,z}}{m}} & S_{5,z} \le k < S_{6,z} \\ \sum_{j=1}^{6} k\binom{6}{j} \frac{1}{m^j} e^{\frac{-6k-6+j+C_{6,z}}{m}} & S_{6,z} \le k < S_{7,z} \\ \sum_{j=1}^{7} k\binom{7}{j} \frac{1}{m^j} e^{\frac{-7k-7+j+C_{7,z}}{m}} & S_{7,z} \le k < S_{8,z} \\ \sum_{j=1}^{8} k\binom{8}{j} \frac{1}{m^j} e^{\frac{-8k-8+j+C_{8,z}}{m}} & S_{8,z} \le k < \infty \end{cases}$$

For the rest of this thesis, I will consider p_z as a continuous function. The following result will be useful in computing $\mathrm{E}(Z(z))$.

Theorem 3.3.1.1. $\int_1^\infty p_z(k)dk - O(1) \le \mathrm{E}(Z(z)) \le \int_1^\infty p_z(k)dk + O(1)$.

Proof. The proof of this will use the following lemma.

Lemma 3.3.2. *$p_z(k)$ is $O(1)$ $\forall k$*

Proof. Let $0 \le z < 1$. Then $\forall\ k < S_{1,z}$, $p_z(k) = 0$, while $\forall\ k \ge S_{1,z}$, $p_z(k) = k\sum_{j=1}^{i}\binom{i}{j}\frac{1}{m^j}e^{\frac{-ik-i+j+C_{i,z}}{m}}$, for some $1 \le i \le 8$. Hence for the purposes of the proof of this lemma, we can assume $k \ge S_{1,z}$. I will state some facts that will make the argument of this proof flow more smoothly.

Fact 1: For k such that $S_{i,z} \le k < S_{i+1,z}$, $e^{\frac{-ik-i+j+C_{i,z}}{m}} \le 1$.
This holds because since $k \ge S_{i,z}$, $ik \ge iS_{i,z} \ge \sum_{j=1}^{i} S_{j,z} = C_{i,z}$. Also, $i \ge j$. Hence $-ik - i + j + C_{i,z} \le 0$, and $e^{\frac{-ik-i+j+C_{i,z}}{m}} \le 1$.

Fact 2: No useful pair of kangaroos can start their walks further than a distance of $6N$ apart on an interval of size N. Also, the average step size is at least $0.06697\sqrt{N}$. Both of these facts will be explained in 'Remark regarding Lemma 3.3.2' on Page 44.

Fact 3: The interval size can be assumed to be greater than 2^{30}. This was explained in the introduction, and can be assumed because one would typically use baby-step giant-step algorithms to solve the IDLP on intervals of size smaller than 2^{30}.

I will show that $p_z(k)$ is $O(1)$ for two separate cases.

Case 1: The case where $S_{8,z} \leq m/8$.
First, consider the situation in this case where $S_{1,z} \leq k \leq m/8$. Let i be such that $S_{i,z} \leq k < S_{i+1,z}$. Then by Fact 1, Fact 2, Fact 3, and the condition that $k \leq m/8$, $p_z(k) = \sum_{j=1}^{i} \binom{i}{j} \frac{k}{m^j} e^{\frac{-ik-i+j+C_{i,z}}{m}} \leq \sum_{j=1}^{i} \binom{i}{j} \frac{k}{m^j} \leq \sum_{j=1}^{i} \binom{i}{j} \frac{\frac{m}{8}}{m^j} \leq \sum_{j=1}^{i} \binom{i}{j} \frac{1}{8(0.06697\sqrt{2^{30}})^{j-1}}$, which is clearly $O(1)$.
Now consider the case where $k > m/8$. Then since $S_{8,z} < m/8$, $k > S_{8,z}$. Hence $p_z(k) = \sum_{j=1}^{8} k\binom{8}{j} \frac{1}{m^j} e^{\frac{-8k-8+j+C_{8,z}}{m}}$.
Hence $\frac{dp_z(k)}{dk} = \left(\sum_{j=1}^{8} \binom{8}{j} \frac{1}{m^j} e^{\frac{-8k-8+j+C_{8,z}}{m}}\right)\left(1 - \frac{8k}{m}\right)$. This is less than 0 for $k > m/8$. Hence $\forall\ k > m/8$, $p_z(k) < p_z(m/8)$. Since $p_z(m/8)$ is $O(1)$, $p_z(k)$ is $O(1)$ for $k > m/8$.

Case 2: The case where $S_{8,z} > m/8$. First, consider the situation where $S_{1,z} \leq k \leq S_{8,z}$. Let i be such that $S_{i,z} \leq k < S_{i+1,z}$ Now Fact 3 and Remark 5 (see Page 29) imply that $S_{8,z} \leq \frac{6N}{0.06697\sqrt{N}} < 90\sqrt{N}$. Hence $k < 90\sqrt{N}$. Therefore, $p_z(k) = \sum_{j=1}^{i} \binom{i}{j} \frac{k}{m^j} e^{\frac{-ik-i+j+C_{i,z}}{m}} \leq \sum_{j=1}^{i} \binom{i}{j} \frac{k}{m^j} \leq \sum_{j=1}^{i} \binom{i}{j} \frac{90\sqrt{N}}{m^j} \leq \sum_{j=1}^{i} \binom{i}{j} \frac{90}{0.06697^j (0.06697\sqrt{2^{30}})^{j-1}}$
$\leq \binom{8}{1} \frac{90}{0.06697} + \sum_{j=2}^{8} o(1)$, which is $O(1)$.
Now when $k > S_{8,z}$, $\frac{dp_z(k)}{dk} = \left(\sum_{j=1}^{8} \binom{8}{j} \frac{1}{m^j} e^{\frac{-8k-8+j+C_{8,z}}{m}}\right)(1 - \frac{8k}{m})$. Since $k > S_{8,z} > m/8$, $\frac{dp_z(k)}{dk} < 0\ \forall\ k > S_{8,z}$. Therefore, $p_z(k) < p_z(S_{8,z})\ \forall k\ > S_{8,z}$. Since $p_z(S_{8,z})$ is $O(1)$, $p_z(k)$ is $O(1)\ \forall\ k > S_{8,z}$.

□

I will now prove the theorem by approximating the sum of $p_z(k)$ over all $k \in \mathbb{N}$ to the integral of $p_z(k)$, over each interval $[S_{i,z}, S_{i+1,z})$. On the interval $[1, S_{1,z})$, $p_z(k) = 0$, so $\sum_{k=1}^{S_{i,z}-1} p_z(k) = \int_1^{S_{1,z}} p_z(k)dk$ **(2)**. Hence for the rest of the proof I will consider $p_z(k)$ on intervals $[S_{i,z}, S_{i+1,z})$, where $i \geq 1$. Now let $p_{i,z}$ be the function such that $p_{i,z}(k) = k\sum_{j=1}^{i} \binom{i}{j} \frac{1}{m^j} e^{\frac{-ik-i+j+C_{i,z}}{m}}$ (so $p_{i,z}(k) = p_z(k) \iff k \in [S_{i,z}, S_{i+1,z})$). Now by differentiating $p_{i,z}$, we obtain $\frac{dp_{i,z}}{dk} = \left(\sum_{j=1}^{i} \binom{i}{j} \frac{1}{m^j} e^{\frac{-ik-i+j+C_{i,z}}{m}}\right)\left(1 - \frac{ki}{m}\right)$. Hence $p_{i,z}$

has a single turning point (at $k = \frac{m}{i}$). Since $p_z(k) = p_{i,z}(k)$ for some i for every $i \geq 1$, on each interval $[S_{i,z}, S_{i+1,z})$, p_z is either only decreasing, only increasing, or $\exists$ a t such that $\forall\ k < t$, p_z is increasing, and $\forall\ k > t$, p_z is decreasing.
In the case where p_z is only decreasing on an interval $[S_{i,z}, S_{i+1,z})$, by applying the integral test for convergence, we can conclude that $\int_{S_{i,z}}^{S_{i+1,z}} p_z(r)dr < \sum_{k=S_{i,z}}^{S_{i+1,z}-1} p_z(k) < \int_{S_{i,z}}^{S_{i+1,z}} p_z(r)dr + p_z(S_{i,z})$ **(3)**, while if p_z is only increasing on $[S_{i,z}, S_{i+1,z})$, $\int_{S_{i,z}}^{S_{i+1,z}} p_z(r)dr - p_z(S_{i,z}) < \sum_{k=S_{i,z}}^{S_{i+1,z}-1} p_z(k) < \int_{k=S_{i,z}}^{S_{i+1,z}} p_z(r)dr$ **(4)**.

Now consider p_z on intervals of the form $[S_{i,z}, S_{i+1,z})$, where there exists a turning point t such that $\forall\ k < t$, p_z is increasing, while $\forall\ k > t$, p_z is decreasing (see figure 3.3(a)). Let j be defined such that $j = \max\{m \in \mathbb{N} | n \leq t - 1\}$. Then by applying the integral for convergence test to the intervals $[S_{i,z}, j + 1]$, and $[j + 2, S_{i+1,z}]$, one can see that $\sum_{k=S_{i,z}}^{j+1} p_z(k) > \int_{k=S_{i,z}}^{j+1} p_z(r)dr$ and $\sum_{k=j+2}^{S_{i+1,z}-1} p_z(k) > \int_{j+2}^{S_{i+1,z}} p_z(r)dr$. Therefore, $\sum_{S_{i,z}}^{S_{i+1,z}-1} p_z(k) + \int_{j+1}^{j+2} p_z(r)dr > \int_{S_{i,z}}^{S_{i+1,z}} p_z(r)dr$. Now $\int_{j+1}^{j+2} p_z(r)dr = \text{Ave}\{p_z(r) | j+1 \leq r \leq j+2\}$, which is $O(1)$ since all $p_z(k)$ are $O(1)$. Hence $\int_{S_{i,z}}^{S_{i+1,z}} p_z(r)dr - O(1) < \sum_{S_{i,z}}^{S_{i+1,z}-1} p_z(k)$ **(5)**.
Now the integral for convergence test, applied again to the intervals $[S_{i,z}, j + 1]$, and $[j + 2, S_{i+1,z}]$ implies that $\sum_{k=S_{i,z}}^{j} p_z(k) < \int_{S_{i,z}}^{j+1} p_z(r)dr$, and $\sum_{k=j+3}^{S_{i+1,z}-1} p_z(k) < \int_{j+2}^{S_{i+1,z}} p_z(r)dr$. Also, it is clear that $p_z(j + 1) + p_z(j + 2) < p_z(j + 1) + p_z(j + 2) + \int_{j+1}^{j+2} p_z(r)dr$. Hence, $\sum_{k=S_{i,z}}^{j} p_z(k) + p_z(j + 1) + p_z(j + 2) + \sum_{k=j+3}^{S_{i+1,z}-1} p_z(k) < \int_{S_{i,z}}^{j+1} p_z(r)dr + \int_{j+1}^{j+2} p_z(r)dr + \int_{j+2}^{S_{i+1,z}} p_z(r)dr$. Therefore, since $p_z(j + 1)$ and $p_z(j + 2)$ are $O(1)$, $\sum_{k=S_{i,z}}^{S_{i+1,z}-1} p(k) < \int_{S_{i,z}}^{S_{i+1,z}} p_z(r)dr + O(1)$ **(6)**.

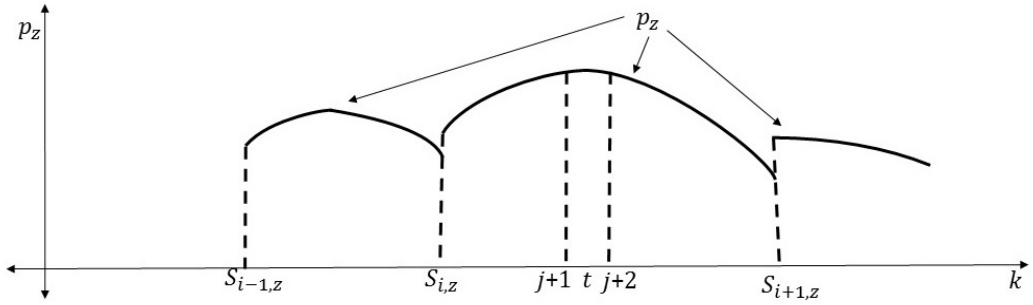


Figure 3.3(a)

From **(2)**,**(3)**,**(4)**, **(5)**, and **(6)**, we can see that for all intervals $[S_{i,z}, S_{i+1,z})$,

$$\int_{S_{i,z}}^{S_{i+1,z}} p_z(r)dr - O(1) < \sum_{S_{i,z}}^{S_{i+1,z}-1} p_z(k) < \int_{S_{i,z}}^{S_{i+1,z}} p_z(r)dr + O(1)$$

. Therefore, we obtain the required result of

$$\int_1^\infty p_z(k)dk - O(1) < \sum_{k=1}^{\infty} p_z(k) = \mathrm{E}(Z(z)) < \int_1^\infty p_z(k)dk + O(1)$$

□

Hence for large interval sizes N, $\int_1^\infty p_z(k)dk$ approximates the expected number of steps until the first collision very well. The following lemma will make the computing of $\int_1^\infty p_z(k)dk$ far more achievable.

Lemma 3.3.3. *$\forall i, j$ where $1 \le i \le 8$, and $j \ge 2$, $\int_{S_{i,z}}^{S_{i+1,z}} k\binom{i}{j}\frac{1}{m^j}e^{\frac{-ik+C_{i,z}-i+j}{m}}$ is $O(1)$.*

Proof. The integral of $k\binom{i}{j}\frac{1}{m^j}e^{\frac{-ik+C_{i,z}-i+j}{m}}$ with respect to k is

$$\frac{\binom{i}{j}(ik+m)e^{\frac{-ik+C_{i,z}-i+j}{m}}}{i^2m^{j-1}}$$

so $\int_{S_{i,z}}^{S_{i+1,z}} k\binom{i}{j}\frac{1}{m^j}e^{\frac{-ik+C_{i,z}-i+j}{m}}dk$ is

$$\binom{i}{j}\frac{\left(e^{\frac{-iS_{i,z}+C_{i,z}-i+j}{m}}(iS_{i,z}+m) - e^{\frac{-iS_{i+1,z}+C_{i,z}-i+j}{m}}(iS_{i+1,z}+m)\right)}{m^{j-1}i^2} \qquad \textbf{(7)}$$

Now in the proof of Lemma 3.3.2, I showed that $e^{\frac{-iS_{i,z}+C_{i,z}-i+j}{m}} \le 1$. Hence **(7)** is less than $\frac{\binom{i}{j}(S_{i,z}+m)}{i^2m^{j-1}}$. In the proof of the same lemma , I also showed that $S_{i,z} < 90\sqrt{N}$, and $c_m \le 0.06697\sqrt{N}$. Hence $\frac{\binom{i}{j}(S_{i,z}+m)}{i^2m^{j-1}} < \frac{(90+0.06697)\sqrt{N}}{(0.06697\sqrt{N})^{j-1}}$, which is $O(1)$ for $j \ge 2$. □

Therefore, from the formulas at the top of page 33, we see that

$$\int_1^\infty p_z(k)dk = \sum_{i=1}^{8}\int_{S_{i,z}}^{S_{i+1,z}} \frac{ik}{m}e^{\frac{-ik+C_{i,z}-i+1}{m}} + O(1) \qquad \textbf{(8)}.$$

Now each $\int_{S_{i,z}}^{S_{i+1,z}} \frac{ik}{m} e^{\frac{-ik+C_{i,z}-i+1}{m}}$ is

$$e^{\frac{-iS_{i,z}+C_{i,z}-i+1}{m}}(\frac{m}{i}+S_{i,z}) - e^{\frac{-iS_{i+1,z}+C_{i,z}-i+1}{m}}(\frac{m}{i}+S_{i+1,z})$$

Substituting this into **(8)**, we obtain

$$\int_1^\infty p_z(k)dk = \sum_{i=1}^{8}\left(e^{\frac{-iS_{i,z}+C_{i,z}-i+1}{m}}(\frac{m}{i}+S_{i,z}) - e^{\frac{-iS_{i+1,z}+C_{i,z}-i+1}{m}}(\frac{m}{i}+S_{i+1,z})\right) + O(1)$$

Now since $C_{i,z} = \sum_{j=1}^{i} S_{j,z}$, $C_{i,z}$ can be expressed as $c_{i,z}\sqrt{N}$, for some $c_{i,z}$ independent of N. Hence

$$\sum_{i=1}^{8}\left(e^{\frac{-iS_{i,z}+C_{i,z}-i+1}{m}}(\frac{m}{i}+S_{i,z}) - e^{\frac{-iS_{i+1,z}+C_{i,z}-i+1}{m}}(\frac{m}{i}+S_{i+1,z})\right)$$

$$= \sum_{i=1}^{8}\left(e^{\frac{(-is_{i,z}+c_{i,z})\sqrt{N}-i+1}{c_m\sqrt{N}}}(\frac{c_m}{i}+s_{i,z})\sqrt{N} - e^{\frac{(-is_{i+1,z}+c_{i,z})\sqrt{N}-i+1}{c_m\sqrt{N}}}(\frac{c_m}{i}+S_{i+1,z})\right)\sqrt{N}$$

$$= \sum_{i=1}^{8} e^{\frac{-i+1}{m}}\left(e^{\frac{(-is_{i,z}+c_{i,z})}{c_m}}(\frac{c_m}{i}+s_{i,z}) - e^{\frac{-is_{i+1,z}+c_{i,z}}{c_m}}(\frac{c_m}{i}+s_{i+1,z})\right)\sqrt{N} \qquad \textbf{(9)}$$

Now for the typical interval sizes over which one uses kangaroo methods to solve the IDLP ($N > 2^{30}$), $e^{\frac{i+1}{m}}$ is extremely close to 1. Hence we can safely ignore the $e^{\frac{-i+1}{m}}$ term in **(9)**. I will state how large the approximation error due to ignoring this $e^{\frac{-i+1}{m}}$ term is when I present my final kangaroo algorithm in section 3.4. Therefore, if we define c_z to be $\sum_{i=1}^{8}\left(e^{\frac{(-is_{i,z}+c_{i,z})}{c_m}}(\frac{c_m}{i}+s_{i,z}) - e^{\frac{-is_{i+1,z}+c_{i,z}}{c_m}}(\frac{c_m}{i}+s_{i+1,z})\right)$, we have $\int_1^\infty p_z(k)dk - O(1) = c_z\sqrt{N}$. Using the result from Theorem 3.3.1.1, we can conclude that

$$\mathrm{E}(Z(z)) = c_z\sqrt{N} \pm O(1) \qquad \textbf{(10)}$$

Hence the expected number of steps until the first collision over all instances of the IDLP (i.e. over all z with $0 \le z < N$) is

$$\mathrm{E}(Z) = c\sqrt{N} \pm O(1) \qquad \textbf{(11)}$$

where $c = \mathrm{Ave}\{c_z | 0 \le z < 1\} = \int_0^1 c_z dz$. Now since at each step, each of the 5 kangaroos make one jump, the expected number of group operations until the first collision occurs across all z is

$$5\left(\int_0^1 c_z dz\right) \pm O(1) \qquad \textbf{(12)}$$

3.3.1.3 Number of group operations required in Stage 3

From the analysis provided in [3], if one sets the probability that a group element is distinguished to be $c\log(N)\sqrt{N}$, for some constant $c > 0$, the expected number of group operations required in stage 3 is $\sqrt{N}/c\log(N) = o(1)\sqrt{N}$.

3.3.1.4 Number of Group operations required in Stage 4

The kangaroos used in a 2 TAME, 2 WILD1, and 1 WILD2 kangaroo method are of the same type as those used in the three kangaroo method. I explained how one may find z from a collision between any of these types of kangaroos when I described the three kangaroo method. In any case, at most 3 addition or subtraction operations modulo $|g|$, while in the case where a WILD1 and a WILD2 kangaroo collides, we are required to find $2^{-1} \pmod{|g|}$. Hence the number of group operations required in this stage is $O(1)$.

Now by summing the number of group operations required in stages 1,2,3 and 4, we obtain the required result that the expected number of group operations required to solve the IDLP by any 2 TAME, 2 WILD1, and 1 WILD2 kangaroo method, is approximately $5\left(\left(\int_0^1 c_z dz\right) + o(1)\right)\sqrt{N} + O(\log(N))$. □

3.3.2 Finding a good assignment of starting positions, and average step size

In this subsection, I will use the formula presented in Theorem 3.3.1 to find the best choice of starting positions and average step size that I could possibly find.
The process I will use to do this will be to first state how the formula of Theorem 3.3.1 can be used compute the running time of an algorithm with particular starting positions and average step size (see Algorithm 1), and then to iterate through various possible starting positions and average step sizes, to find which one minimises the running time. For this purpose, I will define a,b,c,t_1, and t_2 to be universal constants independent of the interval size N, such that on an interval of size N, the 2 WILD1 kangaroos start their walks at the positions $aN + zN$ and $cN + zN$, the WILD2 kangaroo starts his walk at $bN - zN$, and the 2 TAME kangaroos start their walks at t_1 and t_2.

One can see from the formula of Theorem 3.3.1, that finding the running time of

any 5 kangaroo method requires finding $\int_0^1 c_z dz$. Now calculating c_z at any z requires finding $s_{i,z}\ \forall\ i$ with $1 \leq i \leq 8$. The most rigorous approach to finding the average of c_z would be to find a formula for each $s_{i,z}$ across all z, and to plug this into the formula for c_z, and then integrate this across all z. Finding a direct formula for each $s_{i,z}$ that holds for all possible choices for the starting positions and the average step size proved to be too difficult. I therefore proposed the following simulation based approach for computing $\int_0^1 c_z dz$. This is how the COMPUTEAVERAGEc_z function of Algorithm 1 computes $\int_0^1 c_z dz$.

At a particular z, $s_{i,z}$ can be computed by finding the starting positions of all kangaroos on an interval of size $N = 1$ (this means computing $aN + zN = a + z$, $b - z$, $c + z$, t_1 and t_2), and then finding the distances between all useful pairs when the kangaroos start at these positions. By then ranking these distances in the manner done in line 5 of Algorithm 1, one can find $D_{i,z}$ for each i between 1 and 8, for an interval of size $N = 1$. One can then find $s_{i,z}$ using $D_{i,z}/m = d_{i,z}N/c_m\sqrt{N} = s_{i,z}\sqrt{N} = s_{i,z}$. Following this, we can compute all $c_{i,z}$, using $c_{i,z} = \sum_{j=1}^{i} s_{j,z}$. Hence we have all the information we need to compute c_z. By finding the average of c_z for a large number of evenly spaced z in $[0, 1)$ (so for instance, computing c_z for all z in $\{z | z = 10^{-p}k, k \in \mathbb{N}, 0 \leq z < 1\}$, with $p \geq 3$), we can approximate $\int_0^1 c_z dz$. The pseudocode for the MATLAB® function to compute c_z is shown in Algorithm 1.

Algorithm 1 Function for finding the average of c_z

1: **function** COMPUTEAVERAGEc_z(a,b,c,t_1,t_2,c_m,p)
2: $z \longleftarrow 0$
3: sumof$c_z \longleftarrow 0$
4: **while** $z < 1$ **do**
5: distancesarray $\leftarrow$ sort($\{ |(a+z) - t_1|, |(b-z) - t_1|, |(c+z) - t_1|, |(a+z) - t_2|$ $|(b-z) - t_2|, |(c+z) - t_2|, |(a+z) - (b-z)|, |(c+z) - (b-z)| \})$
6: **for** $i \leftarrow 1$ to 8 **do**
7: $s_{i,z} \leftarrow$ distancesarray$[i]/c_m$
8: $c_{i,z} \leftarrow \sum_{j=1}^{i} s_{j,z}$
9: $c_z \leftarrow \sum_{i=1}^{8} \left(e^{\frac{(-is_{i,z}+c_{i,z})}{c_m}} (\frac{c_m}{i} + s_{i,z}) - e^{\frac{-is_{i+1,z}+c_{i,z}}{c_m}} (\frac{c_m}{i} + s_{i+1,z}) \right)$
10: sumof$c_z \leftarrow$ sumof$c_z + c_z$
11: $z \leftarrow z + 10^{-p}$
12: **return** sumof$c_z/(10^p)$

Therefore, if we let O_{pt} denote the output of this function on some specified combination of starting positions and average step size (i.e. values of a,b,c,t_1,t_2 and c_m), then from **(11)**, the expected number of steps until the first collision occurs for this combi-

nation is $O_{pt}\sqrt{N} \pm O(1)$ **(13)**. Also, from the formula of theorem 3.3.1, the expected number of group operations required to solve the IDLP for our specified combination of starting positions and average step size is approximately $(5O_{pt} + o(1))\sqrt{N} \pm O(1)$ **(14)**. I will state how large the error of this approximation is when I present my five kangaroo algorithm in section 3.4.

Therefore, if we let a_{opt},b_{opt},c_{opt},$t1_{opt}$,$t2_{opt}$, and cm_{opt} respectively denote the best values for a,b,c,t_1,t_2 and c_m, then a_{opt},b_{opt},c_{opt},$t1_{opt}$,$t2_{opt}$, and cm_{opt} are the values of a,b,c,t_1,t_2 and c_m for which the output of the COMPUTEAVERAGEc_z function is smallest.

This fact gives rise the to the following algorithm for finding good values for a,b,c,t_1,t_2, and c_m. The pseudocode for this algorithm is shown in Algorithm 2. The idea of the algorithm was to start with a range of values for which the optimal values of a,b,c,t_1,t_2, and c_m lay in. These ranges would be encapsulated in the variables a_{min},a_{max},b_{min},b_{max},c_{min},c_{max},$t1_{min}$,$t1_{max}$,$t2_{min}$,$t2_{max}$,cm_{min} and cm_{max}, so we would have $a_{min} \leq a_{opt} \leq a_{max}$, $b_{min} \leq b_{opt} \leq b_{max}$, $c_{min} \leq c_{opt} \leq c_{max}$, $t1_{min} \leq t1_{opt} \leq t1_{max}$, $t2_{min} \leq t2_{opt} \leq t2_{max}$. I was unable to prove a range of values for which a_{opt},b_{opt},c_{opt}, $t1_{opt}$,$t2_{opt}$, and cm_{opt} were guaranteed to lie in, but in **(A)**,**(B)**,**(C)**,**(D)** and **(E)** (which can be found below), I state and justify some ranges for which a_{opt},b_{opt},c_{opt}, $t1_{opt}$,$t2_{opt}$, and cm_{opt} are likely to lie in. Using the SCANREGION function, I would then find good values for b,c,t_1,t_2 and c_m (by **(A)**, a can be fixed at 0) by computing averagec_z for evenly spaced (separated by the amount defined by the variable 'gap' in Algorithm 2) values of b,c,t_1,t_2, and c_m, between the ranges defined by the variables b_{min},b_{max},c_{min},c_{max},$t1_{min}$,$t1_{max}$,$t2_{min}$,$t2_{max}$,cm_{min} and cm_{max} (see lines 3-10). The variables Best$b$,Best$c$, Bestt_1, Bestt_2, and Bestc_m would represent the values of b,c,t_1,t_2 and c_m for which averagec_z was smallest, across all combinations for which averagec_z was computed for (this is carried out in lines 9-18).
We could then find better values for b,c,t_1,t_2 and c_m than Best$b$, Best$c$, Bestt_1, Bestt_2, and Bestc_m, by running the SCANREGION function on values of b,c,t_1,t_2 and c_m in a smaller region centred around Best$b$, Best$c$, Bestt_1, Bestt_2, and Bestc_m (see lines 25-28), that are separated by a smaller gap (see line 29). By repeating this process multiple times (see lines 24-31), we could keep finding better and better values for b,c,t_1,t_2 and c_m. Eventually however, the interval for which the SCANREGION function was called on would shrink to be zero in size. At this point, the the improvements in the best values for b,c,t_1,t_2 and c_m between iterations would become negligible. Line 24 determines when this occurs.
The algorithm then computes Averagec_z to a higher degree of accuracy for the optimal values of b,c,t_1,t_2 and c_m (see line 32). It does this by setting the variable p in the COMPUTEAVERAGEc_z function to be 6 (p was set to 3 the main loop (see line 10), due

to reasons relating to the practicality of the running time of Algorithm 2).

Estimates of initial bounds for the optimal values of a,b,c,t_1,t_2 and c_m

(A). $a_{opt} = 0$

(B). $\frac{-1}{2} = b_{min} \leq b_{opt} \leq b_{max} = \frac{5}{2}$

(C). $0 = c_{min} \leq c_{opt} \leq c_{max} = 3$, *and* $c \geq a$

(D). $-2 = t_{1_{min}} = t_{2_{min}} \leq t_{1_{opt}} \leq t_{2_{opt}} \leq t_{1_{max}} = t_{2_{max}} = \frac{9}{2}$

(E). $0.06697 \leq c_{m_{opt}} \leq 0.5330$

Justification of (A). Suppose we start our kangaroos walks at $N(d + z)$,$N(b + d - z)$,$N(c + d + z)$,$N(t_1 + d)$, and $N(t_2 + d)$, where $d \in \mathbb{R}$. Then for all z, one can see that the starting distances between all pairs of kangaroos is the same in this case as when the kangaroos start their walks at $N(0+z)$,$N(b-z)$,$N(c+z)$,t_1N and t_2N (that is, if we subtract dN from all kangaroos starting positions). Hence if we use the same average step size in both cases, the running time in both algorithms will be the same. Hence for any algorithm where $a > 0$, there exists an algorithm where $a = 0$ which has the same running time. Hence we only need to check the case where $a = 0$, so we can claim that $a_{opt} = 0$ □

Justification of (B). The bound presented here is very loose. If $b-a > 2.5$, or $b < a-0.5$ then $\forall\ z$, on an interval of size N, the initial distance between the kangaroos that start their walks at $a + z$ and $b - z$ is at least $N/2$ (when $z = 0$, $|(a + z) - (-0.5 - z)| = 0.5$, and when $z = 1$, $|(a + z) - (2.5 - z)| = 0.5$). Now in the three kangaroo method, on an interval of size N, the furthest the initial distance between the closest useful pair of kangaroos could be across all z was $N/5$ (this occurred when z was 0, $2N/5$, $3N/5$ and N). Now in a good five kangaroo method, since there are more kangaroos (than in a three kangaroo method), we can expect to the furthest initial distance between the closest useful pair of kangaroos across all z to be smaller than $N/5$. Hence if a pair of kangaroos in a five kangaroo method always starts their walks at least distance $N/2$ apart, such a pair is highly unlikely to ever collide before the closest useful pair collides, at any z, and will therefore be extremely unlikely to ever be the pair which collides first. In a good 5 kangaroo algorithm, it would be natural to suppose that every useful pair of kangaroos can be the pair whose collision leads to the solving of the IDLP (i.e.

Algorithm 2

function SCANREGION(b_{min},b_{max},c_{min},c_{max},$t1_{min}$,$t1_{max}$,$t2_{min}$,$t2_{max}$,$c_{m_{min}}$,$c_{m_{max}}$,gap)
  minaverage$c_z \leftarrow \infty$
  $b_{values} = \{b_{min} + k \times gap | k \in \mathbb{N}, b_{min} \leq b_{min} + k \times gap \leq b_{max}\}$
  $c_{values} = \{c_{min} + k \times gap | k \in \mathbb{N}, c_{min} \leq c_{min} + k \times gap \leq c_{max}\}$
  $t1_{values} = \{t1_{min} + k \times gap | k \in \mathbb{N}, t1_{min} \leq t1_{min} + k \times gap \leq t1_{max}\}$
  $t2_{values} = \{t2_{min} + k \times gap | k \in \mathbb{N}, t2_{min} \leq t2_{min} + k \times gap \leq t2_{max}\}$
  $c_{m_{values}} = \{c_{m_{min}} + k \times gap | k \in \mathbb{N}, c_{m_{min}} \leq c_{m_{min}} + k \times gap \leq c_{m_{max}}\}$
  Combinations = $b_{values} \times c_{values} \times t1_{values} \times t2_{values} \times c_{m_{values}}$
  for each $\{b, c, t_1, t_2, c_m\} \in$ Combinations **do**
    averagec_z = COMPUTEAVERAGEc_z(0,b,c,t_1,t_2,c_m,3)
    if AVERAGEc_z < minaveragec_z **then**
      minaverage$c_z \leftarrow$ averagec_z
      Best$b \leftarrow b$
      Best$c \leftarrow c$
      Best$t_1 \leftarrow t_1$
      Best$t_2 \leftarrow t_2$
      Best$c_m \leftarrow c_m$
  return minaveragec_z,Best$b$,Best$c$,Bestt_1,Bestt_2,Bestc_m

$b_{min} \leftarrow -1/2$,$b_{max} \leftarrow 5/2$,$c_{min} \leftarrow 0$,$c_{max} \leftarrow 3$,$t1_{min} \leftarrow t2_{min} \leftarrow -2$,$t1_{max} \leftarrow t2_{max} \leftarrow 9/2$,$c_{m_{min}} \leftarrow 0.06697$,$c_{m_{max}} \leftarrow 0.5330$
gap $= 2 \times 10^{-2}$
PreviousBestAverage$c_z \leftarrow \infty$
[CurrentBestAveragec_z,Best$b$,Best$c$,Bestt_1,Bestt_2,Bestc_m] $\leftarrow$ SCANREGION(b_{min},b_{max},c_{min},c_{max},$t1_{min}$,$t1_{max}$,$t2_{min}$,$t2_{max}$,$c_{m_{min}}$,$c_{m_{max}}$,gap)
while PreviousBestAveragec_z - CurrentBestAveragec_z $\leq 10^{-4}$ **do**
  $b_{min} \leftarrow$ Best$b$ - gap, $b_{max} \leftarrow$ Best$b$ + gap, $c_{min} \leftarrow$ Best$c$ - gap,
  $c_{max} \leftarrow$ Best$c$ + gap, $t1_{min} \leftarrow$ Bestt_1 - gap,
  $t1_{max} \leftarrow$ Bestt_1 + gap,$t2_{min} \leftarrow$ Bestt_2 - gap,$t2_{max} \leftarrow$ Bestt_2 + gap,
  $c_{m_{min}} \leftarrow$ Bestc_m - gap, $c_{m_{max}} \leftarrow$ Bestc_m + gap
  gap $\leftarrow$ gap/10
  PreviousBestAverage$c_z \leftarrow$ CurrentBestAveragec_z
  [CurrentBestAveragec_z,Best$b$,Best$c$,Bestt_1,Bestt_2,Bestc_m] $\leftarrow$ SCANREGION(b_{min},b_{max},c_{min},c_{max},$t1_{min}$,$t1_{max}$,$t2_{min}$,$t2_{max}$,$c_{m_{min}}$,$c_{m_{max}}$,gap)
AccurateAveragec_z = COMPUTEAVERAGEc_z(0,Best$b$,Best$c$,Bestt_1,Bestt_2,Bestc_m,6)
return AccurateAveragec_z,Best$b$,Best$c$,Bestt_1,Bestt_2,Bestc_m

can collide first), at some z. Hence it is not desirable for there to be a useful pair which always starts its walk at least distance of $0.5N$ apart on an interval of size N. □

Justification of (C). Firstly let $W_{1,1}$ denote the kangaroo that starts its walk at $N(a+z)$, and $W_{1,2}$ denote the kangaroo that starts at $N(c+z)$. Since these kangaroos are of the same types (they're both WILD1 kangaroos), if we fix c,t_1,t_2 and c_m, then an algorithm where $W1,1$ starts at $a+z$ and $W_{1,2}$ starts at $c+z$, has the same running time as an algorithm where $W_{1,1}$ starts at $c+z$ and $W_{1,2}$ starts at $a+z$, since the 2 WILD1 walks that start at the same positions in both cases. Hence $c_{opt} \geq a_{opt} = 0$. Now $c_{opt} \leq 3$, since if $c > 3$, then on an interval of size N, the distance between the kangaroos that start their walks at $N(c+z)$ and $N(b-z)$ can never be less than $N/2$. I gave evidence that this was not desirable in my justification of **(B)**. Hence we may suppose that $0 \leq c_{opt} \leq 3$.

□

Justification of (D). Firstly, we only need to check values where $t_{2_{opt}} \geq t_{1_{opt}}$, because one can show in a similar way to the one shown in proving $c_{opt} \geq a_{opt}$, that for every case where $t_2 < t_1$, there exists an algorithm with the same running time where $t_1 < t_2$. Now $-2 \leq t_1, t_2 \leq 9/2$, since if t_1 and t_2 are outside this range, then the starting distance between any wild kangaroo and any of the tame kangaroos on an interval of size N is at least $N/2$.

□

Justification of (E). I stated in Section 3.2 that by starting the kangaroos walks at h, $g^{0.7124N}h$, $g^{1.3562N}h^{-1}$,$g^{0.9274N}$ and $g^{0.785N}$, the lower bound for the expected number of group operations until the closest useful pair collides (E_{CP}) of $1.8898\sqrt{N}$ group operations could be realised. Using Lemma 3.2.1 in the proof of Theorem 3.2, the average distance between the closest useful pair across all z when $E_{CP} = 1.8898\sqrt{N}$ group operations is $0.0357N$. In the proof of the same lemma, I also showed that $E_{CP} = 5(\text{Ave}(d(z))/m + m)$, where $\text{Ave}(d(z))$ denotes the average distance between the closest useful pair of kangaroos over all z. Hence $\text{Ave}(d(z)) \geq 0.0357N$ in any 5 kangaroo algorithm. Therefore, $E_{CP} \geq 5({}^{0.0357N}/_{c_m\sqrt{N}} + c_m\sqrt{N})$. One could suppose that there must be some bound B, such that if the expected number of group operations for the closest useful pair to collide in some algorithm is larger than B, then this algorithm could have no chance of being the optimal 5 kangaroo algorithm. A very loose bound on B is $3\sqrt{N}$. For $E_{CP} \leq 3\sqrt{N}$, we require $5(0.0357N/c_m\sqrt{N} + c_m\sqrt{N}) \leq 3\sqrt{N}$.

For this to occur, c_m must range between 0.06697, and 0.5330. Hence we can suppose $0.06697 \leq c_{m_{opt}} \leq 0.5330$.

□

Remark regarding Lemma 3.3.2

Lemma 3.3.2 required an upper bound on how far apart a useful pair of kangaroos can be when they start their walks. When t_2 is 4.5, $b = -0.5$, and $zN = N - 1$, the initial distance between the WILD2 kangaroo that starts its walk at $(b - z)N$, and the TAME kangaroo that starts his walk at t_2N, is $6N - 1 < 6N$. In any method where the variables a, b, c, t_1 and t_2 are within the bounds presented in (A),(B),(C) and (D), no useful pair of kangaroos can start their walks further apart than this.
The same lemma also required a lower bound on the average step size used. (E) shows that one lower bound is $0.06697\sqrt{N}$.

Results

When I ran Algorithm 2 in MATLAB®, the values for a,b,c,t_1,t_2 and c_m which were returned had $a = 0$, $b = 1.3578$, $c = 0.7158$, $t_1 = 0.7930$, $t_2 = 0.9220$, and $c_m = 0.3118$. Algorithm 2 computed averagec_z to be 0.3474 in this case.
Using the result of **(14)** (see the top of page 40), we see that in an algorithm where a,b,c,t_1,t_2 and c_m are defined to be these values requires on average $(5 \times 0.3474 + o(1))\sqrt{N} \pm O(1) = (1.737 + o(1))\sqrt{N} \pm O(1)$ group operations to solve the IDLP.
Formula **(13)** also implies that the expected number of steps until the first collision will be $0.3474\sqrt{N} \pm O(1)$ steps **(15)**.

3.4 Five Kangaroo Method

I now present my five kangaroo algorithm.
If we are solving the IDLP on an interval of size N over a group G, and we start the walks of five kangaroos at the group elements h, $g^{1.3578N}h^{-1}$, $g^{0.7158N}h$, $g^{0.922N}$, and $g^{0.793N}$, and use an average step size of $0.3118\sqrt{N}$, and let the kangaroos walk around the group G in the manner defined in section 3.1, then we can expect to solve the IDLP

in on average $(1.737 + o(1))\sqrt{N} \pm O(1)$ group operations.
Note that $1.3578N$,$0.7158N$,$0.922N$, and $0.793N$ might not be integers, so in practice, the kangaroos would start their walks at h, $g^b h^{-1}$, $g^c h$, g^{t_1}, and g_{t_2}, where b,c,t_1, and t_2 are respectively the closest integers to $1.3578N$,$0.7158N$,$0.922N$, and $0.793N$.
There are two main factors which have not been accounted for in this calculation of the running time. The first is eluded to in the remark, stated just before Lemma 3.3.1. This being, that I calculate the expected running time in the instance where for every z, the back kangaroo takes the expected number of steps to catch up to the starting position of the front kangaroo in every useful pair of kangaroos, and make the assumption that this is proportional to the average expected running time across all possible walks that the kangaroos can make. I was unable to find a bound for how much the approximation error would be increased by because of this assumption. However, in [3], Galbraith, Pollard and Ruprai make the same assumption in computing the running time of the three and four kangaroo methods. They were also unable to find a bound for much the approximation error would be increased by because of this assumption. However, when Galbraith, Pollard and Ruprai gathered experimental results to test how well their heuristic estimates worked in practice, they found that their experimental results matched their estimated results well [3]. Hence I can assume the magnitude of the approximation error is not affected too badly as a result of this assumption.
The other factor that wasn't accounted for in my calculation of the running time, was how I ignore the $e^{\frac{-i+1}{m}}$ term in **(9)** (see page 37). Since $m = 0.3118\sqrt{N}$ in my 5 kangaroo algorithm, and in practice, one would typically use kangaroo methods on intervals of size at least 2^{30}, the multiplicative factor of the approximation error due to this is has a lower bound of $e^{\frac{-i+1}{m}} \geq e^{\frac{-7}{0.3118\sqrt{N}}} \geq e^{\frac{7}{0.3118\sqrt{2^{30}}}} \geq 1 - (7 \times 10^{-4})$, and an upper bound of 1.
This five kangaroo algorithm is therefore a huge improvement on the previously optimal five kangaroo method of Galbraith, Pollard, and Ruprai, which required at least $2\sqrt{N}$ group operations to solve the IDLP on average. This algorithm also beats the running time of the three kangaroo method, so it therefore answers one of the main questions of this dissertation. This being, 'Are five kangaroos worse than three?'.

Chapter 4

Seven Kangaroo Method

Using the same idea that was applied by Galbraith, Pollard and Ruprai in [3] in extending the three kangaroo method to the four kangaroo method, the five kangaroo method can be extended to give a seven kangaroo method, with slightly improved running time. If we are solving the IDLP on an interval of size N, let A,B, and C respectively be the closest *even* integers to 0,1.3578N, and 0.7158N, and let T_1 and T_2 be the closest integers to 0.422N, and 0.793N. Suppose we start the walks of 7 kangaroos at the group elements h, $g^B h^{-1}$, $g^C h$, g^{T_1}, g^{T_1+1}, g^{T_2}, and g^{T_2+1}. Then we are effectively starting two WILD1 kangaroos at positions z, and $C+z$, one WILD2 kangaroo at the position $B-z$, and four TAME kangaroos at the positions T_1, T_1+1, T_2 and T_2+1. Now z, $B-z$ and $C+z$ are either all odd, or all even. Hence all WILD1 and WILD2 kangaroos start their walks an even distance apart. Also, exactly one of T_1 and T_1+1, and exactly one of T_2 and T_2+1 are of the same parity as z, $B-z$ and $C+z$. I will let $T_{1_{useful}}$ and $T_{2_{useful}}$ denote the TAME kangaroos whose starting positions are of the same parity as the starting positions of the WILD1 and WILD2 kangaroos, and $T_{1_{useless}}$ and $T_{2_{useless}}$ denote the two other TAME kangaroos.
Now suppose we make all step sizes even. Then $T_{1_{useless}}$ and $T_{2_{useless}}$ will be unable to collide with any other kangaroo, excluding themselves. However, $T_{1_{useful}}$,$T_{2_{useful}}$, and all WILD1 and WILD2 kangaroos are able to collide, and are all starting their walks at at almost the exact same positions as the five kangaroos do in the five kangaroo method of section 3.4. Hence, assuming the algorithm is arranged so that all kangaroos jump one after the other in some specified order, then these 5 kangaroos ($T_{1_{useful}}$,$T_{2_{useful}}$, the WILD2 kangaroo, and the two WILD1 kangaroos) are effectively performing the five kangaroo method of section 3.4, except over an interval of size $N/2$, since the fact that

all step sizes are even means that only every second group element is being considered in this method. Hence from the statement of **(15)** (see the results section on page 44), the expected number of steps until the first collision occurs is $\left(0.3474\sqrt{N/2} \pm O(1)\right)$ $= 0.2456\sqrt{N} \pm O(1)$ steps. However, since there are now 7 kangaroos jumping at each step, the expected number of group operations until the first collision occurs is $7 \times 0.2456\sqrt{N} \pm O(1) = 1.7195\sqrt{N} \pm O(1)$ group operations. By defining the probability that a point is distinguished in the same way as it was in the five kangaroo method, we can conclude that the seven kangaroo method presented here requires on average an estimated $\left(1.7195 + o(1)\right)\sqrt{N} \pm O(1)$ group operations to solve the IDLP.
As I have already stated, currently the fastest kangaroo method is the four kangaroo method. This has an estimated average running time of $\left(1.714 + o(1)\right)\sqrt{N}$ group operations. Therefore, the seven kangaroo method presented here is very close to being the optimal kangaroo method.

Chapter 5

Conclusion

The main results of this thesis were the following.

- **Result 1:** The presentation of a five kangaroo algorithm which requires on average $(1.737 + o(1))\sqrt{N} \pm O(1)$ group operations to solve the IDLP.
- **Result 2:** The presentation of a seven kangaroo method that requires on average $(1.7195 + o(1))\sqrt{N} \pm O(1)$ group operations to solve the IDLP.

For clarity, I will restate the main questions that this thesis attempted to answer here also.

- **Question 1:** Can we improve kangaroo methods by using larger numbers of kangaroos?
- **Question 2:** Are 5 kangaroos worse than three?

Before this dissertation, the four kangaroo method of Galbraith, Pollard, and Ruprai [3] was by the far the best kangaroo algorithm. This algorithm has an estimated average running time of $(1.714 + o(1))\sqrt{N}$ group operations. It was unknown whether we could beat the running time of the four kangaroo method by using more than four kangaroos. The fastest algorithm that used more than four kangaroos was far slower

than the four kangaroo method, requiring on average at least $2\sqrt{N}$ group operations to solve the IDLP.

The five and seven kangaroo algorithms presented in this report are a significant improvement in kangaroo methods that use more than four kangaroos. Even though the running time of the methods presented in this thesis did not beat the running time of the four kangaroo method, they came very close to doing so. Therefore, even though the main question of this dissertation (Question 1) still remains unanswered, we can now have more confidence that kangaroo methods can be improved by using more than four kangaroos.

Question 2 was also answered in this thesis, since the estimated average running time of the five kangaroo method presented in section 3.4 ($(1.737 + o(1))\sqrt{N} \pm O(1)$ group operations) beat the estimated average running time of the three kangaroo method of [3] ($(1.818 + o(1))\sqrt{N}$ group operations).

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