

Uncountably many groups with the same profinite completion

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January 10, 2018

We use the construction described in Section 13.4 of the book *Subgroup Growth*.

This starts with a rooted tree \mathcal{T} in which each vertex of level $n \geq 1$ has valency $1 + l_n$ (and the root has valency l_0).

Given a sequence $(T_n)_{n \geq 0}$ with $T_n \leq \text{Sym}(l_n)$ for each n we take

$$W = \varprojlim W_n$$

where $W_0 = T_0$ and $W_{n+1} = T_n \wr W_{n-1}$ for $n \geq 0$. We identify W as a subgroup of $\text{Aut}(\mathcal{T})$ in the natural way.

On pages 262-263 of *Subgroup Growth* we define four elements ξ, η, a and b of W , set $\Gamma = \langle \xi, \eta, a, b \rangle$, and prove that under certain conditions, Γ is both dense and satisfies the congruence subgroup property in W . Together, these imply that the natural homomorphism from $\widehat{\Gamma}$ to W is an isomorphism of profinite groups.

The conditions are as follows:

- (i) T_n is a doubly transitive subgroup of $\text{Sym}(l_n)$
- (ii) there exist a two-generator perfect group $P = \langle x, y \rangle$ and for each n an epimorphism $\phi_n : P \rightarrow T_n$
- (iii) the automorphisms ξ, η, a and b are built in a particular way out of the

$$\alpha_n = x\phi_n, \beta_n = y\phi_n \in T_n \leq \text{Sym}(l_n).$$

Now let's get down to specifics. Let $l_n = 5$ and $T_n = \text{Alt}(5)$ for all n . Put

$$\alpha = (123), \beta = (12345).$$

Let $\lambda \in \{0, 1\}^{\mathbb{N}_0}$ and set

$$\begin{aligned} \alpha_n &= \alpha, \beta_n = \beta \text{ if } \lambda_n = 0 \\ \alpha_n &= \beta, \beta_n = \alpha \text{ if } \lambda_n = 1. \end{aligned}$$

Let $P = \text{Alt}(5) \times \text{Alt}(5)$, $x = (\alpha, \beta)$, $y = (\beta, \alpha) \in P$. It's an easy exercise to check that $P = \langle x, y \rangle$, and we define $\phi_n : P \rightarrow T_n$ by

$$\begin{aligned} x\phi_n &= \alpha^{1-\lambda_n} \cdot \beta^{\lambda_n} \\ y\phi_n &= \alpha^{\lambda_n} \cdot \beta^{1-\lambda_n} \end{aligned}$$

(so ϕ_n is simply the projection of $P = \text{Alt}(5) \times \text{Alt}(5)$ onto either the first or the second direct factor).

Let $\Gamma(\lambda) = \langle \xi(\lambda), \eta(\lambda), a(\lambda), b(\lambda) \rangle$ denote the group Γ constructed as above using the sequence λ . There are 2^{\aleph_0} such sequences, so we have constructed 2^{\aleph_0} 4-generator subgroups of $\text{Aut}(\mathcal{T})$ with profinite completion W . (These groups are of course residually finite since they $\text{Aut}(\mathcal{T})$ is.)

Claim: For each sequence λ , the set $S(\lambda) := \{\mu \mid \Gamma(\lambda) \cong \Gamma(\mu)\}$ is countable.

The claim implies that the number of isomorphism classes among the groups $\Gamma(\lambda)$ is still 2^{\aleph_0} .

Sketch proof of claim:

Suppose that $S(\lambda)$ is uncountable. For $\mu \in S(\lambda)$ let $\theta_\mu : \Gamma(\mu) \rightarrow \Gamma(\lambda)$ be an isomorphism. Then θ_μ extends to a continuous automorphism σ_μ of W (universal property of profinite completions).

Now the set

$$\{a(\mu)^\sigma \mid \mu \in S(\lambda), \sigma \in \text{Aut}(W)\} \subseteq \Gamma(\lambda)$$

is countable because $\Gamma(\lambda)$ is a finitely generated group. Hence there exists $c \in \Gamma(\lambda)$ such that the set

$$X := \{\mu \in S(\lambda) \mid a(\mu)^{\sigma_\mu} = c\}$$

is uncountable (all we need is: of cardinality at least 2).

Let $\mu \neq \nu \in X$. Then

$$a(\mu)^{\sigma_\mu \sigma_\nu^{-1}} = a(\nu).$$

Now for some n we have $\mu_n \neq \nu_n$. Say $a(\mu)_n = \alpha$ and $a(\nu)_n = \beta$. The (continuous!) automorphism $\sigma_\mu \sigma_\nu^{-1}$ of W induces an automorphism τ on the quotient

$$W_n = (\text{Alt}(5) \times \text{Alt}(5) \times \text{Alt}(5) \times \text{Alt}(5) \times \text{Alt}(5)) \rtimes W_{n-1}$$

(*exercise!*), sending the coset of $a(\mu)$ to that of $a(\nu)$:

$$(\alpha, 1, 1, 1, 1) \cdot u \xrightarrow{\tau} (\beta, 1, 1, 1, 1) \cdot v$$

in an obvious notation. Examining the structure of $\text{Aut}(W_n)$ (this is not a hard exercise) we find that this forces

$$(\beta, 1, 1, 1, 1) = (*, \dots, *, \alpha^z, *, \dots, *)$$

for some automorphism z of $\text{Alt}(5)$. This implies that $\beta = \alpha^z$, a contradiction since α and β have distinct orders.

Thus $S(\lambda)$ must be countable.

So we have established

Theorem 1 *There are continuously many pairwise non-isomorphic 4-generator residually finite groups all having the iterated wreath product W as their profinite completion.*

Remarks

1. There is lots of flexibility in this construction - we could use any family of finite images of a fixed finitely generated perfect group, provided they each have a primitive faithful permutation representation; these are used for example in my paper ‘The finite images of finitely generated groups’, *Proc. London Math. Soc.* **82** (2001), 597–613.

2. A similar but (to me) more difficult construction was published by V. Nekrashevych in *arXiv:1303.5782v2* [math.GR] 26 Mar 2013.

3. Uncountably many groups having the same product of finite alternating groups as their profinite completion were constructed by Laci Pyber in ‘Groups of intermediate subgroup growth and a problem of Grothendieck’, *Duke Math. J.* **121** (2004), 169–188.