REGULAR ORBITS OF SYMMETRIC AND ALTERNATING GROUPS

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Abstract. Given a finite group G and a faithful irreducible FG-module V where F has prime order, does G have a regular orbit on V? This problem is equivalent to determining which primitive permutation groups of affine type have a base of size 2. In this paper, we classify the pairs (G, V) for which G has a regular orbit on V where G is a covering group of a symmetric or alternating group and V is a faithful irreducible FG-module such that the order of F is prime and divides the order of G.

1. Introduction

Let G be a finite group acting faithfully on a set Ω. A *base* $\mathscr{B}$ for G is a non-empty subset of Ω with the property that only the identity fixes every element of $\mathscr{B}$; if $\mathscr{B} = \{\omega\}$ for some $\omega \in \Omega$, then the orbit $\{\omega g : g \in G\}$ of G on Ω is *regular*. Bases have been very useful in permutation group theory in the past half century, both theoretically in bounding the order of a primitive permutation group in terms of its degree (e.g., [3]) and computationally (cf. [34]). Recently, much work has been done on classifying the finite primitive permutation groups of almost simple and diagonal type with a base of size 2 [8–10, 12]. In this paper, we consider this problem for primitive permutation groups of affine type.

A finite permutation group X is *affine* if its socle is a finite-dimensional $\mathbb{F}_p$-vector space V for some prime p, in which case $X = V : X_0$ and $X_0 \leqslant \mathrm{GL}(V)$, where X_0 denotes the stabiliser of the vector 0 in X. Such a group X is primitive precisely when V is an irreducible $\mathbb{F}_pX_0$-module, in which case we say that X is a *primitive permutation group of affine type*. Note that X has a base of size 2 on V if and only if X_0 has a regular orbit on V. Thus classifying the primitive permutation groups of affine type with a base of size 2 amounts to determining which finite groups G, primes p, and faithful irreducible $\mathbb{F}_pG$-modules V are such that G has a regular orbit on V.

More generally, given a finite group G and a faithful FG-module V where F is any field, we can ask whether G has a regular orbit on V. This problem is of independent interest to representation theorists. Indeed, the classification of the pairs (G, V) for which G has no regular orbits on V where G is a p'-group that normalises a quasisimple group acting irreducibly on the faithful $\mathbb{F}_pG$-module V [16, 25, 26] provided an important contribution to the solution of the famous $k(GV)$-problem [32], which proved part of a well-known conjecture of Brauer concerning defect groups of blocks [7].

However, little work has been done on the regular orbit problem in the case where the characteristic of the field divides the order of the group. Hall, Liebeck and Seitz [19,

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Theorem 6] proved that if G is a finite quasisimple group with no regular orbits on a faithful irreducible FG-module V where F is a field of characteristic p, then either G is of Lie type in characteristic p, or $G = A_n$ where $p \leqslant n$ and V is the fully deleted permutation module (cf. §4.3), or (G, V) is one of finitely many exceptional pairs. These exceptional pairs are not known in general. Motivated by this result, we classify the pairs (G, V) for which G has a regular orbit on V where G is a scalar extension of a covering group H of the symmetric group S_n or the alternating group A_n and V is a faithful irreducible $\mathbb{F}_pH$-module such that $p \leqslant n$. The case where $p > n$ follows from [16, 25].

Let S be a finite group. A finite group L is a *covering group* or *cover* of S if $L/Z(L) \simeq S$ and $Z(L) \leqslant L'$. We say that L is a *proper* covering group when $Z(L) \neq 1$. The proper covering groups of S_n for $n \geqslant 5$ are $2.S_n^+$ and $2.S_n^-$, and these groups are isomorphic precisely when $n = 6$ [20, 33]. The proper covering groups of A_n are $2.A_n$ for $n \geqslant 5$, and $3.A_n$ and $6.A_n$ for $n = 6$ or 7 [20, 33]. The following is our main result.

Theorem 1.1. *Let H be a covering group of S_n or A_n where $n \geqslant 5$. Let G be a group for which $H \leqslant G \leqslant H \circ \mathbb{F}_p^*$ where p is a prime and $p \leqslant n$. Let V be a faithful irreducible $\mathbb{F}_pH$-module, and let $d := \dim_{\mathbb{F}_p}(V)$.*

(i) *If either V or $V \otimes_{\mathbb{F}_p} \mathrm{sgn}$ is the fully deleted permutation module of S_n, then G has a regular orbit on V if and only if $G = A_n$ and $p = n - 1$.*
(ii) *If neither V nor $V \otimes_{\mathbb{F}_p} \mathrm{sgn}$ is the fully deleted permutation module of S_n, then G has a regular orbit on V if and only if (n, p, G, d) is not listed in Table 1.*

n	p	G	d
5	2	A_5, S_5	4
	3	$A_5 \times \mathbb{F}_3^*$, $S_5 \times \mathbb{F}_3^*$	6
		$2.A_5$, $2.S_5^+$, $2.S_5^-$	4
	5	$H = 2.A_5$	2
		$2.S_5^- \circ \mathbb{F}_5^*$, $H = 2.S_5^+$	4
		$2.A_5 \circ \mathbb{F}_5^*$	4
6	2	A_6, S_6	4
		$3.A_6$	6
	3	$H \in \{A_6, S_6\}$	6
		$2.A_6$, $2.S_6$	4
	5	$H \in \{A_6, S_6\}$, $G \neq A_6$	5
		$H = 2.A_6$	4
		$G \neq H = 3.A_6$	6
7	2	A_7	4
		S_7	8, 14
		$3.A_7$	12
	3	$2.A_7$, $2.S_7^+$, $2.S_7^-$	8
7	5	$H = 3.A_7$	6
	7	$H \in \{2.A_7, 2.S_7^-\}$	4
		$H = 3.A_7$	6
8	2	A_8	4, 14
		S_8	8, 14
	3	$2.A_8$, $2.S_8^-$	8
	5	$2.S_8^+ \circ \mathbb{F}_5^*$, $H = 2.S_8^-$	8
9	2	A_9	8, 20
		S_9	16
	3	$2.A_9$, $2.S_9^-$	8
	5	$H = 2.A_9$	8
10	2	A_{10}, S_{10}	16
	3	$2.A_{10}$, $2.S_{10}^-$	16
	5	$H = 2.A_{10}$	8
11	3	$2.A_{11}$	16
12	2	S_{12}	32
	3	$2.A_{12}$	16

TABLE 1. $\mathbb{F}_pG$-modules V on which G has no regular orbits

The fully deleted permutation module is a faithful irreducible $\mathbb{F}_pS_n$-module of dimension $n-1$ when $p \nmid n$ and dimension $n-2$ otherwise; its restriction to A_n is always irreducible (cf. §4.3). The definitions of sgn and $H \circ \mathbb{F}_p^*$ are given in Section 2.

When H is specified in Table 1, we mean that $H \leqslant G \leqslant H \circ \mathbb{F}_p^*$ with no restrictions on G. Also, for certain d listed in Table 1, there exist multiple faithful irreducible $\mathbb{F}_pH$-modules of dimension d, none of which admit regular orbits; this includes the case where $A_n \leqslant H$ and d is the dimension of the fully deleted permutation module.

Theorem 1.1 follows from Theorem 4.1 and Remark 4.2 in the case where H is S_n or A_n, and Theorem 5.1 in the case where H is a proper covering group of S_n or A_n. Moreover, Theorems 4.1 and 5.1 are consequences of more general results concerning regular orbits of central extensions of almost simple groups with socle A_n (cf. Lemmas 4.4 and 5.3).

Observe that when $A_n \leqslant H$ and $n \geqslant 7$, representations only occur in Table 1 for $p = 2$. Moreover, when $2.A_n \leqslant H$, every representation listed in Table 1 is a basic spin module (cf. §5) except when $(n, p, G, d) = (5, 5, 2.A_5 \circ \mathbb{F}_5^*, 4)$.

It is well known that the group algebras of $2.S_n^+$ and $2.S_n^-$ are isomorphic over every field containing a primitive fourth root of unity. Thus, over such fields, the representation theory of $2.S_n^+$ and $2.S_n^-$ is essentially the same, and typically, in order to answer a representation theoretical question, it suffices to consider one of the double covers. However, this is not the case for the regular orbit problem. Indeed, even over a splitting field containing a primitive fourth root of unity, there is an example where only one double cover has a regular orbit; this occurs for $(n, p) = (8, 5)$ in Table 1. Other examples occur for $(n, p) = (5, 5)$, in which case $\mathbb{F}_p$ is not a splitting field but contains a primitive fourth root of unity, and $(n, p) = (7, 7)$, $(8, 3)$, $(9, 3)$ or $(10, 3)$, in which case $\mathbb{F}_p$ does not contain a primitive fourth root of unity.

As an immediate consequence of Theorem 1.1, we obtain a result concerning bases of primitive permutation groups of affine type.

Corollary 1.2. *Let X be a primitive permutation group of affine type with socle $V \simeq \mathbb{F}_p^d$ where p is a prime. Suppose that $H \leqslant X_0 \leqslant H \circ \mathbb{F}_p^*$ where H is a covering group of S_n or A_n for $n \geqslant 5$, and assume that $p \leqslant n$.*

- (i) *If either V or $V \otimes_{\mathbb{F}_p} \mathrm{sgn}$ is the fully deleted permutation module of S_n, then X has a base of size 2 on V if and only if $X_0 = A_n$ and $p = n - 1$.*
- (ii) *If neither V nor $V \otimes_{\mathbb{F}_p} \mathrm{sgn}$ is the fully deleted permutation module of S_n, then X has a base of size 2 on V if and only if (n, p, G, d) is not listed in Table 1 where $G := X_0$.*

In fact, it can be established by routine computations using Magma [6] that for (n, p, X_0, d) listed in Table 1 with $n \geqslant 7$, the affine group X has a base of size 3 with the following exceptions: $(7, 2, A_7, 4)$, $(8, 2, A_8, 4)$, $(8, 2, S_8, 8)$ and $(9, 2, A_9, 8)$, in which case X has a base of minimal size 4, 5, 4 and 4 respectively. When (i) holds, the minimal base size of X cannot be constant in general, for $|X|$ is not bounded above by $|V|^c$ for any absolute constant c. In either case, $d + 1$ is an upper bound on the minimal base size, for any basis of V is a base for X_0.

This paper is organised as follows. In §2 we collect some notation, definitions and basic results, and in §3 we determine some bounds for the dimensions of faithful representations admitting no regular orbits. In §4 we consider the regular orbits of S_n and A_n, and in §5 the regular orbits of the proper covering groups of S_n and A_n. In §6 we briefly comment on our computational methods.

2. Preliminaries

Unless otherwise specified, all groups in this paper are finite, and all homomorphisms and actions are written on the right.

Let G be a finite group. We denote the derived subgroup of G by G', the centre of G by $Z(G)$, the conjugacy class of $g \in G$ by g^G, and the generalised Fitting subgroup of G

by $F^*(G)$ (cf. [1] for a definition). The group G is *quasisimple* if $G = G'$ and $G/Z(G)$ is simple, and *almost quasisimple* if $G/Z(G)$ is almost simple.

Lemma 2.1. *Let G be a finite almost quasisimple group.*

(i) *$F^*(G) = F^*(G)'Z(G)$ and $F^*(G)/Z(G)$ is the socle of $G/Z(G)$.*
(ii) *$F^*(G)'$ is quasisimple and $Z(F^*(G)') = F^*(G)' \cap Z(G)$.*

Proof. Follows from [18, Lemma 2.1] and [1, 31.1]. □

In particular, if G is an almost quasisimple group and the socle of $G/Z(G)$ is A_n, then $F^*(G)'$ is a quasisimple group with $F^*(G)'/Z(F^*(G)') \simeq A_n$. Hence $F^*(G)'$ is a covering group of A_n, so $F^*(G)'$ is one of A_n or $2.A_n$ for $n \geqslant 5$, or $3.A_n$ or $6.A_n$ for $n = 6$ or 7. We will consider the regular orbit problem for almost quasisimple groups G with $F^*(G)' = A_n$ in §4 (cf. Lemma 4.4) and $F^*(G)' = 2.A_n$ in §5 (cf. Lemma 5.3).

Let F be a field. We denote the characteristic of F by $\mathrm{char}(F)$, the multiplicative group of F by F^*, and the group algebra of G over F by FG. All FG-modules in this paper are finite-dimensional, and we denote the dimension of an FG-module V by $\dim_F(V)$. We denote the finite field of order q by $\mathbb{F}_q$.

Let V be an FG-module. We say that V can be *realised over* a subfield K of F if there exists an F-basis $\mathscr{B}$ of V such that the matrix of the F-endomorphism g of V relative to $\mathscr{B}$ has entries in K for every $g \in G$. If F is a finite field and V has character χ, then V can be realised over K if and only if K contains $\chi(g)$ for all $g \in G$ [5, Theorem VII.1.17].

For an extension field E of F and an FG-module V, we denote the extension of scalars of V to E by $V \otimes_F E$ (cf. [11] for a definition). An irreducible FG-module V is *absolutely irreducible* if $V \otimes_F E$ is irreducible for every field extension E of F. Note that V is absolutely irreducible if and only if $\mathrm{End}_{FG}(V) = F$ [11, Theorem 29.13], where $\mathrm{End}_{FG}(V)$ denotes the set of FG-endomorphisms of V. The field F is a *splitting field* for G if every irreducible FG-module is absolutely irreducible.

Let F be a finite field, H a finite group and V a faithful FH-module, and let $S(H)$ be the set of $h \in H$ for which there exists $\lambda_h \in F^*$ such that $vh = \lambda_h v$ for all $v \in V$. Note that $S(H) \leqslant Z(H)$. The *central product* of H and F^*, denoted by $H \circ F^*$, is the quotient $(H \times F^*)/N$ where $N = \{(h, \lambda_h^{-1}) : h \in S(H)\}$. The FH-module V naturally becomes a faithful $F(H \circ F^*)$-module under the action $vN(h, \lambda) := (\lambda v)h$ for all $v \in V$, $h \in H$ and $\lambda \in F^*$. Now V is an irreducible $F(H \circ F^*)$-module if and only if V is an irreducible FH-module. Moreover, if V has dimension d and ρ is the corresponding representation of H in $\mathrm{GL}_d(F)$, then $H \circ F^* \simeq \langle H\rho, F^*\rangle = H\rho F^*$. If V is a faithful irreducible FH-module and $|Z(H)| \leqslant 2$, then $S(H) = Z(H)$, for a central involution must act as -1 on V.

Lemma 2.2. *Let F be a field, G a finite group and V an absolutely irreducible FG-module. For each $g \in Z(G)$, there exists $\lambda_g \in F^*$ such that $vg = \lambda_g v$ for all $v \in V$. If V is faithful, then the map $g \mapsto \lambda_g$ for all $g \in Z(G)$ is an injective homomorphism from $Z(G)$ to F^*.*

Proof. Let $g \in Z(G)$. The F-endomorphism of V defined by $v \mapsto vg$ for all $v \in V$ lies in $\mathrm{End}_{FG}(V) = F$, so the first claim holds. The second is straightforward. □

When F is a finite field and V is a (faithful) irreducible FG-module, we can use the field $k := \mathrm{End}_{FG}(V)$ to construct a (faithful) absolutely irreducible representation of G with the same G-orbits as V. Define k-scalar multiplication on the additive group V to be evaluation, and let G act in the same way. Now V is a (faithful) absolutely irreducible kG-module since $\mathrm{End}_{kG}(V) \subseteq \mathrm{End}_{FG}(V) = k$, and clearly G has a regular orbit on the FG-module V if and only if G has a regular orbit on the kG-module V.

Let H be a subgroup of G, and let V be an FG-module. We denote the restricted module of V from G to H by $V\downarrow H$. We say that $V\downarrow H$ *splits* if it is not irreducible.

Lemma 2.3. *Let G be a finite group with subgroup H, and let F be a field. If W is an irreducible FH-module, then there is an irreducible FG-module V for which $W \leqslant V \downarrow H$.*

Proof. Follows from Frobenius-Nakayama reciprocity [5, Theorem VII.4.5]. □

Let N be a normal subgroup of G, V an FG-module and W an irreducible FN-submodule of V. For $g \in G$, the normality of N implies that the *conjugate* Wg is an irreducible FN-submodule of V. The following is well known from Clifford theory.

Lemma 2.4. *Let G be a finite group, $N \trianglelefteq G$ and F a field. Let V be an irreducible FG-module and W an irreducible FN-submodule of V. Then $V \downarrow N$ is a direct sum of conjugates of W, and if $[G : N] = 2$, then $V \downarrow N = W$ or $W \oplus Wg$ for all $g \in G \setminus N$.*

Proof. Since $\sum_{g\in G} Wg$ is an FG-submodule of V, it is equal to V, and so the first claim holds. If $[G : N] = 2$ and $g \in G \setminus N$, then $V = W + Wg$, so the second claim holds. □

Let N be an index 2 subgroup of G. The *sign module*, denoted by sgn, is the one-dimensional FG-module for which $g \in N$ acts as 1 and $g \in G \setminus N$ acts as -1. For an FG-module V, the *associate* of V is the FG-module $V \otimes_F \text{sgn}$ where $(v\otimes\lambda)g := (vg)\otimes(\lambda g)$ for all $v \in V$, $\lambda \in \text{sgn}$ and $g \in G$.

3. Bounds for dimensions of non-regular representations

In this section, we determine bounds for the dimensions of faithful irreducible representations of almost quasisimple groups that admit no regular orbits. These are obtained using the standard technique of counting fixed points.

Let G be a finite group, F a field, and V an FG-module. We define $C_V(g) := \{v \in V : vg = v\}$ for all $g \in G$. For $X \subseteq G$, we define $[V, X] := \text{span}\{v - vg : v \in V, g \in X\}$, and when $X = \{g\}$, we write $[V, g]$. Note that $\dim_F(V) = \dim_F(C_V(g)) + \dim_F([V, g])$ for all $g \in G$. For $v \in V$, we denote the stabiliser of v in G by $C_G(v)$.

The following is a simple but crucial result.

Lemma 3.1. *Let G be a finite group and F a field. Let V be a faithful FG-module. If G has no regular orbits on V, then $V = \bigcup_{g\in G\setminus\{1\}} C_V(g)$.*

Proof. If $v \in V$ and $v \notin C_V(g)$ for all $1 \neq g \in G$, then v lies in a regular orbit of G. □

Lemma 3.1 implies that if V is a faithful FG-module where G is finite and F is infinite, then G has a regular orbit on V, for no vector space over an infinite field is a finite union of proper subspaces. Moreover, Lemma 3.1 gives us a bound for the size of V that is easily computed using Magma. To see this, we need the following useful observation about fixed points of central elements.

Lemma 3.2. *Let G be a finite group and F a field. Let V be a faithful irreducible FG-module. If $1 \neq g \in Z(G)$, then $C_V(g) = 0$.*

Proof. This follows from the fact that $C_V(g)$ is a proper FG-submodule of V. □

Now we provide the bound mentioned above.

Lemma 3.3. *Let G be a finite group and F a finite field. Let V be a faithful irreducible FG-module. If G has no regular orbits on V, then*

$$|V| \leqslant \sum_{g\in X} |g^G||C_V(g)|,$$

where X is a set of representatives for the conjugacy classes of non-central elements of prime order in G.

Proof. Let $0 \neq v \in V$. Now $v \in C_V(g_0)$ for some $g_0 \in G \setminus Z(G)$ by Lemmas 3.1 and 3.2. This implies that $C_G(v)$ is a non-trivial group, so there exists $h_0 \in C_G(v)$ of prime order and $v \in C_V(h_0)$. In particular, $h_0 \notin Z(G)$ by Lemma 3.2. Since $|C_V(g)| = |C_V(h^{-1}gh)|$ for all $g, h \in G$, the result follows. □

Let G be an almost quasisimple group where $G/Z(G)$ has socle T, and let $g \in G \setminus Z(G)$. Now $\langle T, Z(G)g\rangle$ is generated by the T-conjugates of $Z(G)g$, so we may define $r(g)$ to be the minimal number of T-conjugates of $Z(G)g$ generating $\langle T, Z(G)g\rangle$.

The following result appears in various incarnations in the literature; the version given here, which is essentially [18, Lemma 3.2], is the one most suited to our purposes; see also [26, Lemma 2] and the proof of [19, Theorem 6].

Lemma 3.4. *Let G be an almost quasisimple group and F a field. Let V be a faithful irreducible FG-module. Then*

$$\dim_F(C_V(g)) \leqslant \dim_F(V)\left(1 - \frac{1}{r(g)}\right)$$

for all $g \in G \setminus Z(G)$.

Proof. Let $\overline{g}$ denote the coset $Z(G)g$ for $g \in G$, and let $\overline{N} := N/Z(G)$ where $N/Z(G)$ is the socle of $G/Z(G)$. Fix $g \in G \setminus Z(G)$. Let $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_r$ be conjugates of $\overline{g} = \overline{g}_1$ that generate $\langle \overline{N}, \overline{g}\rangle$ where $r := r(g)$ and the representatives $g_2, \ldots, g_r$ are chosen to be conjugates of g in G. By this choice, $|C_V(g)| = |C_V(g_i)|$ for $1 \leqslant i \leqslant r$. Let $W := [V, \langle g_1, \ldots, g_r\rangle] = \operatorname{span}\{[V, g_i] : 1 \leqslant i \leqslant r\}$. Now W is spanned by $r(g)\dim_F([V, g])$ elements. Observe that $[V, N']$ is a subspace of $[V, \langle g_1, \ldots, g_r\rangle]$, for $N \leqslant \langle g_1, \ldots, g_r\rangle Z(G)$, and so $N' \leqslant \langle g_1, \ldots, g_r\rangle$. But $1 \neq N' \trianglelefteq G$ and V is faithful, so $[V, N']$ is a non-zero FG-submodule of V. Thus $[V, N'] = [V, \langle g_1, \ldots, g_r\rangle] = V$, so $r(g)\dim_F([V, g]) \geqslant \dim_F(V)$. Now $\dim_F(V) - \dim_F(C_V(g)) \geqslant \dim_F(V)/r(g)$, and the result follows. □

The next result is a natural generalisation of part of the proof of [19, Theorem 6].

Lemma 3.5. *Let G be an almost quasisimple group and V a faithful irreducible $\mathbb{F}_qG$-module where q is a power of a prime. If G has no regular orbits on V, then*

$$\dim_{\mathbb{F}_q}(V) \leqslant r(G)\log_q|G|,$$

where $r(G) := \max\{r(g) : g \in G \setminus Z(G)\}$.

Proof. By Lemmas 3.1, 3.2 and 3.4,

$$q^{\dim_{\mathbb{F}_q}(V)} = |V| \leqslant \sum_{g \in G\setminus Z(G)} q^{\dim_{\mathbb{F}_q}(C_V(g))} \leqslant |G|q^{\dim_{\mathbb{F}_q}(V)\left(1-\frac{1}{r(G)}\right)}.$$

Now $q^{\dim_{\mathbb{F}_q}(V)/r(G)} \leqslant |G|$, and so $\dim_{\mathbb{F}_q}(V) \leqslant r(G)\log_q|G|$. □

Now we give some more specific bounds for the case where the socle of $G/Z(G)$ is A_n.

Lemma 3.6. *Let G be an almost quasisimple group, and suppose that the socle of $G/Z(G)$ is A_n where $n \geqslant 5$. Let V be a faithful irreducible $\mathbb{F}_qG$-module where q is a power of a prime. If G has no regular orbits on V, then*

$$\dim_{\mathbb{F}_q}(V) \leqslant (n-1)\log_q|G|, \tag{1}$$

and if $n \geqslant 7$, then

$$\dim_{\mathbb{F}_q}(V) \leqslant \max\{(n-1)\log_q(n(n-1)|Z(G)|), \tfrac{n}{2}\log_q(2n!|Z(G)|)\}. \tag{2}$$

If $n \geqslant 7$ and $|Z(G)| \leqslant n$, then

$$\dim_{\mathbb{F}_q}(V) \leqslant \tfrac{n}{2}\log_q(2n!|Z(G)|). \tag{3}$$

Proof. If $g \in G \setminus Z(G)$, then $r(g) \leqslant n-1$ for $n \geqslant 5$ by [17, Lemma 6.1], so equation (1) follows from Lemma 3.5.

Suppose that $n \geqslant 7$. Then $G/Z(G) = S_n$ or A_n. Let $g \in G \setminus Z(G)$, and write $\overline{g}$ for the coset $Z(G)g$. If $\overline{g}$ is not a transposition, then $r(g) \leqslant n/2$ by [17, Lemma 6.1]. Let S_1 be the set of $g \in G$ for which $\overline{g}$ is a transposition, and let S_2 be the set of $g \in G \setminus Z(G)$ for which $\overline{g}$ is not a transposition. It follows from Lemmas 3.1, 3.2 and 3.4 that

$$|V| \leqslant \sum_{g\in S_1}|C_V(g)| + \sum_{g\in S_2}|C_V(g)| \leqslant |S_1|q^{\dim_{\mathbb{F}_q}(V)\left(1-\frac{1}{n-1}\right)} + |S_2|q^{\dim_{\mathbb{F}_q}(V)\left(1-\frac{2}{n}\right)},$$

and since $q^{\dim_{\mathbb{F}_q}(V)} = |V|$, we obtain that

$$1 \leqslant 2\max\{|S_1|q^{-\frac{1}{n-1}\dim_{\mathbb{F}_q}(V)}, |S_2|q^{-\frac{2}{n}\dim_{\mathbb{F}_q}(V)}\}.$$

If $1 \leqslant 2|S_1|q^{-\dim_{\mathbb{F}_q}(V)/(n-1)}$, then $\dim_{\mathbb{F}_q}(V) \leqslant (n-1)\log_q(2|S_1|)$. Similarly, if $1 \leqslant 2|S_2|q^{-2\dim_{\mathbb{F}_q}(V)/n}$, then $\dim_{\mathbb{F}_q}(V) \leqslant (n/2)\log_q(2|S_2|)$. Thus

$$\dim_{\mathbb{F}_q}(V) \leqslant \max\{(n-1)\log_q(2|S_1|), \tfrac{n}{2}\log_q(2|S_2|)\}.$$

Since $2|S_1| = n(n-1)|Z(G)|$ and $|S_2| \leqslant |G| \leqslant n!|Z(G)|$, we have proved equation (2).

Suppose in addition that $|Z(G)| \leqslant n$. First we claim that $n^5 \leqslant 2n!$ for $n \geqslant 8$. Note that $(n+1)^4 \leqslant 5n^4 \leqslant n^5$, so if $n^5 \leqslant 2n!$, then $(n+1)^5 \leqslant n^5(n+1) \leqslant 2(n+1)!$. Thus the claim holds by induction, and so $(n(n-1)|Z(G)|)^2 \leqslant 2n!|Z(G)|$ for $n \geqslant 8$. Now

$$(n-1)\log_q(n(n-1)|Z(G)|) \leqslant \tfrac{n}{2}\log_q(2n!|Z(G)|),$$

and so $\dim_{\mathbb{F}_q}(V) \leqslant (n/2)\log_q(2n!|Z(G)|)$ when $n \geqslant 8$ by equation (2). Now suppose that $n = 7$. It suffices to show that $(42|Z(G)|)^{12/7} \leqslant 2\cdot 7!|Z(G)|$ when $|Z(G)| \leqslant 7$, and this is true since $42^{12/7}|Z(G)|^{12/7-1} \leqslant 42^{12/7}7^{12/7-1} \leqslant 2\cdot 7!$. $\square$

Motivated by equations (2) and (3) of Lemma 3.6, we finish this section with a technical observation.

Lemma 3.7. *If C is an absolute constant where $C \geqslant 5$, then $\log_q(C(q-1))$ is a decreasing function in q for $q \in \mathbb{R}$ and $q \geqslant 2$.*

Proof. Let $f(q) := \log_q(C(q-1))$. Now $f'(q) < 0$ precisely when $q\log q < (q-1)\log(C(q-1))$. Subtracting $(q-1)\log q$ from both sides, we obtain $\log q < (q-1)\log(C(q-1)/q)$, so it suffices to prove that $q < C^{q-1}(1-1/q)^{q-1}$. But $C \geqslant 5$ and $q \geqslant 2$, so $q < (C/2)^{q-1} \leqslant C^{q-1}(1-1/q)^{q-1}$, as desired. $\square$

4. Symmetric and alternating groups

Our notation for this section follows that of James [21]. For a partition μ of n, let M^μ_F denote the permutation module of S_n on a Young subgroup for μ over a field F, and let S^μ_F denote the *Specht module* for μ over F, which is the submodule of M^μ_F spanned by the polytabloids. Let $<,>$ denote the unique S_n-invariant symmetric non-degenerate bilinear form on M^μ_F for which the natural basis of M^μ_F is orthonormal, and write $S^{\mu\perp}_F$ for the orthogonal complement of S^μ_F with respect to this form. Define

$$D^\mu_F := S^\mu_F/(S^\mu_F \cap S^{\mu\perp}_F).$$

When context permits, we omit the subscript F and write M^μ, S^μ, or D^μ.

It is well known that for a field F of characteristic p, the D^μ_F afford a complete list of non-isomorphic irreducible FS_n-modules as μ ranges over the p-regular partitions of n [21, Theorem 11.5]; recall that a partition μ is *p-regular* for p prime if no part of μ is repeated p times, and always 0-regular for convenience. In particular, when $p > n$ (or when $p = 0$), the S^μ_F afford a complete list of non-isomorphic irreducible FS_n-modules as μ ranges over the partitions of n. Every field is a splitting field for S_n [21, Theorem 11.5], and every field containing $\mathbb{F}_{p^2}$ for p prime is a splitting field for A_n (cf. [28, Corollary 5.1.5] or [29]).

For each p-regular partition μ of n, there exists a unique p-regular partition λ for which $D^\lambda \simeq D^\mu \otimes_F \mathrm{sgn}$, and we denote this partition by $m(\mu)$. Note that $m(\mu) = \mu$ when $p = 2$, so we omit $m(\mu)$ from Table 2 below. Moreover, given an irreducible FA_n-module V, there exists a p-regular partition μ for which $V \leqslant D^\mu \downarrow A_n$ by Lemma 2.3.

In this section, we prove the following theorem.

Theorem 4.1. *Let H be S_n or A_n where $n \geqslant 5$. Let G be a group for which $H \leqslant G \leqslant H \times \mathbb{F}_p^*$ where p is a prime and $p \leqslant n$. Let V be a faithful irreducible $\mathbb{F}_pH$-module and μ a p-regular partition of n for which $V \leqslant D^\mu \downarrow H$. Let $d := \dim_{\mathbb{F}_p}(V)$.*

(i) *If either μ or $m(\mu)$ is $(n-1,1)$, then G has a regular orbit on V if and only if $G = A_n$ and $p = n-1$.*

(ii) *If neither μ nor $m(\mu)$ is $(n-1,1)$, then G has a regular orbit on V if and only if (n,p,μ,G,d) is not listed in Table 2.*

n	p	μ	G	d	$m(\mu)$
5	2	(3,2)	A_5, S_5	4	-
	3	(3,1,1)	$A_5 \times \mathbb{F}_3^*$, $S_5 \times \mathbb{F}_3^*$	6	(3,1,1)
6	2	(4,2)	A_6, S_6	4	-
	3	(4,1,1)	$H \in \{A_6, S_6\}$	6	(4,1,1)
	5	(3,3)	$H \in \{A_6, S_6\}$, $G \neq A_6$	5	(2,2,2)
		(2,2,2)	$H \in \{A_6, S_6\}$, $G \neq A_6$	5	(3,3)
7	2	(4,3)	A_7	4	-
			S_7	8	-
		(5,2)	S_7	14	-
8	2	(5,3)	A_8	4	-
			S_8	8	-
		(6,2)	A_8, S_8	14	-
9	2	(5,4)	A_9	8	-
			S_9	16	-
		(5,3,1)	A_9	20	-
10	2	(6,4)	A_{10}, S_{10}	16	-
12	2	(7,5)	S_{12}	32	-

TABLE 2. $\mathbb{F}_pG$-modules V on which G has no regular orbits

For a partition μ of n, the dimension of D^μ is the rank of the Gram matrix with respect to a basis of S^μ. However, there is no formula that computes this rank in general, in contrast to the Specht module S^μ, whose dimension is given by the characteristic-independent hook formula [21, Theorem 20.1]. Thus we require lower bounds for the dimension of D^μ. These we obtain using a method of James [22], which requires the following notation.

Let F be a field of characteristic p. For each non-negative integer m, write $R_n(m)$ for the class of irreducible FS_n-modules V such that for some p-regular partition μ of n,

(i) $\mu_1 \geqslant n - m$ where μ_1 is the largest part of μ, and
(ii) $V \simeq D^\mu$ or $V \simeq D^\mu \otimes_F \mathrm{sgn}$.

Now [22, Lemma 4] and [22, Appendix Table 1] enable us to construct functions $f(n)$ with the property that for every irreducible FS_n-module V, either $V \in R_n(2)$ or $\dim_F(V) > f(n)$ (cf. Lemma 4.3).

Thus the proof of Theorem 4.1 divides into two cases. Suppose we are given a faithful irreducible $\mathbb{F}_pS_n$-module V on which G has no regular orbits. If $V \notin R_n(2)$, then $\dim_{\mathbb{F}_p}(V)$ is bounded below by $f(n)$ and above by functions of §3, and this is usually a contradiction. Otherwise $V \in R_n(2)$, in which case the functions of §3 are useless since $\dim_{\mathbb{F}_p}(V) \leqslant n^2$, so we use constructive methods instead. Note that for a field F, the only non-faithful irreducible FS_n-modules are the trivial module $D^{(n)}$ and the sign module $D^{(n)} \otimes_F \mathrm{sgn}$, and so an irreducible FS_n-module V is faithful if and only if $V \notin R_n(0)$.

We will often make use of the known Brauer character tables of the symmetric and alternating groups. The Brauer Atlas [23] contains the Brauer character tables of S_n and A_n for $n \leqslant 12$ and $p \leqslant n$, while GAP [13] in conjunction with the SpinSym package [27] contains the Brauer character tables of S_n and A_n for $n \leqslant 17$ and $p \leqslant n$, as well as $n = 18$ when $p = 2, 3, 5$ or 7, and $n = 19$ when $p = 2$. Moreover, for those character tables in [13], SpinSym provides a function to determine the corresponding partitions.

Remark 4.2. If $H = S_n$ or A_n and (n, p, G, d) is listed in Table 1, then (n, p, μ, G, d) is listed in Table 2 by [23]. Hence Theorem 1.1 follows from Theorem 4.1 for such H.

This section is organised as follows. In §4.1 we consider modules that are not in $R_n(2)$, and in §4.2 and §4.3 we consider modules that are in $R_n(2) \setminus R_n(1)$ and $R_n(1)$ respectively. Lastly, in §4.4 we prove Theorem 4.1.

4.1. **Modules not in $R_n(2)$.** The following lemma is the key tool for this case. It relies significantly on [22]. We include the case $p > n$ for completeness.

Lemma 4.3. *Let F be a field of positive characteristic p. Let V be an irreducible FS_n-module where $n \geqslant 15$ when $p = 2$ and $n \geqslant 11$ when p is odd. Let*

$$f(n) := \tfrac{1}{6}(n^3 - 9n^2 + 14n - 6).$$

For $p = 2$, let $f_p(n)$ be defined by $f_p(n) = f(n)$ for $n \geqslant 23$ and

$$\begin{aligned} f_p(15) = f_p(16) &= 127, \\ f_p(17) = f_p(18) &= 253, \\ f_p(19) = f_p(20) &= 505, \\ f_p(21) = f_p(22) &= 930. \end{aligned}$$

For odd p, let $f_p(n)$ be defined by $f_p(n) = f(n)$ for $n \geqslant 16$ and

$$\begin{aligned} f_p(11) &= 54, \\ f_p(12) &= 88, \\ f_p(13) &= 107, \\ f_p(14) &= 175, \\ f_p(15) &= 213. \end{aligned}$$

Then $V \in R_n(2)$ or $\dim_F(V) > f_p(n)$.

Proof. Suppose that there is a function $g : \mathbb{N} \to \mathbb{R}$ and a positive integer N for which:

(i) $2g(n) > g(n+2)$ for all $n \geqslant N$.

(ii) For $n = N$ or $N+1$, if U is an irreducible FS_n-module, then $U \in R_n(2)$ or $\dim_F(U) > g(n)$.

(iii) For all $n \geqslant N$, if $U \in R_n(4) \setminus R_n(2)$, then $\dim_F(U) > g(n)$.

Then [22, Lemma 4] implies that for all $n \geqslant N$, either $V \in R_n(2)$ or $\dim_F(V) > g(n)$. Thus it suffices to show that $f_p(n)$ satisfies conditions (i)-(iii) with $N = 15$ when $p = 2$ and $N = 11$ otherwise. Note that $2f_2(n) > f_2(n+2)$ for all $n \geqslant 15$, and if p is odd, then $2f_p(n) > f_p(n+2)$ for all $n \geqslant 11$. Moreover, using the lower bounds of [22, Appendix Table 1], it is routine to verify that if $U \in R_n(4) \setminus R_n(2)$ and $n \geqslant 11$, then $\dim_F(U) > f(n)$ unless U is $D^{(7,4)}$ or its associate, in which case $\dim_F(U) \geqslant 55 > f_p(11)$ for all odd p. Since $f(n) \geqslant f_p(n)$ for all p and $n \geqslant 11$, it remains to check condition (ii).

Let U be an irreducible FS_n-module, and suppose that U is not in $R_n(2)$. To begin, suppose that $p = 2$. If $n = 15$ or 16, then $\dim_F(U) > (n-1)(n-2)/2$ by [22, Theorem 7] since $U \notin R_n(2)$. Using the Brauer character table of S_n [13], we check that $\dim_F(U) \geqslant 128 > f_2(n)$. Thus condition (ii) holds with $N = 15$.

Now suppose that p is an odd prime and $n = 11$ or 12. First assume that $p \leqslant n$. Since $\dim_F(U) > (n-1)(n-2)/2$ by [22, Theorem 7], $\dim_F(U) \geqslant 55$ when $n = 11$ and $\dim_F(U) \geqslant 89$ when $n = 12$ by [23]. Thus $\dim_F(U) > f_p(n)$, as desired. Assume instead that $p > n$. Now $U \simeq S^\mu$ for some partition μ of n. The dimensions of the Specht modules are listed in the decomposition matrices in [21, Appendix]: $\dim_F(U) \geqslant 55$ when $n = 11$ and $\dim_F(U) \geqslant 89$ when $n = 12$. Thus condition (ii) holds with $N = 11$. $\square$

Note that the dimension of $D_F^{(n-3,3)}$ for a field F of positive characteristic is precisely $f(n)+1$ for infinitely many n by [22, Appendix Table 1], so Lemma 4.3 provides a tight lower bound for $\dim_F(V)$ for $V \notin R_n(2)$.

Let F be an arbitrary field. By [22, Theorem 5], there are only finitely many n for which $D^\mu \notin R_n(3)$ and $\dim_F(D^\mu) \leqslant n^3$. Motivated by classifying these exceptional modules, Müller [30] determined the dimensions of the irreducible FS_n-modules of dimension at most n^3 for $\mathrm{char}(F) \in \{2,3\}$ along with the corresponding partitions; we will use this information whenever character tables are not available.

We begin with a reduction for almost quasisimple groups G with $F^*(G)' \simeq A_n$.

Lemma 4.4. *Let G be an almost quasisimple group where $N := F^*(G)' \simeq A_n$ and $n \geqslant 11$. Let F be a finite field. Let V be a faithful irreducible FG-module, $k := \mathrm{End}_{FG}(V)$ and $q := |k|$. Let W be an irreducible kN-submodule of V and μ a $\mathrm{char}(F)$-regular partition of n for which $W \leqslant D^\mu \downarrow N$. If G has no regular orbits on V and $D^\mu \notin R_n(2)$, then $n \leqslant 20$, and if $\mathrm{char}(F) \leqslant n$ and q is odd, then $(n,q) = (11,5)$ and $\dim_k(W) = \dim_k(D^\mu) = 55$. Moreover, if $n \geqslant 15$ and q is even, then $(n, \mu, q, \dim_k(W), \dim_k(D^\mu))$ is listed in Table 3.*

In fact, $\dim_k(V) = \dim_k(W)$ or $2\dim_k(W)$ by Lemma 2.4, for W is an irreducible $kF^*(G)$-submodule of V by Lemmas 2.1 and 2.2, and $[G : F^*(G)] \leqslant 2$ by Lemma 2.1.

Proof of Lemma 4.4. Suppose that G has no regular orbits on V and $D^\mu \notin R_n(2)$. Let $p := \mathrm{char}(F)$. Since V is a faithful absolutely irreducible kG-module, Lemma 2.2 implies that $Z(G) \leqslant k^*$, and so $|Z(G)| \leqslant q-1$. Let

$$g(q,n) := \max\{(n-1)\log_q(n(n-1)(q-1)), \tfrac{n}{2}\log_q(2n!(q-1))\}.$$

Now equation (2) of Lemma 3.6 implies that $\dim_k(V) \leqslant \lfloor g(q,n) \rfloor$. Since $\dim_k(D^\mu)$ is equal to $\dim_k(W)$ or $2\dim_k(W)$ by Lemma 2.4, it follows that $\dim_k(D^\mu) \leqslant 2\lfloor g(q,n)\rfloor$. Note that if n is fixed, then $g(q,n)$ is a decreasing function in q by Lemma 3.7.

n	μ	q	$\dim_k(W)$	$\dim_k(D^\mu)$
15	$(8,7)$	2, 4, 8, 16, 32	64	128
16	$(9,7)$	2, 4, 8, 16, 32, 64	64	128
	$(13,3)$	2	336	336
17	$(9,8)$	2, 4, 8	128	256
18	$(10,8)$	2	256	256
19	$(10,9)$	2	512	512
		4	256	512
20	$(11,9)$	2	512	512
		4	256	512

TABLE 3. Possible $\dim_k(W)$ and $\dim_k(D^\mu)$ when $n \geqslant 15$ and q is even

To begin, suppose that q is odd. Recall the function $f_p(n)$ defined in Lemma 4.3. Since $n \geqslant 11$ and $D^\mu \notin R_n(2)$ by assumption, it follows from this lemma that $f_p(n) < \dim_k(D^\mu)$. Thus $f_p(n) < 2\lfloor g(q,n)\rfloor$. However, if $n \geqslant 21$, then $2\lfloor g(q,n)\rfloor \leqslant 2\lfloor g(3,n)\rfloor \leqslant f_p(n)$, a contradiction. Thus $n \leqslant 20$ for odd q, as claimed. Similarly, if $q \geqslant 121$, then we obtain a contradiction for all $n \geqslant 11$, so $q < 121$. Moreover, if $q \geqslant 5$, then $n \leqslant 15$; if $q \geqslant 9$, then $n \leqslant 14$; if $q \geqslant 11$, then $n \leqslant 13$; if $q \geqslant 25$, then $n \leqslant 12$; and if $q \geqslant 27$, then $n \leqslant 11$.

Hence if we assume that $p \leqslant n$, then (n, q) is listed in Table 4. Suppose that $q = 3$ and $n = 19$ or 20. Since $\dim_k(D^\mu) \leqslant 2g(3,n) \leqslant n^3$, we apply [30] to determine the dimensions of those D^μ for which $f_p(n) < \dim_k(D^\mu) \leqslant 2\lfloor g(3,n)\rfloor$. However, there are no such D^μ when $n = 20$, and when $n = 19$, the only possible dimension is 647, in which case $W = D^\mu \downarrow N$ since 647 is odd, so $\dim_k(D^\mu) \leqslant \dim_k(V) \leqslant \lfloor g(3,19)\rfloor = 352$, a contradiction. Similarly, for each remaining (n,q) besides $(11,5)$, we use the Brauer character tables in [13, 23, 27] to determine that if there exists a p-regular partition μ for which $f_p(n) < \dim_k(D^\mu) \leqslant 2\lfloor g(3,n)\rfloor$, then $W = D^\mu \downarrow N$ and $\lfloor g(q,n)\rfloor < \dim_k(D^\mu)$, a contradiction. Thus $(n,q) = (11,5)$, and by a similar argument, $\dim_k(W) = \dim_k(D^\mu) = 55$.

n	q
11	3, 5, 7, 9, 11, 25, 27, 49, 81
12	3, 5, 7, 9, 11, 25
13	3, 5, 7, 9, 11, 13
14	3, 5, 7, 9
15	3, 5, 7
$16 \leqslant n \leqslant 20$	3

TABLE 4. Possible odd q when $n \geqslant 11$

We may assume for the remainder of the proof that q is even and $n \geqslant 15$. Recall the function $f_2(n)$ defined in Lemma 4.3. As for odd q, it follows that $f_2(n) < \dim_k(D^\mu)$, and so $f_2(n) < 2\lfloor g(q,n)\rfloor$. However, if $n \geqslant 31$, then $2\lfloor g(q,n)\rfloor \leqslant 2\lfloor g(2,n)\rfloor \leqslant f_2(n)$, and if $n \geqslant 21$ and $q \geqslant 4$, then $2\lfloor g(q,n)\rfloor \leqslant 2\lfloor g(4,n)\rfloor \leqslant f_2(n)$, both contradictions. Thus either $n \leqslant 20$, or $q = 2$ and $21 \leqslant n \leqslant 30$.

Suppose for a contradiction that $q = 2$ and $21 \leqslant n \leqslant 30$. Since $\dim_k(D^\mu) \leqslant 2g(2,n) \leqslant n^3$, we apply [30] to determine the dimensions of those D^μ for which $f_2(n) < \dim_k(D^\mu) \leqslant 2\lfloor g(2,n)\rfloor$; these are listed in Table 5. Moreover, if $\dim_k(D^\mu) \neq 1024$, then $\mu = (n-3,3)$, and if $\dim_k(D^\mu) = 1024$, then $\mu = (11,10)$ or $(12,10)$. If $(n, \dim_k(D^\mu)) \neq (21, 1024)$, then $W = D^\mu \downarrow N$ by [4, Theorem 1.1], so $\dim_k(D^\mu) \leqslant \dim_k(V) \leqslant \lfloor g(2,n)\rfloor$. However, it can be verified that $\lfloor g(2,n)\rfloor < \dim_k(D^\mu)$ in each case, a contradiction. Similarly, if $n = 21$

and $\dim_k(D^\mu) = 1024$, then $D^\mu \downarrow A_n$ is irreducible over $k = \mathbb{F}_2$ by [4, Theorems 5.1 and 6.1], in which case $\dim_k(D^\mu) \leqslant \dim_k(V) \leqslant \lfloor g(2,21) \rfloor = 697$, a contradiction.

n	21, 22	23, 24	25, 26	28	30
$\dim_k(D^\mu)$	1024, 1120	1496	2000	2548	3248

TABLE 5. Possible $\dim_k(D^\mu)$ when $q = 2$ and $n \geqslant 21$

Thus q is even and $15 \leqslant n \leqslant 20$. Note that if $q \geqslant 128$, then $2\lfloor g(q,n) \rfloor \leqslant 2\lfloor g(128,n) \rfloor \leqslant f_2(n)$, a contradiction. Moreover, if $q \geqslant 8$, then $n \leqslant 18$; if $q \geqslant 16$, then $n = 18$ or $n \leqslant 16$; if $q \geqslant 32$, then $n \leqslant 16$; and if $q \geqslant 64$, then $n = 16$. Hence (n, q) is listed in Table 6.

n	q
15	2, 4, 8, 16, 32
16	2, 4, 8, 16, 32, 64
17	2, 4, 8
18	2, 4, 8, 16
19, 20	2, 4

TABLE 6. Possible even q when $15 \leqslant n \leqslant 20$

First suppose that $n = 20$. Since $\dim_k(D^\mu) \leqslant 2g(2,n) \leqslant n^3$, we apply [30] to determine that the only D^μ for which $f_2(n) < \dim_k(D^\mu) \leqslant 2\lfloor g(q,n) \rfloor$ are those with dimension 512 or 780 when $q = 2$ and dimension 512 when $q = 4$. Moreover, if $\dim_k(D^\mu) = 512$, then $\mu = (11,9)$, and if $\dim_k(D^\mu) = 780$, then $\mu = (17,3)$. If $q = 2$ and $\dim_k(D^\mu) = 780$, then $W = D^\mu \downarrow N$ by [4, Theorem 1.1], and so $\dim_k(D^\mu) \leqslant \dim_k(V) \leqslant \lfloor g(2,20) \rfloor = 620$, a contradiction. Thus $\dim_k(D^\mu) = 512$ and $q = 2$ or 4, in which case $D^\mu \downarrow A_n$ is irreducible if and only if $q = 2$ by [4, Theorems 5.1 and 6.1]. Thus either $q = 2$ and $\dim_k(W) = 512$, or $q = 4$ and $\dim_k(W) = 256$.

Similarly, using the Brauer character tables in [13, 23, 27], we determine for each remaining (n, q) in Table 6 that if there exists a 2-regular partition μ for which $f_2(n) < \dim_k(D^\mu) \leqslant 2\lfloor g(q,n) \rfloor$ and either $D^\mu \downarrow N$ splits, or $W = D^\mu \downarrow N$ and $\lfloor g(q,n) \rfloor \geqslant \dim_k(D^\mu)$, then $(n, \mu, q, \dim_k(W), \dim_k(D^\mu))$ is listed in Table 3. □

Now we are in a position to determine the regular orbits of $S_n \times \mathbb{F}_p^*$ on $\mathbb{F}_p S_n$-modules not in $R_n(2)$. We also prove some results for $\mathbb{F}_p S_n$-modules in $R_n(2) \setminus R_n(1)$ when n is small, as the inclusion of these cases simplifies the proof.

Proposition 4.5. *Let G be a group for which $S_n \leqslant G \leqslant S_n \times \mathbb{F}_p^*$ where $n \geqslant 7$ and p is a prime such that $p \leqslant n$. Let μ be a p-regular partition of n and V the $\mathbb{F}_p S_n$-module D^μ.*

(i) *If $D^\mu \notin R_n(2)$, then G has no regular orbits on V if and only if $p = 2$ and $\mu = (\lfloor n/2 \rfloor + 1, \lfloor (n-1)/2 \rfloor)$ for $7 \leqslant n \leqslant 10$ or $n = 12$.*

(ii) *If $D^\mu \in R_n(2) \setminus R_n(1)$ where either $n \leqslant 11$, or $12 \leqslant n \leqslant 14$ and $p = 2$, then G has no regular orbits on V if and only if $p = 2$ and $\mu = (n-2, 2)$ for $7 \leqslant n \leqslant 8$.*

Proof. We will prove (i) and (ii) simultaneously. Therefore, we will assume throughout this proof that either $D^\mu \notin R_n(2)$, or $D^\mu \in R_n(2) \setminus R_n(1)$ and either $n \leqslant 11$, or $12 \leqslant n \leqslant 14$ and $p = 2$. In particular, D^μ is faithful. Note that $\mathrm{End}_{\mathbb{F}_p G}(V) = \mathbb{F}_p$.

Suppose that G does not have a regular orbit on V. First consider the case where $p = 2$ and $n \geqslant 15$. Lemma 4.4 implies that μ and $\dim_{\mathbb{F}_2}(D^\mu)$ are listed in Table 3. If μ is $(9,7)$, $(13,3)$, $(10,8)$ or $(11,9)$, then using MAGMA (cf. §6 for further details), we determine that $S_n \times \mathbb{F}_p^*$ has a regular orbit on V, a contradiction. Otherwise, μ is $(8,7)$, $(9,8)$ or $(10,9)$, in which case $D^\mu = D^\lambda \downarrow S_n$ where λ is $(9,7)$, $(10,8)$ or $(11,9)$ respectively by [21, Theorem 9.3], so G has a regular orbit on V, a contradiction.

Next suppose that either $p = 2$ and $n \leqslant 14$, or p is odd. We claim that $(n, p, \dim_{\mathbb{F}_p}(D^\mu))$ is listed in Table 7. Note that if p is odd, then $n \leqslant 11$ by Lemma 4.4. Let

$$g(p,n) := \tfrac{n}{2} \log_p(2n!(p-1)).$$

Since $Z(G) \leqslant \mathbb{F}_p^*$ by Lemma 2.2 and $p \leqslant n$, equation (3) of Lemma 3.6 implies that $\dim_{\mathbb{F}_p}(V) \leqslant \lfloor g(p,n) \rfloor$. Note that if U is an irreducible $\mathbb{F}_p S_n$-module such that $U \in R_n(1)$ but U does not have dimension 1, then $\dim_{\mathbb{F}_p}(U)$ is either $n-2$ when $p \mid n$, or $n-1$ when $p \nmid n$ (cf. §4.3). Hence by [13, 23], the dimensions of those D^μ for which $D^\mu \notin R_n(1)$ and $\dim_{\mathbb{F}_p}(V) \leqslant \lfloor g(p,n) \rfloor$ are precisely those listed in Table 7, proving the claim.

n	p	$\dim_{\mathbb{F}_p}(D^\mu)$	n	p	$\dim_{\mathbb{F}_p}(D^\mu)$
7	2	$8^\times$, $14^\times$, 20	10	2	$16^\times$, 26, 48
	3	13, 15, 20		3	34, 36, 41
	5	8, 13, 15, 20		5	28, 34, 35
	7	10, 14		7	35, 36, 42
8	2	$8^\times$, $14^\times$, 40, 64	11	2	32, 44,100,144
	3	13, 21, 28, 35		3	34, 45
	5	13, 20, 21		5	43, 45, 55
	7	14, 19, 21		7	44, 45
9	2	$16^\times$, 26, 40, 48, 78		11	36, 44
	3	21, 27, 35, 41	12	2	$32^\times$, 44, 100, 164
	5	21, 27, 28, 34	13	2	64, 208
	7	19, 28	14	2	64, 208

TABLE 7. Possible $\dim_{\mathbb{F}_p}(D^\mu)$

Suppose that $\dim_{\mathbb{F}_p}(D^\mu)$ is listed in Table 7 with no adjacent $^\times$. Using MAGMA, we determine that $S_n \times \mathbb{F}_p^*$ has a regular orbit on V, a contradiction. All of these computations are routine except for $n = 12$ and $\dim_{\mathbb{F}_2}(V) = 44$; in this case we use ORB [31]. Also, we do not require MAGMA when $p = 2$ and $(n, \dim_{\mathbb{F}_p}(D^\mu))$ is one of $(11, 44)$, $(11, 100)$, or $(13, 208)$, for in these cases $D^\mu = D^\lambda \downarrow S_n$ where λ is a p-regular partition of $n+1$ such that $\dim_{\mathbb{F}_p}(D^\lambda)$ is listed in Table 7 by [21, Theorem 9.3].

Thus $\dim_{\mathbb{F}_p}(D^\mu)$ is listed in Table 7 with an adjacent $^\times$. From the decomposition matrices in [21], $\mu = (\lfloor n/2 \rfloor + 1, \lfloor (n-1)/2 \rfloor)$ when $\dim_{\mathbb{F}_2}(D^\mu) = 2^{\lfloor (n-1)/2 \rfloor}$ for $7 \leqslant n \leqslant 10$ or $n = 12$, and $\mu = (n-2, 2)$ when $\dim_{\mathbb{F}_2}(D^\mu) = 14$ for $7 \leqslant n \leqslant 8$, as desired.

Conversely, suppose that $p = 2$ and μ is either $(n-2, 2)$ for $7 \leqslant n \leqslant 8$, or $(\lfloor n/2 \rfloor + 1, \lfloor (n-1)/2 \rfloor)$ for $7 \leqslant n \leqslant 10$ or $n = 12$. Note that $G = S_n$. If $\mu = (6,2)$ or $(\lfloor n/2 \rfloor + 1, \lfloor (n-1)/2 \rfloor)$ for $7 \leqslant n \leqslant 10$, then $|V| < |G|$, so G has no regular orbits on V. If $\mu = (5,2)$, then no orbit is regular by MAGMA, and if $\mu = (7,5)$, then no orbit is regular by ORB [31]. □

Now we consider the regular orbits of $A_n \times \mathbb{F}_p^*$.

Proposition 4.6. *Let G be a group for which $A_n \leqslant G \leqslant A_n \times \mathbb{F}_p^*$ where $n \geqslant 7$ and p is a prime such that $p \leqslant n$. Let V be a faithful irreducible $\mathbb{F}_p A_n$-module, and let μ be a p-regular partition of n for which $V \leqslant D^\mu \downarrow A_n$. Suppose that $D^\mu \notin R_n(2)$.*

(i) *If $V \neq D^\mu \downarrow A_n$, then G has no regular orbits on V if and only if $p = 2$ and μ is $(5,3,1)$ or $(\lfloor n/2 \rfloor + 1, \lfloor (n-1)/2 \rfloor)$ for $7 \leqslant n \leqslant 9$.*

(ii) *If $V = D^\mu \downarrow A_n$, then G has no regular orbits on V if and only if $p = 2$ and $\mu = (6,4)$.*

Proof. (i) By Lemma 2.4, $D^\mu \downarrow A_n = V \oplus Vg$ for every $g \in S_n \setminus A_n$. Now $\mathrm{End}_{\mathbb{F}_pG}(V) = \mathbb{F}_p$ since for every field F of characteristic p, the irreducible FS_n-module $D^\mu \otimes_{\mathbb{F}_p} F$ restricted to A_n is $(V \otimes_{\mathbb{F}_p} F) \oplus (V \otimes_{\mathbb{F}_p} F)g$, and so $V \otimes_{\mathbb{F}_p} F$ must be irreducible by Lemma 2.4. Note that G has a regular orbit on V if and only if G has a regular orbit on Vg.

Suppose that G does not have a regular orbit on V. We claim that $(n, p, \dim_{\mathbb{F}_p}(D^\mu))$ is listed in Table 8. First suppose that $p = 2$ and $n \geqslant 15$. Lemma 4.4 implies that $\dim_{\mathbb{F}_2}(D^\mu)$ is listed in Table 3 where $q = 2$, $W = V$ and $k = \mathbb{F}_2$, so the claim holds.

(n,p)	(7,2)	(8,2)	(9,2)	(9,5)	(10,2)	(10,5)	(15,2)	(16,2)	(17,2)
$\dim_{\mathbb{F}_p}(D^\mu)$	$8^\times$	$8^\times$, 40	$16^\times$, $40^\times$	70	128	70	128	128	256

TABLE 8. Possible $\dim_{\mathbb{F}_p}(D^\mu)$ when $V \neq D^\mu \downarrow A_n$

Now suppose that either $p = 2$ and $n \leqslant 14$, or p is odd. Note that if p is odd, then $n \leqslant 11$ by Lemma 4.4. Let

$$h(p, n) := n \log_p(n!(p-1)/2).$$

By Lemma 2.2, $Z(G) \leqslant \mathbb{F}_p^*$, so Lemma 3.5 implies that $\dim_{\mathbb{F}_p}(V) \leqslant r(G)\log_p(n!(p-1)/2)$. Since $r(G) = r(A_n) \leqslant n/2$ by [17, Lemma 6.1], and since $\dim_{\mathbb{F}_p}(V) = \dim_{\mathbb{F}_p}(D^\mu)/2$, we obtain that $\dim_{\mathbb{F}_p}(D^\mu) \leqslant \lfloor h(p, n)\rfloor$. By [13, 23, 27], the dimensions of those D^μ for which $D^\mu \downarrow A_n$ splits (over $\mathbb{F}_p$) are precisely those listed in Table 8, proving the claim.

Suppose that $\dim_{\mathbb{F}_p}(D^\mu)$ is listed in Table 8 with no adjacent $^\times$. Using MAGMA, we determine that $A_n \times \mathbb{F}_p^*$ has a regular orbit on V, a contradiction. Thus $\dim_{\mathbb{F}_p}(D^\mu)$ is listed in Table 8 with an adjacent $^\times$. Using the decomposition matrices in [21], we determine that μ is $(5, 3, 1)$ when $\dim_{\mathbb{F}_2}(D^\mu) = 40$ and $(\lfloor n/2\rfloor + 1, \lfloor (n-1)/2\rfloor)$ when $\dim_{\mathbb{F}_2}(D^\mu) = 2^{\lfloor (n-1)/2\rfloor}$ for $7 \leqslant n \leqslant 9$, as desired.

Conversely, suppose that $p = 2$ and $\mu = (\lfloor n/2\rfloor + 1, \lfloor (n-1)/2\rfloor)$ for $7 \leqslant n \leqslant 9$ or $(5, 3, 1)$. Note that $G = A_n$. If $\mu \neq (5, 3, 1)$, then $|V| < |G|$, so G does not have a regular orbit on V, and if $\mu = (5, 3, 1)$, then we use MAGMA to check no orbit is regular.

(ii) If G has no regular orbits on V, then $S_n \times \mathbb{F}_p^*$ has no regular orbits on D^μ, so $p = 2$ and $\mu = (\lfloor n/2\rfloor + 1, \lfloor (n-1)/2\rfloor)$ for $7 \leqslant n \leqslant 10$ or $n = 12$ by Proposition 4.5. In particular, $G = A_n$. Since $D^\mu \downarrow A_n$ is irreducible, the only possibilities for μ are $(6, 4)$ or $(7, 5)$. If $\mu = (7, 5)$, then we use MAGMA to find a regular orbit of G on V, a contradiction. Hence $\mu = (6, 4)$, in which case $|V| < |G|$, so G does not have a regular orbit on V. □

4.2. **Modules in $R_n(2) \setminus R_n(1)$.** In this section, it is natural to work over an arbitrary field, and we obtain the following more general result for modules in $R_n(2) \setminus R_n(1)$.

Proposition 4.7. *Let H be S_n or A_n where $n \geqslant 5$. Let $G := H \times A$ where F is a field and A is a finite subgroup of F^*. Let V be a faithful irreducible FH-module where $V \leqslant D^\mu \downarrow H$ and $D^\mu \in R_n(2) \setminus R_n(1)$. If $n \geqslant 12$, then G has a regular orbit on V.*

Proposition 4.7 extends Step 5 of the proof of [19, Theorem 6], which exhibits a regular orbit of A_n on modules in $R_n(2) \setminus R_n(1)$ for $n > 30$. Our methods of proof are similar. Although Proposition 4.7 is trivial when F is an infinite field by Lemma 3.1, we include such F here since Lemma 3.1 only guarantees the existence of a regular orbit, whereas our proof is constructive.

For modules in $R_n(2) \setminus R_n(1)$, we are primarily concerned with the partitions $(n-2, 2)$ and $(n-2, 1, 1)$. For these partitions, the modules M^μ and S^μ can be understood most readily using graphs. We assume a familiarity with basic terminology from graph theory throughout this section.

If $\mu = (n-2, 2)$, then M^μ is the permutation module of S_n on the set of unordered pairs from $\{1, \dots, n\}$, so the set of simple undirected graphs on n vertices with edges weighted by field elements is isomorphic to M^μ if we identify each unordered pair $\{i, j\}$ with the edge whose ends are i and j. With this viewpoint, the Specht module S^μ is spanned by the alternating 4-cycles, which are graphs of the form $\{i, j\} - \{j, k\} + \{k, l\} - \{l, i\}$ for distinct $i, j, k, l \in \{1, \dots, n\}$. Observe that the sum of $\{1, 2\} - \{2, 3\} + \{3, 4\} - \{4, 1\}$ and $\{1, 4\} - \{4, 5\} + \{5, 6\} - \{6, 1\}$ is the alternating 6-cycle $\{1, 2\} - \{2, 3\} + \{3, 4\} - \{4, 5\} + \{5, 6\} - \{6, 1\}$. Continuing in this way, we conclude that S^μ contains every alternating $2m$-cycle for $m \geqslant 2$.

Similarly, if $\mu = (n-2, 1, 1)$, then M^μ is the permutation module of S_n on the set of ordered pairs from $\{1, \dots, n\}$, so the set of simple directed graphs on n vertices with edges weighted by field elements is isomorphic to M^μ if we identify each ordered pair (i, j) with the edge whose tail is i and head is j. With this viewpoint, the Specht module S^μ is spanned by the directed 3-cycles, which are graphs of the form $(i, j) - (j, i) + (j, k) - (k, j) + (k, i) - (i, k)$ for distinct $i, j, k \in \{1, \dots, n\}$. Observe that the sum of $(1, 2) - (2, 1) + (2, 3) - (3, 2) + (3, 1) - (1, 3)$ and $(1, 3) - (3, 1) + (3, 4) - (4, 3) + (4, 1) - (1, 4)$ is the directed 4-cycle $(1, 2) - (2, 1) + (2, 3) - (3, 2) + (3, 4) - (4, 3) + (4, 1) - (1, 4)$. Continuing in this way, we conclude that S^μ contains every directed m-cycle for $m \geqslant 3$.

Lemma 4.8. *Let F be a field, and suppose that $n \geqslant 7$. If V is an FS_n-module in $R_n(2) \setminus R_n(1)$, then $V \downarrow A_n$ is irreducible.*

Proof. For $n > 30$, this is proved in Step 5 of the proof of [19, Theorem 6]. We reproduce this proof here in order to deal with smaller n. Since the modules in $R_n(2) \setminus R_n(1)$ have the form D^μ or $D^\mu \otimes_F \mathrm{sgn}$ where μ is $(n-2, 2)$ or $(n-2, 1, 1)$, and since $D^\mu \downarrow A_n \simeq D^\mu \otimes_F \mathrm{sgn} \downarrow A_n$, it suffices to assume that $V = D^\mu$ where μ is $(n-2, 2)$ or $(n-2, 1, 1)$.

Suppose for a contradiction that $D^\mu \downarrow A_n$ is not irreducible, and let W be an irreducible FA_n-submodule of D^μ. Lemma 2.4 implies that $D^\mu = W \oplus Wg$ where $g = (12) \in S_n$. Recall that $\dim_F([W, g]) = \dim_F(W) - \dim_F(C_W(g))$. But $C_W(g) = 0$ since $W \cap Wg = 0$, and $(n^2 - 5n + 2)/2 \leqslant \dim_F(D^\mu)$ by [22, Appendix Table 1], so $(n^2 - 5n + 2)/4 \leqslant \dim_F(D^\mu)/2 = \dim_F([W, g])$. Moreover, $\dim_F([W, g]) \leqslant \dim_F([M^\mu, g]) = \dim_F(M^\mu) - \dim_F(C^\mu)$ where $C^\mu := C_{M^\mu}(g)$, and $\dim_F(M^\mu)$ is either $n(n-1)/2$ or $n(n-1)$ when μ is $(n-2, 2)$ or $(n-2, 1, 1)$ respectively. Hence $\dim_F(C^{(n-2,2)}) \leqslant (n^2 + 3n - 2)/4$ and $\dim_F(C^{(n-2,1,1)}) \leqslant (3n^2 + n - 2)/4$.

Now we consider the dimension of C^μ and compare it to the upper bounds above to obtain a contradiction for all but the smallest n. Suppose that $\mu = (n-2, 2)$. The graphs $\{1, 2\}$, $\{i, j\}$ and $\{1, i\} + \{2, i\}$ are fixed by g for all $i, j \notin \{1, 2\}$, and these form a linearly independent set in M^μ, so $\dim_F(C^\mu) \geqslant 1 + (n-2)(n-3)/2 + (n-2) = (n^2 - 3n + 4)/2$. But this is impossible unless $n = 7$. Next suppose that $\mu = (n-2, 1, 1)$. The graphs (i, j), $(1, 2) + (2, 1)$, $(1, i) + (2, i)$, and $(i, 1) + (i, 2)$ are fixed by g for all $i, j \notin \{1, 2\}$, and again these form a linearly independent set, so $\dim_F(C^\mu) \geqslant (n-2)(n-3) + 1 + (n-2) + (n-2) = n^2 - 3n + 3$. But this is impossible unless $n \leqslant 11$.

Hence either $n = 7$ when $\mu = (n-2, 2)$, or $7 \leqslant n \leqslant 11$ when $\mu = (n-2, 1, 1)$. If $n = 11$ and $\mathrm{char}(F) = 11$, then $\dim_F(D^\mu) = 36$, and so $D^\mu \downarrow A_n$ is irreducible by [23]. Otherwise, $D^\mu \downarrow A_n$ is irreducible by [13, 27] (including the case $\mathrm{char}(F) > n$ or $\mathrm{char}(F) = 0$). $\square$

Note that when $n = 5$ or 6, there are examples of FS_n-modules V in $R_n(2) \setminus R_n(1)$ for which $V \downarrow A_n$ is not irreducible.

We will need the following technical result about graphs. For a graph Γ, we denote the vertex set of Γ by $V\Gamma$ and the edge set of Γ by $E\Gamma$, and for $u \in V\Gamma$, we denote the valency of u by $|u|$ and the set of vertices adjacent to u by $\Gamma(u)$. We denote the complete graph with ℓ vertices by K_ℓ and the complete bipartite graph with parts of size ℓ and ℓ' by $K_{\ell,\ell'}$.

Lemma 4.9. *Let Γ be a finite simple undirected graph. Suppose that $|V\Gamma| \geqslant 12$ and $1 \leqslant |E\Gamma| \leqslant 2|V|+8$, and suppose that the maximal valency of Γ is at most 8. Either there exist distinct $v_1, v_2, v_3, v_4 \in V\Gamma$ such that $\{v_1, v_2\} \in E\Gamma$ but $\{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\} \notin E\Gamma$, or $|V\Gamma| = 12$ and Γ is one of $K_{4,8}$, $K_6 \cup K_6$ or $K_5 \cup K_7$.*

Proof. Let $a \in V\Gamma$ have minimal non-zero valency. Note that if $u \in V\Gamma$ has valency 0, then since $|V\Gamma| \geqslant 12$ and $|a| \leqslant 8$, we may take $v_4 \in V\Gamma \setminus (\Gamma(a) \cup \{a, u\})$ along with $v_1 = a$, $v_2 \in \Gamma(a)$ and $v_3 = u$. Thus we may assume that every vertex has non-zero valency. In particular, since $2|E\Gamma| = \sum_{v \in V\Gamma} |v|$, it follows that $2|E\Gamma| \geqslant |V\Gamma||a|$. Thus $|a| \leqslant 5$, or else $6|V\Gamma| \leqslant 2|E\Gamma| \leqslant 2(2|V\Gamma| + 8)$, and so $|V\Gamma| \leqslant 8$, a contradiction. Choose $b \in V\Gamma \setminus \Gamma(a)$ with maximal valency. Let $A := \Gamma(a) \setminus \Gamma(b)$, let $B := \Gamma(b) \setminus \Gamma(a)$, and let $C := V\Gamma \setminus (\Gamma(a) \cup \Gamma(b) \cup \{a, b\})$.

Suppose first that $C \neq \varnothing$. If $A \neq \varnothing$, then let $v_1 = a$, $v_3 = b$ and choose $v_2 \in A$ and $v_4 \in C$. By the symmetry of this argument, we may assume that $\Gamma(a) = \Gamma(b)$. Now $|a| = |b|$, but a has minimal valency and b has maximal valency in $V\Gamma \setminus \Gamma(a)$, so every vertex of C has the same valency as a and b. If there is an edge whose ends are both in C, then we may take v_1 and v_2 to be the ends of this edge along with $v_3 = b$ and $v_4 = a$. Otherwise, every vertex of C is adjacent to every vertex of $\Gamma(a)$. Now every vertex of $\Gamma(a)$ has valency at least $|C| + 2$, so $|C| \leqslant 6$, but $|a| \leqslant 5$, so $|V\Gamma| = 2 + |a| + |C| \leqslant 13$. If $|V\Gamma| = 13$, then Γ must contain a subgraph isomorphic to $K_{5,8}$, but $K_{5,8}$ has 40 edges, so Γ has at least 40 edges, contradicting our assumption that $|E\Gamma| \leqslant 34$. Similarly, if $|V\Gamma| = 12$, then Γ must contain a subgraph isomorphic to $K_{5,7}$ or $K_{4,8}$, but $K_{5,7}$ has 35 edges, $K_{4,8}$ has 32 edges, and $|E\Gamma| \leqslant 32$, so Γ must be $K_{4,8}$.

Thus we may assume that C is empty. Note that $|B| = |V\Gamma| - |a| - 2 \geqslant 12 - 5 - 2 = 5$. This implies that $|\Gamma(a) \cap \Gamma(b)| \leqslant 3$, so $|A| + |B| = |V\Gamma| - |\Gamma(a) \cap \Gamma(b)| - 2 \geqslant 12 - 3 - 2 = 7$. Moreover, $A \neq \varnothing$ since $|b| \leqslant 8$ and $|V\Gamma| \geqslant 12$. If there is an edge that has one end in A and the other in B, then we may take these ends to be v_1 and v_2 respectively along with $v_3 = a$ and $v_4 = b$, so we assume that there is no such edge. Suppose further that there exists $u \in \Gamma(a) \cap \Gamma(b)$. The vertex u cannot be adjacent to every vertex of A and B or else $|u| \geqslant 9$. If u is not adjacent to some vertex of A, then we take $v_1 = a$, $v_2 = u$, $v_3 \in A \setminus \Gamma(u)$, and $v_4 \in B$. Thus by symmetry we may assume that $\Gamma(a) \cap \Gamma(b)$ is empty, so that Γ consists of exactly two connected components.

If the component containing a is not complete, then we may choose distinct non-adjacent vertices $v_1, v_4 \in A$ and take $v_2 = a$ and $v_3 = b$. By symmetry, we may assume that the components of Γ are $K_{|A|+1}$ and $K_{|B|+1}$. Since $|A| \leqslant 5$ and $|B| \leqslant 8$, the initial assumptions on $|V\Gamma|$ and $|E\Gamma|$ imply that $|V\Gamma| = 12$, so Γ is $K_6 \cup K_6$ or $K_5 \cup K_7$. $\square$

For $s \in S^\mu$, the *underlying graph* of s is either the graph s with weights removed when $\mu = (n-2, 2)$, or the graph s with weights, direction and multiple edges removed when $\mu = (n-2, 1, 1)$. Thus the underlying graph of $s \in S^\mu$ is always a finite simple undirected graph. Recall that $D^\mu = S^\mu/(S^\mu \cap S^{\mu\perp})$.

Lemma 4.10. *Let μ be $(n-2, 2)$ or $(n-2, 1, 1)$, and suppose that $n \geqslant 12$. Let F be a field for which μ is* $\mathrm{char}(F)$*-regular, and let A be a finite subgroup of F^*. If there exists $s \in S^\mu$ whose underlying graph has trivial automorphism group, maximal valency at most 4, and at most $n + 4$ edges when $n \geqslant 13$ or at most 14 edges when $n = 12$, then $S_n \times A$ has a regular orbit on D^μ and $D^\mu \otimes_F$ sgn.*

Proof. We claim it suffices to prove that $s-\lambda sg \notin S^{\mu\perp}$ for all $1\neq g\in S_n$ and $\lambda\in F^*$. Suppose that this occurs. Then $s\notin S^{\mu\perp}$ since $S^{\mu\perp}$ is an FS_n-submodule of M^μ. If $(s+S^\mu\cap S^{\mu\perp})g\lambda = s+S^\mu\cap S^{\mu\perp}$ for some $g\in S_n$ and $\lambda\in A$, then $s-\lambda sg\in S^{\mu\perp}$, so $g=1$. But $s\notin S^{\mu\perp}$, so $\lambda=1$. Hence $S_n\times A$ has a regular orbit on D^μ. Moreover, if $(s+S^\mu\cap S^{\mu\perp}\otimes 1)g\lambda = s+S^\mu\cap S^{\mu\perp}\otimes 1$ for some $g\in S_n$ and $\lambda\in A$, then either $g\in A_n$ and $s-\lambda sg\in S^{\mu\perp}$, or $g\in S_n\setminus A_n$ and $s+\lambda sg\in S^{\mu\perp}$. If the latter holds, then $g=1$, but this is ridiculous since $g\notin A_n$, so the former holds. Again $g=1$, so $\lambda=1$. Hence $S_n\times A$ has a regular orbit on $D^\mu\otimes_F \mathrm{sgn}$, and the claim is proved.

Fix $1\neq g\in S_n$ and $\lambda\in F^*$. Now $s-\lambda sg\neq 0$, or else g is a non-trivial automorphism of the underlying graph of s. Moreover, the underlying graph Γ of $s-\lambda sg$ has at most $2n+8$ edges when $n\geqslant 13$ or at most 28 edges when $n=12$, and its vertices have valency at most 8. Note that if $n=12$, then Γ cannot be $K_{4,8}$, $K_6\cup K_6$ or $K_5\cup K_8$, as these graphs have too many edges. Hence Lemma 4.9 implies that there exist distinct vertices i,j,k,l such that $\{i,j\}$ is an edge of Γ but $\{j,k\}$, $\{k,l\}$ and $\{l,i\}$ are not edges of Γ. Let

$$s' := \begin{cases} \{i,j\}-\{j,k\}+\{k,l\}-\{l,i\} & \text{if } \mu=(n-2,2),\\ (i,j)-(j,i)+(j,k)-(k,j) & \\ \qquad +(k,l)-(l,k)+(l,i)-(i,l) & \text{if } \mu=(n-2,1,1),\end{cases}$$

so that $s'\in S^\mu$. We claim that $< s-\lambda sg, s' >\neq 0$, in which case $s-\lambda sg\notin S^{\mu\perp}$, as desired. Certainly this holds if $\mu=(n-2,2)$ since $< s-\lambda sg, s' >$ is the weight of the edge $\{i,j\}$ in $s-\lambda sg$, so we assume that $\mu=(n-2,1,1)$. Observe that (u,v) is an edge of $t\in S^\mu$ if and only if (v,u) is an edge of t. Also, if (u,v) has weight δ in t, then (v,u) has weight $-\delta$ in t. Let δ be the weight of (i,j) in $s-\lambda sg$. Now $<s-\lambda sg,s'>=<\delta(i,j)-\delta(j,i),(i,j)-(j,i)>=2\delta\neq 0$ since μ is $\mathrm{char}(F)$-regular. □

Proof of Proposition 4.7. By Lemma 4.8, it suffices to show that $S_n\times A$ has a regular orbit on V, where V is D^μ or $D^\mu\otimes_F\mathrm{sgn}$ and μ is $(n-2,2)$ or $(n-2,1,1)$.

Suppose first that $n\geqslant 13$. Let $m:=2\lfloor n/2\rfloor$. If $\mu=(n-2,2)$, then define

$$\begin{aligned} s_1 &:= \{1,2\}-\{2,4\}+\{4,5\}-\{5,1\},\\ s_2 &:= \{2,3\}-\{3,4\}+\{4,6\}-\{6,2\},\\ s_3 &:= \{5,6\}-\{6,7\}+\cdots+\{m-1,m\}-\{m,5\},\end{aligned}$$

and if $\mu=(n-2,1,1)$, then define s_1, s_2 and s_3 by replacing each weighted edge $\pm\{i,j\}$ above by $(i,j)-(j,i)$. Note that s_1, s_2 and s_3 are in S^μ in either case. Let $s:=s_1+s_2+s_3$. Now the underlying graph of s has $m+4$ edges and maximal valency 4. Moreover, it is routine to verify that the underlying graph of s has a trivial automorphism group. Thus $S_n\times A$ has a regular orbit on D^μ and $D^\mu\otimes_F\mathrm{sgn}$ for $n\geqslant 13$ by Lemma 4.10.

Now suppose that $n=12$ and $|F|\neq 2$. We may choose non-zero elements λ_1 and λ_2 of F such that $\lambda_1+\lambda_2\neq 0$. For $\mu=(n-2,2)$, define

$$\begin{aligned} s_1 &:= \lambda_1(\{1,2\}-\{2,3\}+\{3,4\}-\{4,1\}),\\ s_2 &:= \lambda_2(\{3,4\}-\{4,5\}+\{5,6\}-\{6,7\}+\{7,8\}-\{8,3\}),\\ s_3 &:= \lambda_1(\{7,8\}-\{8,9\}+\{9,10\}-\{10,11\}+\{11,12\}-\{12,7\}),\end{aligned}$$

and for $\mu=(n-2,1,1)$, define s_1, s_2 and s_3 by replacing each weighted edge $\pm\lambda_k\{i,j\}$ by $\lambda_k(i,j)-\lambda_k(j,i)$. Now $s:=s_1+s_2+s_3\in S^\mu$ and the underlying graph of s has 14 edges, maximal valency 3 and trivial automorphism group, so we are done by Lemma 4.10.

Lastly, if $n=12$ and $|F|=2$, then S_n has a regular orbit on V by Proposition 4.5(ii). □

4.3. **Modules in $R_n(1)$.** Now we find the only infinite class of faithful irreducible modules on which S_n has no regular orbits. Neither module in $R_n(0)$ is faithful for $n \geqslant 5$, so we are only concerned with modules in $R_n(1) \setminus R_n(0)$. In particular, we are primarily concerned with the partition $(n-1,1)$.

Let F be a field and $\mu = (n-1,1)$. The FS_n-module M^μ is the permutation module F^n where S_n acts on F^n by permuting the coordinates. The *deleted permutation module* $S^\mu = \{(a_1,\dots,a_n) \in F^n : \sum_{i=1}^n a_i = 0\}$ and has dimension $n-1$. Moreover, $S^{\mu\perp} = \{(a,\dots,a) \in F^n\}$, and clearly $S^\mu \cap S^{\mu\perp}$ is either 0 when $p \nmid n$, or $S^{\mu\perp}$ when $p \mid n$, so the *fully deleted permutation module* D^μ has dimension $n-1$ if $p \nmid n$ and dimension $n-2$ if $p \mid n$. Note that $D^\mu \downarrow A_n$ is irreducible for $n \geqslant 5$.

The regular orbits of $S_n \times \mathbb{F}_q^*$ on $S^{(n-1,1)}$ were determined by Gluck [15], and also Schmid [32] for $\mathrm{char}(\mathbb{F}_q) > n$. We now determine the regular orbits on $D^{(n-1,1)}$.

Proposition 4.11. *Let V be the $\mathbb{F}_pS_n$-module $D^{(n-1,1)}$ where $n \geqslant 5$ and p is a prime such that $p \leqslant n$.*

(i) *If $S_n \leqslant G \leqslant S_n \times \mathbb{F}_p^*$, then G does not have a regular orbit on V or $V \otimes_{\mathbb{F}_p} \mathrm{sgn}$.*
(ii) *If $A_n \leqslant G \leqslant A_n \times \mathbb{F}_p^*$, then G has a regular orbit on V if and only if $G = A_n$ and $p = n-1$.*

Proof. Let $\mu := (n-1,1)$ and $W := S^\mu \cap S^{\mu\perp}$. Let $H \leqslant G \leqslant H \times \mathbb{F}_p^*$ where $H = S_n$ or A_n. We prove (i) and (ii) simultaneously by considering the various possibilities for p in relation to n.

Suppose that $p \leqslant n-2$. Clearly every n-tuple of elements from $\mathbb{F}_p$ must contain either three repeated entries or two pairs of repeated entries, so every element of V is fixed by some non-trivial element of A_n. But if A_n has no regular orbits on V, then G has no regular orbits on V or $V \otimes_{\mathbb{F}_p} \mathrm{sgn}$, so this case is complete.

Suppose that $p = n$. Again, it suffices to prove that A_n has no regular orbits on V. Let $v + W \in V$. Note that if v has exactly one pair of repeated entries, then there is exactly one $b \in \mathbb{F}_p$ that does not appear in v, but the sum of the elements of $\mathbb{F}_p$ vanishes, as does the sum of the coordinates of v, so b must be the repeated entry, a contradiction. Moreover, if v has at least two pairs of repeated entries or a triple of repeated entries, then $v + W$ is certainly fixed by a non-trivial element of A_n. Hence we may assume that v is of the form $(v_1,\dots,v_p)$ where $v_i \neq v_j$ for all $i \neq j$. Let $g \in S_n$ be the permutation for which $vg = (v_1+1,\dots,v_p+1)$. Now g has no fixed points and fixes $v + W$. Moreover, g must be a p-cycle, for if $(i_1 \cdots i_k)$ is a cycle of g for some $k \in \{2,\dots,p\}$, then $v_{i_k} = v_{i_1} + 1$ and $v_{i_j} = v_{i_{j+1}} + 1$ for all $j \in \{1,\dots,k-1\}$, and it follows that $v_{i_1} = v_{i_1} + k$. Thus $k = p$ and $g \in A_n$, as desired.

Suppose that $p = n-1$. Then $V = S^\mu$. First we claim that if $1 \neq \lambda \in \mathbb{F}_p^*$ and v is an element of V with exactly one pair of repeated entries, then there exists $1 \neq g \in A_n$ such that $vg\lambda = v$. Write $v = (v_1,\dots,v_n)$, and let i and j be the unique pair of distinct indices for which $v_i = v_j$. Since the sum of the elements in $\mathbb{F}_p$ is zero, it follows that $v_i = 0$. Thus the non-zero entries of v are the $p-1$ distinct elements of $\mathbb{F}_p^*$, and so the set of entries of v contains a transversal T for the cosets of $\langle\lambda\rangle$ in $\mathbb{F}_p^*$. Let m be the order of λ in $\mathbb{F}_p^*$. As in the proof of [15, Lemma 2], each coset of $\langle\lambda\rangle$ has the form $\{v_{i_0},\dots,v_{i_{m-1}}\}$ for some $v_{i_0} \in T$, where $v_{i_j} = \lambda^j v_{i_0}$ for $1 \leqslant j \leqslant m-1$. Define $g \in A_n$ to be the product of the disjoint cycles $(i_0,\dots,i_{m-1})$ and $(i\ j)$ if needed. Now $g \neq 1$ and $vg\lambda = v$, as claimed.

If $G = A_n$, then $(1,2,\dots,p-1,0,0)$ lies in a regular orbit of G, so we may assume that $G \neq A_n$. We claim that G does not have a regular orbit on V or $V \otimes_{\mathbb{F}_p} \mathrm{sgn}$. Let $0 \neq v \in V$. If v has a triple of repeated entries or two pairs of repeated entries, then some $g \in A_n$ fixes both v and $v \otimes 1$, so we may assume that v has exactly one pair of repeated entries, say with indices i and j. Suppose that $H = S_n$. Some element of S_n will fix v, so G has no regular orbits on V. Moreover, by the claim there exists $1 \neq g \in A_n$ such that $vg = -v$, so $g(i\ j)$ fixes $v \otimes 1$, and we conclude that G has no regular orbits on $V \otimes_{\mathbb{F}_p} \mathrm{sgn}$. Thus $H = A_n$. Since $A_n < G$, there exists $1 \neq \lambda \in G \cap \mathbb{F}_p^*$. Now there exists $1 \neq g \in A_n$ such that $vg\lambda = v$ by the claim, and $g\lambda \in G$, so G has no regular orbits on V. □

4.4. **Proof of Theorem 4.1.** Let $d := \dim_{\mathbb{F}_p}(V)$. There exists a non-negative integer m for which $D^\mu \in R_n(m)$. Since V is faithful as an $\mathbb{F}_pH$-module, D^μ is faithful as an $\mathbb{F}_pS_n$-module, so $D^\mu \notin R_n(0)$. If $D^\mu \in R_n(1) \setminus R_n(0)$, then (i) holds by Proposition 4.11, so we assume that $D^\mu \notin R_n(1)$, in which case the condition of (ii) holds.

Suppose that $n \geqslant 7$. Note that for μ listed in Table 2, $\dim_{\mathbb{F}_p}(V)$ is as listed by [21, 23]. If $D^\mu \notin R_n(2)$, then we are done by Propositions 4.5(i) and 4.6, so we assume that $D^\mu \in R_n(2) \setminus R_n(1)$, and if $H = S_n$, then we are done by Propositions 4.5(ii) and 4.7. Hence we assume that $H = A_n$, in which case we are done by Proposition 4.7 unless $n \leqslant 11$. Recall that $V \downarrow A_n = D^\mu$ by Lemma 4.8, and suppose that G does not have a regular orbit on V. Then $S_n \times \mathbb{F}_p^*$ does not have a regular orbit on D^μ, so Proposition 4.5(ii) implies that $p = 2$ and $\mu = (n-2, 2)$ for $7 \leqslant n \leqslant 8$. If $\mu = (5, 2)$, then G has a regular orbit on V by Magma, so $\mu = (6, 2)$, in which case G does not have a regular orbit on V since $|V| < |G|$.

Thus $n = 5$ or 6. Using [13, 27], we determine the possibilities for μ and d. If (n, p, μ, G, d) is not listed in Table 2, then G has a regular orbit on V by Magma, so we may assume that (n, p, μ, G, d) is listed in Table 2. If μ is $(3, 2)$ or $(4, 2)$, then $|V| < |G|$, so G does not have a regular orbit on V. Otherwise, we determine that G has no regular orbits on V using Magma. □

5. Covering groups

Recall that the proper covering groups of S_n and A_n are $2.S_n^+$ and $2.S_n^-$ for $n \geqslant 5$, $2.A_n$ for $n \geqslant 5$, and $3.A_n$ and $6.A_n$ for $n = 6$ or 7. We focus on the double covers of S_n and A_n for most of this section, as $3.A_n$ and $6.A_n$ will be dealt with computationally. For $n \geqslant 5$,

$$2.S_n^+ := \langle z, s_1, \ldots, s_{n-1} : z^2 = 1, s_i^2 = (s_is_{i+1})^3 = 1, zs_i = s_iz, (s_is_j)^2 = z \text{ if } |i-j| > 1\rangle,$$
$$2.S_n^- := \langle z, t_1, \ldots, t_{n-1} : z^2 = 1, t_i^2 = (t_it_{i+1})^3 = z, zt_i = t_iz, (t_it_j)^2 = z \text{ if } |i-j| > 1\rangle.$$

The centre of each group is $\{1, z\}$, and they are isomorphic precisely when $n = 6$, in which case we write $2.S_6$. We also write $2.S_n^\varepsilon$ when no distinction between the two covers needs to be made. The cover $2.A_n$ is the derived subgroup of $2.S_n^\varepsilon$ and has centre $\{1, z\}$.

Let G be $2.S_n^\varepsilon$ or $2.A_n$ where $n \geqslant 5$, and let z be the unique central involution of G. Let F be a field, and let V be an irreducible FG-module. Now z must act as 1 or -1 on V. (Indeed, this is the case for a central involution in any finite group.) Since every non-trivial normal subgroup of G contains z, it follows that V is faithful precisely when z acts as -1. In particular, G has no faithful irreducible representation over a field of characteristic 2. In this section, we prove the following theorem.

Theorem 5.1. *Let H be a proper covering group of S_n or A_n where $n \geqslant 5$. Let G be a group for which $H \leqslant G \leqslant H \circ \mathbb{F}_p^*$ where p is a prime and $p \leqslant n$. Let V be a faithful irreducible $\mathbb{F}_pH$-module. Let $d := \dim_{\mathbb{F}_p}(V)$. The group G has a regular orbit on V if and only if (n, p, G, d) is not listed in Table 1.*

Let G be $2.S_n^\varepsilon$ or $2.A_n$ where $n \geqslant 5$. We will be primarily interested in the so-called basic spin modules of G, for these have minimal dimension among the faithful irreducible modules by [24] (cf. Theorem 5.2). In fact, non-basic spin modules have such large dimension that they almost always have regular orbits. Indeed, there is only one non-basic spin module listed in Table 1; it arises for $n = p = 5$ when $G = 2.A_5 \circ \mathbb{F}_5^*$ and has dimension 4.

Over the complex numbers, the irreducible representations of G can be indexed by certain partitions of n [33], and the complex basic spin modules of G are those representations corresponding to the partition (n). For an algebraically closed field of positive characteristic p, a *basic spin module* is a composition factor of the reduction modulo p of a complex basic spin module, and by [35], this reduction is irreducible except when $p \mid n$ and either n is odd for $G = 2.S_n^\varepsilon$, or n is even for $G = 2.A_n$. Moreover, there are at most two basic spin modules, and when there are two, they are either associates or conjugates. Lastly, every basic spin module of $2.A_n$ arises as a submodule of a basic spin module of $2.S_n^\varepsilon$.

Let p be a prime. As in [24], define

$$\delta(G) := \begin{cases} 2^{\left\lfloor \frac{1}{2}(n-1-\kappa(p,n)) \right\rfloor} & \text{if } G = 2.S_n^\varepsilon \\ 2^{\left\lfloor \frac{1}{2}(n-2-\kappa(p,n)) \right\rfloor} & \text{if } G = 2.A_n, \end{cases}$$

where $\kappa(p, n) := 1$ if $p \mid n$ and 0 otherwise. Now $\delta(G)$ is the dimension of a basic spin module of G over a splitting field of characteristic p [35]. Moreover, Kleshchev and Tiep [24] provide a lower bound for the dimensions of faithful irreducible representations of G in terms of $\delta(G)$, which we now state.

Theorem 5.2. *Let G be $2.S_n^\varepsilon$ or $2.A_n$ where $n \geqslant 8$, and let F be an algebraically closed field of positive characteristic. If V is a faithful irreducible FG-module for which $\dim_F V < 2\delta(G)$, then V is a basic spin module and $\dim_F V = \delta(G)$.*

Theorem 5.2 can be applied to any finite field in the following way. Let V be a faithful irreducible FG-module where F is a finite field and G is $2.S_n^\varepsilon$ or $2.A_n$. Let $k := \mathrm{End}_{FG}(V)$. Recall that V is a faithful absolutely irreducible kG-module. If V is the realisation over k of a basic spin module of G, then we also refer to V as a basic spin module. For $n \geqslant 8$, Theorem 5.2 implies that either $\dim_k(V) = \delta(G)$, in which case V is a basic spin module, or $2\delta(G) \leqslant \dim_k(V) \leqslant \dim_F(V)$.

Unlike S_n, not every field is a splitting field for $2.S_n^\varepsilon$. However, every field containing $\mathbb{F}_{p^2}$ for p an odd prime is a splitting field for $2.S_n^\varepsilon$ and $2.A_n$ (cf. [28, Corollary 5.1.5]). Note that there are instances where $\mathbb{F}_p$ is a splitting field for $2.A_n$ but not for $2.S_n^\varepsilon$.

The Brauer character tables of $2.S_n^+$ and $2.A_n$ for $p \leqslant n$ and $5 \leqslant n \leqslant 12$ may be found in [23] and also in [13] for $p \leqslant n$ and $5 \leqslant n \leqslant 13$. The Brauer character tables of $2.S_n^-$ for $5 \leqslant n \leqslant 18$ and $p \in \{3, 5, 7\}$ may be found in GAP [13] via the SpinSym package [27]. We can convert the character table of one double cover to that of the other using GAP. When the reduction modulo p of an ordinary irreducible representation of $2.S_n^\varepsilon$ or $2.A_n$ is irreducible, the Brauer character is the ordinary character restricted to p-regular elements and can therefore be accessed using the generic character tables in GAP.

We begin with a reduction for almost quasisimple groups G with $F^*(G)' \simeq 2.A_n$.

Lemma 5.3. *Let G be an almost quasisimple group where $N := F^*(G)' \simeq 2.A_n$ and $n \geqslant 8$. Let F be a finite field. Let V be a faithful irreducible FG-module, $k := \mathrm{End}_{FG}(V)$ and $q := |k|$. Let W be an irreducible kN-submodule of V. If G has no regular orbits on V, then $n \leqslant 20$, and if $\mathrm{char}(F) \leqslant n$, then the following hold.*

(i) *If $n \geqslant 13$, then W is a basic spin module and (n, q) is listed in Table 9. If there is no $*$ next to q, then $V \downarrow N = W$ and W is an absolutely irreducible kN-module.*

(ii) *If $n \leqslant 12$ and W is not a basic spin module, then $V \downarrow N = W$ and W is an absolutely irreducible kN-module where $(n, q, \dim_k(W))$ is listed in Table 10.*

n	q
13	$3^*, 5^*, 7^*, 9^*, 11^*, 13^*, 25$
	$27, 49, 81, 121, 169, 243$
14	$3^*, 5, 7^*, 9, 11, 13, 49$
15	$3^*, 5^*, 9, 11, 13, 25, 27$
16	$3, 5, 7$
17	$3^*, 7, 9, 11$
18	$3^*, 9$
19	3
20	5

TABLE 9. Possible q when $n \geqslant 13$

n	q	$\dim_k(W)$
8	9	24
	7	16
9	3	48
10	3, 5	48
11	5	56

TABLE 10. Possible q and $\dim_k(W)$ when $n \leqslant 12$ and W is non-basic

Proof. Suppose that G has no regular orbits on V. Let $p := \mathrm{char}(F)$. Since V is a faithful absolutely irreducible kG-module, Lemma 2.2 implies that $Z(G) \leqslant k^*$, and so $|Z(G)| \leqslant q-1$. Let

$$g(q,n) := \max\{(n-1)\log_q(n(n-1)(q-1)), \tfrac{n}{2}\log_q(2n!(q-1))\}.$$

Now equation (2) of Lemma 3.6 implies that $\dim_k(V) \leqslant \lfloor g(q,n) \rfloor$. Note that if n is fixed, then $g(q,n)$ is a decreasing function in q by Lemma 3.7.

Since $Z(N) \leqslant Z(G)$ by Lemma 2.1, the central involution of N must act as -1 on V. Thus $p \neq 2$ and W is a faithful kN-module, so $\delta(N) \leqslant \dim_k(W)$ by Theorem 5.2. In particular, $\delta(N) \leqslant \lfloor g(q,n) \rfloor$. If $n \geqslant 21$, then $\lfloor g(q,n) \rfloor \leqslant \lfloor g(3,n) \rfloor < 2^{\lfloor (n-3)/2 \rfloor} \leqslant \delta(N)$, a contradiction. Thus $n \leqslant 20$, as claimed.

We assume for the remainder of the proof that $p \leqslant n$. Let $E := \mathrm{End}_{kN}(W)$. Suppose that $n \geqslant 13$. First we claim that either (n,q) is listed in Table 9 or $(n,q) \in P$ where

$$P := \{(13,125), (13,343), (14,343), (15,7), (17,5)\}.$$

If $q \geqslant 7$ and $n \geqslant 19$, then $\lfloor g(q,n) \rfloor \leqslant \lfloor g(7,n) \rfloor < 2^{\lfloor (n-3)/2 \rfloor} \leqslant \delta(N)$, a contradiction. Hence if $q \geqslant 7$, then $n \leqslant 18$. Similarly, if $q \geqslant 17$, then $n \leqslant 16$; if $q \geqslant 49$, then either $n = 16$ or $n \leqslant 14$; if $q \geqslant 121$, then $n \leqslant 14$; and if $q \geqslant 625$, then either $n = 14$ or $n \leqslant 12$. If $n = 14$ where $q \geqslant 25$ and $p \neq 7$, then $\lfloor g(q,14) \rfloor \leqslant \lfloor g(25,14) \rfloor < 64 = \delta(N)$, a contradiction, and if $n = 14$ and $7^4 \mid q$, then $\lfloor g(q,14) \rfloor \leqslant \lfloor g(7^4,14) \rfloor < 32 = \delta(N)$, a contradiction. Similarly, if $n = 16$, then $\kappa(p,n) = 0$, so we obtain a contradiction for $q \geqslant 9$. In fact, if (n,q) is one of $(20,3)$, $(19,5)$, $(17,13)$, or $(18,q)$ where $q \in \{5,7,11,13\}$, then $\kappa(p,n) = 0$, and we obtain contradictions. The claim follows.

Let Q be the set of (n,q) listed in Table 9 with an adjacent $*$. By the claim, $2\delta(N) \leqslant \lfloor g(q,n) \rfloor$ if and only if $(n,q) \in Q$. This has several consequences.

Firstly, W is a basic spin module, or else Theorem 5.2 implies that $2\delta(N) \leqslant \dim_E(W) \leqslant \dim_k(W) \leqslant \lfloor g(q,n) \rfloor$, and so $(n,q) \in Q$, but for such (n,q), there is no faithful irreducible EN-module whose dimension lies between $2\delta(N)$ and $\lfloor g(q,n) \rfloor$ by [13, 27], a contradiction. Secondly, if $(n,q) \notin Q$, then W is an absolutely irreducible kN-module, for if not, then

$\dim_k(W) \geqslant 2\dim_E(W) = 2\delta(N)$, so $2\delta(N) \leqslant \lfloor g(q,n)\rfloor$, a contradiction. Thirdly, (n,q) is listed in Table 9, for if not, then $(n,q) \in P$ by the claim, but this implies that W is not an absolutely irreducible kN-module by [13, 27], so $(n,q) \in Q$, a contradiction. Lastly, if $(n,q) \notin Q$, then $V \downarrow N$ is irreducible, or else $2\dim_k(W) \leqslant \dim_k(V)$ by Lemma 2.4, so $2\delta(N) \leqslant \lfloor g(q,n)\rfloor$, a contradiction. Thus we have proved (i).

Henceforth, we may assume that $n \leqslant 12$ and W is not a basic spin module. Now $2\delta(N) \leqslant \dim_E(W) \leqslant \lfloor g(p,n)\rfloor$. For each (n,p), we use [13, 23, 27] to determine the possibilities for $\dim_E(W)$. Either these are the dimensions given in Table 10, or $n = 8$, $p = 5$ and $\dim_E(W) = 24$. Since $\dim_E(W) \leqslant \lfloor g(q,n)\rfloor$, it follows that either q is listed in Table 10, or $n = 8$ and $q = 3$ or 5. If $n = 8$ and $p = 3$ or 5, then no faithful irreducible $\mathbb{F}_{p^2}N$-module of dimension 24 can be realised over $\mathbb{F}_p$ [23], so $\dim_k(W) = 48$ when $q = 3$ or 5, while $\lfloor g(q,8)\rfloor < 48$, a contradiction. Thus $(n,q,\dim_E(W))$ is listed in Table 10, in which case W is an absolutely irreducible kN-module [23], so $E = k$. Lastly, if $V \downarrow N \neq W$, then $2\dim_k(W) \leqslant \dim_k(V) \leqslant \lfloor g(q,n)\rfloor$ by Lemma 2.4, a contradiction. □

Next we consider the double covers of the symmetric group.

Proposition 5.4. *Let $H := 2.S_n^\varepsilon$ where $n \geqslant 8$, and let G be such that $H \leqslant G \leqslant H \circ \mathbb{F}_p^*$ where p is a prime and $p \leqslant n$. Let V be a faithful irreducible $\mathbb{F}_pH$-module.*

(i) *If $\varepsilon = -$, then G has no regular orbits on V if and only if $\dim_{\mathbb{F}_p}(V) = \delta(H)$ and (n,p) is one of $(8,3)$, $(8,5)$, $(9,3)$, or $(10,3)$.*
(ii) *If $\varepsilon = +$, then G has no regular orbits on V if and only if $\dim_{\mathbb{F}_p}(V) = \delta(H)$ and $(n,p) = (8,5)$ and $G = 2.S_n^+ \circ \mathbb{F}_p^*$.*

Proof. We will prove (i) and (ii) simultaneously. Suppose that G does not have a regular orbit on V. Lemma 5.3 implies that $n \leqslant 20$. Let $k := \mathrm{End}_{\mathbb{F}_pG}(V)$ and $q := |k|$. Let W be an irreducible kN-submodule of V where $N := F^*(G)' = 2.A_n$.

First we claim that q is either p or p^2. Let χ be the character of the kG-module V and $\mathbb{F}_p(\chi)$ the subfield of k generated by $\mathbb{F}_p$ and the image of χ. By [5, Theorem VII.1.16], the $\mathbb{F}_pG$-module V is a direct sum of $[k : \mathbb{F}_p(\chi)]$ irreducible $\mathbb{F}_pG$-modules, so $k = \mathbb{F}_p(\chi)$. Since χ is also the character of the irreducible $\overline{k}G$-module $V \otimes_k \overline{k}$, where $\overline{k}$ denotes the algebraic closure of k, it follows from [5, Theorem VII.2.6] that k is contained in the unique smallest splitting field for G in $\overline{k}$. Since $\mathbb{F}_{p^2}$ is a splitting field for H, the claim follows.

Suppose that $n \geqslant 12$. We claim that $\dim_k(V) = \delta(H)$ and that (n,p,ε) is listed in Table 11. By Lemma 5.3, W is a basic spin module and (n,q) is listed in Table 9 for $n \geqslant 13$. Suppose that either $n = 12$ or (n,q) is such that (n,p) has an adjacent $*$ in Table 9. Then $\dim_k(V) = \delta(H)$ by [13, 23, 27], and $\dim_{\mathbb{F}_p}(V) = 64$ when $(n,p,\epsilon) = (12,11,-)$, but equation (3) of Lemma 3.6 implies that $\dim_{\mathbb{F}_p}(V) \leqslant 57$, a contradiction. Hence the claim holds in this case.

n	12	12	13	14	14	15	16	16	17, 18
p	3, 5, 7	11	$p \leqslant n$	3, 7	11	3, 5	3	7	3
ε	$\pm$	$+$	$\pm$	$\pm$	$-$	$\pm$	$+$	$-$	$\pm$

TABLE 11. Possible p and ε when $n \geqslant 12$

We may therefore assume that (n,q) is such that (n,p) has no adjacent $*$ in Table 9. Now $q = p$, $W = V \downarrow N$ and W is a faithful absolutely irreducible kN-module by Lemma 5.3. Thus $\dim_k(V) = \dim_k(W) = \delta(N)$. But if either n is even and $p \mid n$, or n is odd and $p \nmid n$, then $\delta(H) = 2\delta(N)$, and so $\dim_k(V) < \delta(H)$, contradicting Theorem 5.2.

We conclude that either n is even and $p \nmid n$, or n is odd and $p \mid n$. Now $\dim_k(V) = \dim_k(W) = \delta(N) = \delta(H)$. If $n = 14$ and (p, ε) is one of $(5, \pm)$, $(11, +)$ or $(13, \pm)$, or $n = 16$ and (p, ε) is one of $(3, -)$, $(5, \pm)$ or $(7, +)$, then $k = \mathbb{F}_{p^2}$ by [13, 27], a contradiction.

Thus (n, p, ε) is listed in Table 11 and $\dim_k(V) = \delta(H)$. Using Magma, we determine that $H \circ \mathbb{F}_p^*$ has a regular orbit on V, a contradiction.

Hence $n \leqslant 11$. First suppose that W is not a basic spin module. Lemma 5.3 implies that $V \downarrow N = W$ and $\dim_k(W)$ is listed in Table 10. Using [13, 23, 27], we determine that if (n, q) is one of $(8, 9)$, $(10, 5)$ or $(11, 5)$, then there is no faithful irreducible kH-module of dimension 24, 48 or 56 respectively, a contradiction. Thus (n, q) is one of $(8, 7)$, $(9, 3)$ or $(10, 3)$. If $\varepsilon = +$, then $k = \mathbb{F}_{p^2}$ by [13], a contradiction, and if $\varepsilon = -$, then using Magma, we determine that $H \circ \mathbb{F}_p^*$ has a regular orbit on V, a contradiction.

Thus W is a basic spin module, in which case $\dim_k(V) = \delta(H)$ by [13, 23, 27]. Moreover, $(n, q, \varepsilon, \dim_{\mathbb{F}_p}(V))$ is one of $(8, 3, -, 8)$, $(8, 5, \pm, 8)$, $(9, 3, -, 8)$, $(10, 3, -, 16)$, or else $H \circ \mathbb{F}_p^*$ has a regular orbit on V by Magma. Note that $\dim_{\mathbb{F}_p}(V) = \delta(H)$ since $k = \mathbb{F}_p$. Now (i) holds, so we may assume that $\varepsilon = +$. If $G = 2.S_8^+$, then G has a regular orbit on V by Magma, a contradiction, so $G = 2.S_8^+ \circ \mathbb{F}_5^*$, as desired.

Conversely, suppose that $\dim_{\mathbb{F}_p}(V) = \delta(H)$ and either $\varepsilon = -$ and (n, p) is one of $(8, 3)$, $(8, 5)$, $(9, 3)$ or $(10, 3)$, or $\varepsilon = +$ and $(n, p) = (8, 5)$ and $G = 2.S_n^+ \circ \mathbb{F}_p^*$. If (n, p) is $(8, 3)$ or $(9, 3)$, then $|V| < |G|$, and so G has no regular orbits on V. Otherwise, we use Magma to check that no orbit is regular. □

Using Proposition 5.4, we now consider the double cover of the alternating group.

Proposition 5.5. *Let $H := 2.A_n$ where $n \geqslant 8$, and let G be such that $H \leqslant G \leqslant H \circ \mathbb{F}_p^*$ where p is a prime and $p \leqslant n$. Let V be a faithful irreducible $\mathbb{F}_pH$-module. Then G has no regular orbits on V if and only if $\dim_{\mathbb{F}_p}(V) = \delta(H)$ and either $p = 3$ and $n \in \{8, 9, 10, 11, 12\}$, or $p = 5$ and $n \in \{9, 10\}$.*

Proof. Suppose that G has no regular orbits on V. Let $k := \mathrm{End}_{\mathbb{F}_pG}(V)$ and $q := |k|$. As in the proof of Proposition 5.4, q is either p or p^2 since $\mathbb{F}_{p^2}$ is a splitting field for H. For $\varepsilon \in \{+, -\}$, let V^ε be an irreducible $\mathbb{F}_p(2.S_n^\varepsilon)$-module for which $V \leqslant V^\varepsilon \downarrow H$, which exists by Lemma 2.3. Since $r(G) = r(A_n) \leqslant n/2$ by [17, Lemma 6.1], Lemma 3.5 implies that

$$\dim_{\mathbb{F}_p}(V) \leqslant r(G) \log_p |G| \leqslant \tfrac{n}{2} \log_p(n! \tfrac{p-1}{2}) =: h(p, n).$$

Suppose that $n \geqslant 13$. Lemma 5.3 implies that V is a basic spin module and (n, q) is listed in Table 9. We claim that $(n, q) \in P$ where

$$P := \{(13, 3), (13, 13), (14, 5), (14, 13), (15, 5), (16, 5), (20, 5)\},$$

in which case $H \circ \mathbb{F}_p^*$ has a regular orbit on V by Magma, a contradiction. If $(n, q) = (17, 11)$, then $128 = \delta(H) = \dim_{\mathbb{F}_p}(V) \leqslant \lfloor h(p, n) \rfloor = 124$, a contradiction. In addition, if (n, q) is one of $(15, 11)$, $(15, 13)$, $(17, 7)$ or $(19, 3)$, then $q = p^2$ by [13], a contradiction.

We may assume that V^ϵ is a basic spin module. If $V = V^\varepsilon \downarrow 2.A_n$ for some $\varepsilon \in \{+, -\}$, then G has a regular orbit on V by Proposition 5.4, a contradiction. Thus $V^\varepsilon \downarrow 2.A_n = V \oplus Vg$ for every $g \in 2.S_n^\varepsilon \setminus 2.A_n$ and $\varepsilon \in \{+, -\}$. If (n, p) is one of $(13, 5)$, $(13, 7)$, $(13, 11)$, $(14, 7)$, $(14, 11)$, $(16, 7)$, $(17, 3)$ or $(18, 3)$, then $V^- \downarrow H$ does not split by [13, 27], a contradiction. Similarly, if $p = 3$ and $14 \leqslant n \leqslant 16$, then $V^+ \downarrow H$ does not split by [13, 27], a contradiction. Lastly, if (n, p) is one of $(13, 3)$, $(13, 13)$ or $(15, 5)$, then $q = p$ by [13, 27]. Thus $(n, q) \in P$, proving the claim.

Hence $n \leqslant 12$. First suppose that V is not a basic spin module. Then (n, q) is listed in Table 10. If $(n, q) = (8, 9)$, then $48 = \dim_{\mathbb{F}_p}(V) \leqslant \lfloor h(p, n) \rfloor = 38$, a contradiction. If (n, q) is one of $(8, 7)$, $(9, 3)$ or $(10, 3)$, then $V = V^- \downarrow H$, so G has a regular orbit on V by

Proposition 5.4, a contradiction. Lastly, if (n,q) is $(10,5)$ or $(11,5)$, then we determine that $H \circ \mathbb{F}_p^*$ has a regular orbit on V using MAGMA, a contradiction.

Thus V is a basic spin module, and we may assume that V^ε is also a basic spin module. First suppose that $V \neq V^\varepsilon \downarrow H$ for both $\varepsilon \in \{+,-\}$. Using [23], we determine that (n,p) is one of $(9,5)$, $(9,7)$, $(10,5)$, $(11,3)$, $(11,5)$ or $(12,3)$. If (n,p) is $(9,7)$ or $(11,5)$, then $H \circ \mathbb{F}_p^*$ has a regular orbit on V by MAGMA, a contradiction. Thus (n,p) is one of $(9,5)$, $(10,5)$, $(11,3)$ or $(12,3)$, in which case $q = p$ by [23], so $\dim_{\mathbb{F}_p}(V) = \delta(H)$.

Lastly, if $V = V^\varepsilon \downarrow H$ for some $\varepsilon \in \{+,-\}$, then $2.S_n^\varepsilon \circ \mathbb{F}_p^*$ has no regular orbits on V^ε, so $\dim_{\mathbb{F}_p}(V) = \delta(H)$ and (n,p) is one of $(8,3)$, $(8,5)$, $(9,3)$ or $(10,3)$ by Proposition 5.4. If $(n,p) = (8,5)$, then $H \circ \mathbb{F}_p^*$ has a regular orbit on V by MAGMA, a contradiction.

Conversely, suppose that $\dim_{\mathbb{F}_p}(V) = \delta(H)$ and either $p = 3$ and $n \in \{8,9,10,11,12\}$, or $p = 5$ and $n \in \{9,10\}$. If (n,p) is one of $(8,3)$, $(9,3)$, $(10,5)$ or $(12,3)$, then $|V| < |G|$, so G has no regular orbits on V. Otherwise, no orbit is regular by MAGMA. □

Proof of Theorem 5.1. Let $d := \dim_{\mathbb{F}_p}(V)$. If $n \geqslant 8$, then we are done by Propositions 5.4 and 5.5, so we may assume that $n \leqslant 7$. Using [13, 23, 27], we determine the possibilities for d. If (n,p,G,d) is not listed in Table 1, then we use MAGMA to prove that G has a regular orbit on V. Thus we may assume that (n,p,G,d) is listed in Table 1. If either $(n,p,d) = (7,3,8)$ and $G = 2.A_7$, or $(n,p,d) = (5,5,4)$ and $H = 2.S_5^+$ or $G = 2.S_5^- \circ \mathbb{F}_5^*$ or $G = 2.A_5 \circ \mathbb{F}_5^*$, then no regular orbits exist by MAGMA. Similarly, if $H = 3.A_6 \neq G$ and $(n,p,d) = (6,5,6)$, or if $H = 3.A_7$ and $(n,p,d) = (7,5,6)$ or $(7,7,6)$, then no regular orbits exist by MAGMA. Otherwise, $|V| < |G|$, so G has no regular orbits on V. □

6. Comments on computations

We used functions from [27] to construct representations for covering groups of S_n and A_n. Various representations are also available via the ATLAS package [36]. MAGMA has an implementation of the Burnside algorithm to construct all faithful irreducible representations of a finite permutation group over a given finite field. We used this to construct representations, either all or those of specified degree, for certain small degree permutation groups. We use our implementation of the algorithm of [14] to rewrite a representation over a smaller field.

We used the ORB package [31] to prove that a 44-dimensional representation of S_{12} over $\mathbb{F}_2$ has a regular orbit and a 32-dimensional representation of S_{12} over $\mathbb{F}_2$ has no regular orbits. We used Lemma 3.3 extensively to decide whether a group G has a regular orbit. Its realisation assumes knowledge of conjugacy classes of G. While these can often be readily computed, we used the infrastructure of [2] for these computations with covering groups for S_n and A_n where $n > 11$. Most remaining computations reported here are routine and were performed using MAGMA. Records of these are available at http://www.math.auckland.ac.nz/~obrien/regular.

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