A computer-assisted analysis of some matrix groups

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Dedicated to Charles Leedham-Green on the occasion of his 65th birthday

Abstract

We use algorithms developed recently for the study of linear groups to investigate a sequence of matrix groups defined over GF(2); these are images of representations of certain finitely presented groups considered by Soicher in a study of simplicial complexes related to the Suzuki sequence graphs.

1 Introduction

In [25], Soicher considered a sequence of simplicial complexes known as Γ_n-*complexes*, and classifies a more restricted type, known as Γ_n^*-complexes, for $n \leq 8$. These complexes arise naturally in connection with a sequence Γ_n of graphs, the *Suzuki sequence* graphs. The automorphism groups of these graphs are well-known for $n \leq 6$. For example, $\mathrm{Aut}(\Gamma_6) = \mathrm{Suz}:2$, where Suz is the sporadic simple group of Suzuki. Our notation for the structure of finite groups follows that of [8].

Soicher established that the automorphism group of a Γ_n-complex is a quotient of the finitely-presented $(n+2)$-generator group

$$\begin{aligned} U_n \;:=\; & \langle a, u_0, u_1, \ldots, u_n \mid a^2,\ u_i^2\ (0 \leq i \leq n),\ (au_0)^3,\ (u_0u_1)^3,\ (u_1u_2)^8, \\ & (u_iu_{i+1})^3\ (2 \leq i < n),\ (au_i)^2\ (i \geq 1),\ (u_iu_j)^2\ (i+1 < j),\ a^{-1}(u_1u_2)^4 \rangle. \end{aligned}$$

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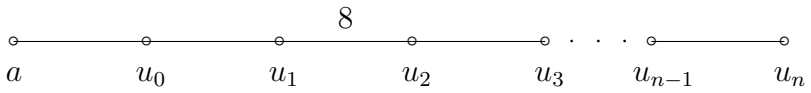
Alternatively, U_n is the group defined by the Coxeter diagram



with the extra relation $a = (u_1u_2)^4$. He proved that $U_2 \cong L_3(2):2$ and $U_3 \cong (3 \times U_3(3)):2$.

The automorphism group of a Γ_n^*-complex is a quotient of the group U_n^*, which is the quotient of U_n obtained by adjoining the additional relator $(u_0u_1u_2u_3)^8$. (This element generates the normal subgroup of order 3 in U_3.) In [24], Soicher proved that, for $3 \leq n \leq 8$, U_n^* is isomorphic to $U_3(3):2$, $J_2:2$, $G_2(4):2$, $3 \cdot \mathrm{Suz}:2$, $\mathrm{Co}_1 \times 2$, and $2 \cdot (\mathrm{Co}_1 \wr 2)$, respectively.

In unpublished work, Richard Parker described a construction for two linear representations of U_n over GF(2): these are identical for even n but distinct for odd $n > 1$. They are also representations of U_n^* for $n \geq 3$. For $n \geq 2$, the representations have degree $6 \cdot 2^{m-2}$ and $6 \cdot 2^{m-1}$ when $n = 2m - 1$, and degree $6 \cdot 2^{m-1}$ when $n = 2m$. We describe the construction in Section 2. We denote the images of the representations of U_n^* as G_n and, when n is odd, G_n^l, respectively. (The superscript l means 'large'.)

One of the major projects in computational group theory during the past 15 years has been the development of effective algorithms for analysing the structure of linear groups defined by generating matrices over a finite field. For a recent survey of this project, see O'Brien [21]. Implementations of some of the resulting algorithms are available in Magma [4] as part of a package developed by Leedham-Green and O'Brien. Although the machinery is not yet complete, it frequently allows us to identify the composition factors of a given linear group.

As a case study, we used this machinery to study Parker's groups G_n and G_n^l for $n \leq 20$, hence for groups of degree at most 3072. This motivated us to develop some additional machinery and to exploit some (reasonably well-known) representation-theoretic results.

Some of the algorithms used in our computations are Monte Carlo or Las Vegas: they rely on random selection. A *Monte Carlo* algorithm may return an incorrect answer to a decision question, but the probability of this event is less than some specified value. If one of the answers given is always correct, then it is *one-sided*. A *Las Vegas* algorithm never returns an incorrect answer, but may report failure with probability less than some specified value. At appropriate points we indicate the nature of our reliance on the outcome of such algorithms.

The outline of the paper is the following. We first describe the matrix represen-

tations of Parker. In Section 3 we consider some basic computations with linear groups. We then discuss a geometric-based approach to the study of linear groups, and introduce the concept of a *composition tree* (see [15] or [21]) whose leaves are the composition factors of a group. In Section 6 we record some module-theoretic results which assist in our structural analysis. In Section 7 we report the structure of the groups, commenting on the individual cases. In Section 8 we show that our results establish the existence of a Γ_n^*-complex for $n = 10$. Finally we consider briefly the finitely-presented groups.

2 The representations

We now describe generating sequences X_n and X_n^l for the matrix groups G_n and G_n^l over GF(2) for $n \geq 1$, where X_n^l and G_n^l are defined only for odd n.

For $n \geq 2$, if we map the generating sequence $[a, u_0, \ldots, u_n]$ of U_n (or of U_n^* when $n \geq 3$) to X_n or X_n^l, then we obtain Parker's representations of U_n (or of U_n^* when $n \geq 3$).

For $(m \times m)$– and $(n \times n)$–matrices α and β, the *Kronecker product* $K(\alpha, \beta)$ of α and β is defined to be the $(mn \times mn)$–matrix in which the entry in position $((i-1)n+k, (j-1)n+l)$ is equal to $\alpha_{ij}\beta_{kl}$, for $1 \leq i, j \leq m$, $1 \leq k, l \leq n$. Note that $K(\alpha, \beta)K(\gamma, \delta) = K(\alpha\gamma, \beta\delta)$ provided that all of the matrices involved are defined.

Define

$$\alpha := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \beta := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \delta := \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \epsilon := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

where all matrices are over GF(2), and let I_k denote the $k \times k$ identity matrix over GF(2).

We define $X_1 = X_1^l = [(\beta\gamma')^4, \alpha, \beta]$ where $\gamma' := K(\gamma, I_3)$. For $n > 1$, we define X_n and X_n^l recursively. If $n = 2m$ is even, then we set $X_n[i] := X_{n-1}^l[i]$ for $1 \leq i \leq n+1$, and $X_n[n+2] := K(\gamma, I_{3 \cdot 2^{m-1}})$.

If $n = 2m - 1$ is odd, then we define X_n by $X_n[i] := X_{n-1}[i]$ for $1 \leq i \leq n+1$, $X_n[n+2] := K(\epsilon, I_{3\cdot 2^{m-2}})$, and X_n^l by $X_n^l[i] := K(I_2, X_n[i])$ for $1 \leq i \leq n+1$, $X_n^l[n+2] := K(\delta, I_{3\cdot 2^{m-2}})$.

That these define representations of U_n and U_n^* is readily verified. For $n \leq 3$ this can be checked directly and, for larger n, it can be proved by induction on n by making use of the matrix identities $(\gamma\epsilon)^3 = I_2$, $(\delta K(\gamma, I_2))^3 = (\delta K(I_2, \gamma))^3 = I_4$, $(K(\delta, I_2)K(I_2, \delta))^2 = I_8$.

3 Some basic computations

If we are given $G \leq \mathrm{GL}(d, q)$, a natural question is: what is the order of G? The Schreier-Sims algorithm, first introduced for permutation groups by Sims [22], can sometimes provide an answer. We review this briefly; for a detailed discussion, see [9, Chapter 4].

Let a group G act faithfully on $\Omega = \{1, \ldots, n\}$. Recall that a *base* for G is a sequence of points $B = [\beta_1, \beta_2, \ldots, \beta_k]$ such that the sequence stabiliser $G_{\beta_1,\beta_2,\ldots,\beta_k} = 1$. This determines a chain of stabilisers

$$G = G^{(1)} \geq G^{(2)} \geq \cdots \geq G^{(k)} \geq G^{(k+1)} = 1,$$

where $G^{(i)} = G_{\beta_1,\beta_2,\ldots,\beta_{i-1}}$. A *strong generating set* for B is a subset S of G such that $G^{(i)} = \left\langle S \cap G^{(i)} \right\rangle$, for $i = 1, \ldots, k$.

The main task in setting up such a data structure is the construction of *basic orbits* – the orbit B_i of β_{i+1} under $G^{(i)}$. Observe that $|G^{(i)} : G^{(i+1)}| = |B_i|$. Sims used Schreier's Lemma to obtain a deterministic algorithm to construct the strong generating sets. By contrast, Leon's random Schreier-Sims [17] used random elements of G. It is usually significantly faster, giving smaller strong generating sets. Its results can be verified; see, for example, [9, Section 6.3].

If we simply exploit the natural faithful action of $G \leq \mathrm{GL}(d, q)$ on the vectors in $V = \mathrm{GF}(q)^d$, then the basic orbits are usually very large; if G is simple, the first orbit length is often $|G|$. By choosing base points having shorter basic orbits, we extend significantly the range of application of the Schreier-Sims. Butler [5] developed the algorithm for linear groups, choosing as base points the one-dimensional subspaces of V. A general strategy to select good base points was introduced by Murray & O'Brien [18].

Despite various limitations imposed by the basic orbit sizes, the algorithm and its variations underpin most of the long-standing machinery for computing with

linear groups. The implementations in MAGMA are very effective for "moderate" degree representations defined over "small" fields.

Celler & Leedham-Green [7] presented a deterministic algorithm to compute the order of $g \in \mathrm{GL}(d, q)$. In summary, from a consideration of the minimal polynomial of g, they first obtain a "good" multiplicative upper bound for $|g|$ and then use a "divide-and-conquer" strategy to obtain the order.

Many of the algorithms developed recently for linear groups rely on random selections, and the analysis of their performance assumes that we select uniformly distributed random elements. MAGMA uses the *product replacement algorithm* of Celler *et al.* [6]. Leedham-Green & O'Brien [16] presented a variation to construct random elements of a normal subgroup, described by a normal generating set.

In the same paper they describe an algorithm to decide if a group G is perfect. By taking commutators of generators, we construct a normal generating set for G', the derived subgroup of G. For each generator g of G, we compute the orders o_i of elements gh_i for randomly chosen elements h_i of G'. If the greatest common divisor of the o_i is 1 (and we check this after each choice of h_i), then we have *proved* that $g \in G'$.

More generally, this algorithm can decide membership in an arbitrary normal subgroup N of G. In particular, Babai & Shalev [2] proved that if N is simple and non-abelian, then we can test membership in N in Monte Carlo polynomial time.

4 A geometric approach

A classification of the maximal subgroups of the classical groups by Aschbacher [1] underpins the *geometric approach* to the study of linear groups.

Let Z denote the group of scalar matrices of G. Then G is *almost simple modulo scalars* if there is a non-abelian simple group T such that $T \leq G/Z \leq \mathrm{Aut}(T)$, the automorphism group of T. In summary, Aschbacher's classification implies that a linear group preserves some natural linear structure in its action on the underlying vector space V and has a normal subgroup related to this structure, or it is almost simple modulo scalars.

In more detail, if G is a maximal subgroup of a classical group, then it is in at least one of the following *Aschbacher categories*.

C1. G acts reducibly.

C2. G acts imprimitively.

C3. G acts on V as a group of semilinear automorphisms of a (d/e)-dimensional space over the extension field $\mathrm{GF}(q^e)$, for some $e > 1$, and so G embeds in $\Gamma L(d/e, q^e)$.

C4. G preserves a decomposition of V as a tensor product $U \otimes W$ of spaces of dimensions $d_1, d_2 > 1$ over F.

C5. G is definable modulo scalars over a subfield.

C6. For some prime r, $d = r^m$ and G/Z is contained in the normaliser of an extraspecial group of order r^{2m+1}, or of a group of order 2^{2m+2} and symplectic-type.

C7. G is tensor-induced.

C8. G normalises a classical group in its natural representation.

C9. Otherwise G is almost simple modulo scalars.

The first seven categories have a normal subgroup associated with a decomposition. The C9-class consists of absolutely irreducible, tensor-indecomposable, primitive groups which are almost simple modulo scalars, cannot be defined over a proper subfield, and are not classical in their natural representation.

In broad outline, this theorem suggests that a first step in investigating a linear group is to determine (at least one of) its categories in the Aschbacher classification. If a category is recognised, then we investigate the group structure more completely using algorithms designed for this category. Usually, we have reduced the size and nature of the problem. For example, if $G \leq \mathrm{GL}(d, q)$ acts imprimitively, then we obtain a permutation representation of degree dividing d for G. If a proper normal subgroup N exists, we recognise N and G/N recursively, ultimately obtaining a composition series for G. Many questions about the structure of G can then be answered by consideration of its composition factors.

5 The composition tree

In ongoing work, Leedham-Green and O'Brien have developed the concept of a *composition tree*, which seeks to realise and exploit the Aschbacher classification. Leedham-Green [15] provided a detailed description of this concept and its practical realisation. Here we summarise it briefly.

A composition series for a group G can be viewed as a labelled rooted binary tree. The nodes correspond to sections of G, the root node to G. A node that corresponds to a section K of G, and is not a leaf, has a left descendant corresponding to a proper normal subgroup N of K and a right descendant corresponding to K/N. The right descendant is an image under a homomorphism; usually these arise naturally from an Aschbacher category of the group, but we also exploit additional ones for unipotent and soluble groups. The left descendant of a node is the kernel of the chosen homomorphism.

The tree is constructed in *right depth-first order*. Namely, we process the node associated with K: if K is not a leaf, construct recursively the subtree rooted at its right descendant I, then the subtree rooted at its left descendant N. Each leaf is a composition factor of the root group G.

It is easy to construct I, since it is the image of K under a homomorphism ϕ. We generate a random element of N as follows. Let $K = \langle x_1, \ldots, x_m \rangle$, and let $I = \phi(K) = \langle \overline{x_1}, \ldots, \overline{x_m} \rangle$. Choose random $k \in K$, and evaluate $\phi(k) \in I$. If we establish that $\phi(k) = w(\overline{x_1}, \ldots, \overline{x_m})$, then $k \cdot w(x_1, \ldots, x_m)^{-1} \in N$. By selecting sufficient random elements of K, we construct with *high probability* a generating set for N.

Observe that this strategy assumes that we can write an arbitrary element of I as a word in its defining generators. A major ongoing goal is to develop *constructive recognition algorithms* which perform such a task. Currently they are available for certain classes of groups; see [21] for details.

If N is nontrivial, we can usually find *some* elements randomly, by computing k^r, where k is a random element of K and r is the order of kN in K/N.

If we know presentations for K/N and N, then we can construct one for K; see [15] for details. If so, we can decide that we have constructed a generating set for N – and not just one for a *proper subgroup* of N.

5.1 Identifying the composition factors

A natural question is: identify the non-abelian composition factors of G. A *non-constructive recognition algorithm* names a simple group G. (More precisely, it may establish that G *contains* a particular named group as a subgroup.)

Neumann & Praeger [19] presented a one-sided Monte Carlo algorithm to decide whether or not a subgroup of $\mathrm{GL}(d, q)$ contains $\mathrm{SL}(d, q)$. Niemeyer & Praeger [20] answered the corresponding question for an arbitrary classical group in the natural representation; their algorithm is available in Magma. A *positive* answer

– that the input group is classical – is guaranteed to be correct. Our applications of these algorithms in Section 7 rely on positive answers only.

Babai *et al.* [3] presented a Monte Carlo algorithm to name a black-box group of Lie type in known defining characteristic. In 2001 Malle and O'Brien developed a practical implementation of this algorithm in MAGMA. It also includes identification procedures for the other quasisimple groups. If the non-abelian composition factor is sporadic, then we identify it by considering the (projective) orders of random elements. Similar methods can be used to deduce the degree of an alternating group.

5.2 Membership in other categories

We briefly mention the algorithms used to decide membership in other categories relevant to this paper. MAGMA uses the MEATAXE, a Las Vegas algorithm, to decide if G acts reducibly on its underlying vector space; it also uses a Las Vegas algorithm to test isomorphism of modules. See [9, Chapter 7] for details of both algorithms. Holt *et al.* [11, 12] present algorithms, implemented in MAGMA, to decide if an absolutely irreducible group acts imprimitively or semilinearly; if the answer is positive, then it is demonstrably correct.

6 A module argument

We now consider a situation which arises frequently in our analysis of these groups and exploit module structure to obtain more detailed structural information.

Let $F := \mathrm{GF}(q)$, and let M be the d-dimensional right module over F on which $G \leq \mathrm{GL}(d, q)$ acts. Suppose that G acts reducibly on M with submodule M_1 of dimension d_1 and quotient $M_2 := M/M_1$ of dimension d_2, where $d_1 + d_2 = d$. We make a basis change to bring the elements of G into the block form

$$\begin{pmatrix} A & \mathbf{0} \\ C & B \end{pmatrix} \tag{6.1}$$

where the diagonal blocks A and B are $(d_1 \times d_1)$– and $(d_2 \times d_2)$–matrices, respectively.

Thus G has the structure $N.H$, where N is the subgroup of G consisting of matrices of the form

$$\begin{pmatrix} I & \mathbf{0} \\ C & I \end{pmatrix} \tag{6.2}$$

and $H \cong G/N$ is isomorphic to the group having elements of the form

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$$

induced in the obvious way from the elements of G. (We are not of course claiming that the matrices of this form necessarily constitute a subgroup of G.)

Since N acts trivially on M_1 and M_2, we may also regard M_1 and M_2 as modules over H. Let H_1 and H_2 be the images of the projections of H onto the upper and lower diagonal blocks, respectively (so H is a subdirect product of H_1 and H_2). The following allows us to obtain readily some structural information about H.

Lemma 6.1. (i) *Let $h \in H$, and let h_1 and h_2 be the projections of h onto H_1 and H_2. If $|h| > |h_i|$ for $i = 1$ or 2, then $|H| > |H_i|$.*

(ii) *If M_1 is isomorphic as FH-module to either M_2 or to the dual of M_2, then $H \cong H_1 \cong H_2$.*

Proof. (i) is clear. Let K_1 and K_2 be the kernels of the actions of H on M_1 and M_2, respectively. Then K_1 and K_2 consist of those elements of H that induce the identity on the upper and lower diagonal blocks, respectively, and $H/K_i \cong H_i$ for $i = 1, 2$. If M_1 is FH-isomorphic to M_2 or to its dual, then $K_1 = K_2$, and so $K_1 = K_2 = 1$ and the result follows. □

Let L be the elementary abelian group of order $q^{d_1 d_2}$ consisting of all matrices that have the form 6.2 defined above. Then $H_1 \times H_2$ acts by conjugation on L, thereby making it into a module for $H_1 \times H_2$ over $F = \mathrm{GF}(q)$. Conjugating an element of form 6.2 by one of form 6.1 results in the $(d_2 \times d_1)$–matrix C being replaced by $B^{-1}CA$. In particular, if we denote the matrix of form 6.2 in which C is a matrix with a single one in position (i, j) by e_{ij}, and let $A = (\alpha_{ij})$ and $(B^{-1})^{\mathrm{T}} = (\overline{\beta}_{ij})$, then e_{ij} is conjugated to $\sum_{k=1}^{d_2} \sum_{l=1}^{d_1} \overline{\beta}_{ik} \alpha_{jl} e_{kl}$. This demonstrates that, as an $F(H_1 \times H_2)$-module, $L \cong M_2^* \otimes_F M_1$, where M_2^* denotes the dual of the module M_2. By [14, VII, Lemma 8.8 b)], we also have $L \cong \mathrm{Hom}_F(M_2, M_1)$.

The subgroup N of L can be regarded in the same manner as a module for H under the conjugation action. However, it is not in general a $\mathrm{GF}(q)$-submodule of the restriction of L to H, but only a $\mathrm{GF}(p)$-submodule, where $q = p^e$ is a power of the prime p. Then N has $\mathrm{GF}(p)$-dimension k for some k with $0 \leq k \leq ed_1d_2$.

In practice, this is not very useful if ed_1d_2 is very large. Since N has potentially large order (and consequently many generators), constructing its generating set remains a challenging open problem. Recall from Section 5 that we can construct some elements of N. If ed_1d_2 is not too large (the current limit of practicality

in MAGMA is about 80000 for $q = 2$), then we can compute (deterministically) the $\mathrm{GF}(p)H$-submodule that they generate, and thereby obtain a lower bound for $|N|$.

The following theoretical result is sometimes applicable. Since $q = p = 2$ in the examples of Section 7, we shall avoid complications arising from the fact that, in general, N is only a $\mathrm{GF}(p)$-submodule of L, by assuming that $q = p$.

Lemma 6.2. *If N is nontrivial, $H = H_1 \times H_2$, $F = \mathrm{GF}(p)$, and M_1 and M_2 are absolutely irreducible FH-modules, then $N = L$.*

Proof. Since H_2 acts trivially on M_1 and H_1 acts trivially on M_2, we may regard M_1 and M_2 as absolutely irreducible F-modules for H_1 and H_2, respectively. By [13, V, Satz 10.3 b)], the tensor product of irreducible modules V_1 and V_2 over an algebraically closed field for finite groups A_1 and A_2 is irreducible as an $(A_1 \times A_2)$-module. The same result holds for absolutely irreducible modules over an arbitrary field, since such modules remain irreducible when we extend to the algebraic closure of the field. Hence $N \cong M_2^* \otimes_F M_1$ is an irreducible FH-module, and the result follows. □

Remark. More generally, if M_1 and M_2 are irreducible FH_i-modules over an arbitrary field F of non-zero characteristic, and $E_i = \mathrm{End}_{FG}(M_i)$ for $i = 1, 2$, then $M_1 \otimes_F M_2$ is an irreducible $F(H_1 \times H_2)$-module if and only if $|E_1 : F|$ and $|E_2 : F|$ are coprime. We are grateful to L.G. Kovács for pointing this out to us.

Knowledge of presentations for $H = G/N$ and N would allow us to verify conclusively that we have constructed N, rather than a proper subgroup. In the absence of a presentation, we know of no general method to obtain an upper bound for the order of N. But, as we shall see in the examples below, we can sometimes use specialised knowledge to deduce such.

7 The groups G_n and G_n^l

We summarise the results of our investigations into the structure of the groups G_n and G_n^l in Table 1. The times given are in seconds, and are the totals for all MAGMA commands executed for the computations involving that group. These can vary considerably from run to run. They were carried out using MAGMA 2.12 on a 400MHz Ultrasparc with 4GB of memory. By combining the results of our computations and the theoretical results presented in Section 6, we were able to prove these results in all cases.

n	Dim	G_n	time	Dim	G_n^l	time
1	6	S_4	0			
2	6	$L(2,7):2$	0			
3	6	$U(3,3):2$	0	12	$U(3,3):2$	0
4	12	$J_2:2$	1			
5	12	$G_2(4):2$	2	24	$G_2(4):2$	2
6	24	$3\cdot\mathrm{Suz}:2$	14			
7	24	Co_1	129	48	$2\times\mathrm{Co}_1$	95
8	48	$\mathrm{Co}_1\wr 2$	208			
9	48	$\mathrm{Co}_1\wr 2$	213	96	$2^{24^2}.(\mathrm{Co}_1\wr 2)$	348
10	96	$\mathrm{SL}(48,2).2$	9			
11	96	$\Omega^+(96,2)$	12	192	$2\times\Omega^+(96,2)$	43
12	192	$\Omega^+(96,4).2$	59			
13	192	$\Omega^+(96,4).2$	57	384	$2^{96^2}.\Omega^+(96,4).2$	20239
14	384	$\mathrm{SU}(192,2).2$	69			
15	384	$\Omega^+(384,2)$	274	768	$2\times\Omega^+(384,2)$	1211
16	768	$\Omega^+(384,2)\wr 2$	993			
17	768	$\Omega^+(384,2)\wr 2$	1040	1536	$2^{384^2}.(\Omega^+(384,2)\wr 2)$	2480
18	1536	$\mathrm{SL}(768,2).2$	4560			
19	1536	$\Omega^+(1536,2)$	12637	3072	$2\times\Omega^+(1536,2)$	129643
20	3072	$\Omega^+(1536,4).2$	589724			

Table 1: The structure of the groups G_n and G_n^l

For $n \geq 10$, there is evidence of a pattern emerging with period 8: namely G_n and G_n^l have similar structures to G_{n+8} and G_{n+8}^l. But the evidence is too limited to justify a conjecture about their structure for arbitrary n.

Five of the nine Aschbacher categories arise when analysing the structure of these groups. These are C1, C2, C3, C8 and C9.

The C9-groups that arise are $L(2,7):2$, $U(3,3):2$, J_2, $G_2(4):2$, $3\cdot\mathrm{Suz}:2$, and Co_1. They are sufficiently small for us to recognise them constructively using the base and strong generating set method described in Section 3. The simple socles of the two largest examples, $3\cdot\mathrm{Suz}$ and Co_1, arise as subgroups of $\mathrm{SL}(12,4)$ and $\mathrm{SL}(24,2)$, respectively. By sampling (projective) orders of random elements as discussed in Section 5.1, we were able to identify their isomorphism types with a high probability of correctness. Using the method described in Section 3, we established that both groups were perfect, so it remained only to verify their orders deterministically to complete their identification. In applying the Schreier-Sims, we chose base points appropriate to these representations, and so the remaining computations were efficient. For $3\cdot\mathrm{Suz}$, we completed the verification using the

matrix representation; for Co_1, we constructed a faithful permutation representation of degree 98280, and used this to confirm the order of the group.

We now discuss the groups in Table 1 individually. Let $F := \mathrm{GF}(2)$. The groups G_n for $2 \leq n \leq 7$ are C9-groups, whereas G_{11}, G_{15} and G_{19} are C8-groups.

For $1 \leq n \leq 5$, we immediately established using the standard MAGMA functions `Order` and `ChiefFactors` (both using variations of the Schreier-Sims algorithm), that G_n, and also G_n^l when n is odd, are isomorphic to S_4, $L(2,7):2$, $U_3(3):2$, $J_2:2$, $G_2(4):2$, respectively. This, together with the identification of G_6 as $3 \cdot \mathrm{Suz}:2$, confirms certain results of [25].

The C8-groups arising are $\mathrm{SL}(48,2)$, $\mathrm{SL}(768,2)$, $\Omega^+(96,2)$, $\Omega^+(96,4)$, $\Omega^+(384,2)$, $\Omega^+(1536,2)$, $\Omega^+(1536,4)$, and $\mathrm{SU}(192,2)$. We readily identified these groups using the algorithm mentioned in Section 5.1. We confirmed that the orthogonal groups are of type Ω^+ rather than SO^+, by proving that they are perfect using the algorithm outlined in Section 3.

The groups G_6, G_{12}, G_{13}, G_{14} and G_{20} are C3-groups. Hence they have a normal subgroup N that acts irreducibly but not absolutely irreducibly, and so N can be rewritten as a group acting absolutely irreducibly in smaller dimension over a larger field. The elements outside of N act as field automorphisms on N. In each of these examples, $|G_n:N| = 2$, and N can be rewritten as a group of degree half the original dimension over $\mathrm{GF}(4)$. For G_6, we identified N as $3 \cdot \mathrm{Suz}$; otherwise, N is a C8-group; in all cases these were recognised as described above.

We commented in Section 5 upon the difficulty of obtaining generators of normal subgroups N of G that arise in the composition tree program. However, in these examples $|G/N| = 2$, and we readily calculated Schreier generators for N.

The groups G_8, G_9, G_{10}, G_{16}, G_{17} and G_{18} are C2-groups, with two blocks of imprimitivity. Again we have a normal subgroup N of index 2 for which we found Schreier generators, but here N acts decomposably with two components of degree half of the original. The restricted actions on the components are C8-groups in each case, and they were recognised as before.

Since these C8-groups S are simple, there are only two possibilities for the structure of N: either $N \cong S$ or $N \cong S \times S$. We used Lemma 6.1 to distinguish between these possibilities. For G_8, G_9, G_{16} and G_{17}, we found elements in N for which the restrictions onto the two components have different orders. Thus $N \cong S \times S$ in these examples and, by [10, Theorem 3] for example, G is isomorphic to the wreath product $S \wr 2$. For G_{10} and G_{18}, the FN-modules corresponding to the actions on the two components were dual to each other. Hence $N \cong S$; so G_{10} and G_{18} are $\mathrm{SL}(48,2)$ and $\mathrm{SL}(768,2)$, respectively, extended by the duality automorphism.

The remaining examples, G_n^l for odd $3 \leq n \leq 19$, are C1-groups with two irreducible constituents each having dimension $d/2$, where d is the dimension of G_n^l. From the discussion in Section 6, we learn that $G_n^l \cong N_n.H_n$, where N_n is an elementary abelian group of order 2^k with $0 \leq k \leq d^2/4$, and $H_n \leq \mathrm{GL}(d/2,2) \times \mathrm{GL}(d/2,2)$. In each of these examples, the FH_n-modules (referred to as M_1 and M_2 in Section 6) corresponding to the actions on the two components of H_n are isomorphic. Hence, by Lemma 6.1, $H_n \leq \mathrm{GL}(d/2,2)$, and H_n acts faithfully on each of the two components. We analysed the structure of H_n, and found that $H_n \cong G_n$ in each case.

For G_3^l and G_5^l we verified immediately with the MAGMA function `ChiefSeries` that $G_n^l \cong H_n$, so N_n is trivial.

For G_7^l, G_{11}^l, G_{15}^l and G_{19}^l, the derived group $[G_n^l, G_n^l]$ acts decomposably with two components of dimension $d/2$. Since H_n is perfect, this implies that $[G_n^l, G_n^l] \cong H_n$, and $G_n^l \cong N_n \times H_n$. By using Schreier generators, as described earlier, we were able to show that $|G_n^l : H_n| = 2$, so $G_n^l \cong 2 \times H_n$. Alternatively, the fact that N_n is nontrivial and $G_n^l \cong N_n \times H_n$ enables us to deduce theoretically that $|N_n| = 2$, as follows. We saw in Section 6 that $L_n \cong \mathrm{Hom}_F(M_2, M_1)$, where L_n is the module for H_n consisting of all matrices of the form 6.2. But M_1 and M_2 are isomorphic absolutely irreducible modules for H_n, and so the submodule of fixed points of L_n under the action of H_n corresponds to $\mathrm{Hom}_{FH_n}(M_2, M_1)$ which has dimension 1 over F. Since N_n clearly lies in the fixed point submodule in these examples, we conclude $|N_n| \leq 2$.

In the remaining examples, G_9^l, G_{13}^l and G_{17}^l, the subgroup N_n is much larger. For G_9^l and G_{13}^l, we constructed in MAGMA the FH_n-module L_n defined in Section 6, and then constructed the submodule generated by a few randomly chosen elements of N_n to prove that $|N_9| \geq 2^{24^2}$ and $|N_{13}| \geq 2^{96^2}$. For G_{17}^l, L_n has dimension $384^2 = 147456$; although we could define L_n in MAGMA, we were unable to construct its submodules.

For G_9^l, we used the permutation representation of degree 98280 of Co_1 to obtain a presentation of $H_9 \cong \mathrm{Co}_1 \wr 2$, which we could then use to prove that $|N_9| = 2^{24^2}$. We also applied an alternative approach to G_9^l, however, which proved more generally applicable, in particular to the other two examples. We first found generators for the subgroup $N_9.H_9'$ of index 2 in G_9^l, where $H_9' = \mathrm{Co}_1 \times \mathrm{Co}_1$. We then found that the restriction of the natural module M for G_9^l to $N_9.H_9'$ is decomposable with two isomorphic components of dimension $d/2 = 48$. Hence, by Lemma 6.1, $N_9.H_9'$ acts faithfully on each of these components, so we can restrict to the action on one of them. This restricted action is reducible with two irreducible constituents of degree $d/4 = 24$, from which it is clear that $|N_9| \leq 2^{24^2}$.

We were able to find the corresponding decomposition for G_{17}^l, and thereby de-

duce in the same way that $|N_{17}| \leq 2^{384^2}$, but in this case we could not find a lower bound for $|N_{17}|$. However, by the theory described in Section 6, N_{17} is a submodule of the tensor product $M_2^* \otimes M_1$ for the direct product $\Omega^+(384,2) \times \Omega^+(384,2)$, where M_1 and M_2 are both equal to the natural module for $\Omega^+(384,2)$. Since this natural module is absolutely irreducible, it follows from Lemma 6.2 that $|N_{17}| = 2^{384^2}$. We can apply the same argument to G_9^l.

At first sight, it seems not possible to apply a similar argument to find an upper bound for $|N_{13}|$, since H_{13} is semilinear rather than imprimitive. However, if we regard G_{13}^l as a subgroup of $\mathrm{GL}(384,4)$ rather than of $\mathrm{GL}(384,2)$, then H_{13} is imprimitive. We now find the analogous decomposition to that of G_9^l and G_{17}^l and so deduce that N_{13} has dimension at most 96^2 as a vector space over $\mathrm{GF}(4)$. But the $\mathrm{GF}(4)$-dimension of N_{13} is just the $\mathrm{GF}(4)$-dimension of $N_{13} \otimes_F \mathrm{GF}(4)$, which is equal to the dimension of N_{13} over $F = \mathrm{GF}(2)$, and so $|N_{13}| \leq 2^{96^2}$.

8 The existence of Γ_n^*-complexes

We now consider what our analysis of Parker's representations says about the existence of Γ_n^*-complexes for larger values of n.

Theorem 4 of [25] provides conditions sufficient for the construction of a Γ_n-complex from a known Γ_{n-1}-complex. Suppose that we have an epimorphism $\phi : U_n \to G$ for a finite group G and let $H_i := \phi(\langle a, u_0, \ldots, u_i \rangle)$ for $1 \leq i \leq n$. Then the restriction of ϕ to $\langle a, u_0, \ldots, u_{n-1} \rangle$ induces an epimorphism $\psi : U_{n-1} \to H_{n-1}$. We call ψ a Γ_{n-1}-*map* if $H_{n-1} = \mathrm{Aut}(L)$ for some Γ_{n-1}-complex L, and if certain other technical conditions are satisfied; see [25, Definition 4] for a precise definition. Further ϕ is a Γ_n-map if H_{n-1} is a core-free subgroup of G and $H_{n-1} \cap H_{n-1}^{\phi(u_n)} = H_{n-2}$. Observe that H_{n-2} is always a subgroup of $H_{n-1} \cap H_{n-1}^{\phi(u_n)}$; if H_{n-1} is a core-free subgroup of G, then it cannot be normalised by $\phi(u_n)$, so a sufficient condition for equality is that H_{n-2} is a maximal subgroup of H_{n-1}. Corresponding assertions apply with U_n replaced by U_n^* and Γ_n by Γ_n^*.

Let $\upsilon_n : U_n \to U_{n+1}$ be the homomorphism induced by mapping each generator of U_n to the generator of U_{n+1} with the same name. If we denote Parker's representations by $\phi_n : U_n^* \to G_n$ and $\phi_n^l : U_n^* \to G_n^l$, then it is immediately clear from the definitions of ϕ_n and ϕ_n^l in Section 2 that there are embeddings $\iota_n : G_n \to G_{n+1}^l$ (n even) and $\iota_n : G_n^l \to G_{n+1}$ (n odd) such that $\phi_n \iota_n = \upsilon_n \phi_{n+1}^l$ (n even) and $\phi_n^l \iota_n = \upsilon_n \phi_{n+1}$ (n odd).

Consider the case when $n = 9$. Theorem 6 of [25] shows that $\psi = \phi_8$ is a Γ_8-map, $H_8 = \iota_8(G_8)$ is a core-free subgroup of G_9^l, and $H_7 = \iota_8\iota_7(G_7^l) = H_8 \cap H_8^{\phi(u_9)}$ since $\iota_7(G_7^l)$ is a maximal subgroup of G_8. As Soicher comments at the end of Section

4 of [25], this shows that $\phi = \phi_9^l$ is a Γ_9^*-map, and establishes the existence of a Γ_9^*-complex with automorphism group isomorphic to $G_9^l = 2^{24^2} . (\mathrm{Co}_1 \wr 2)$.

Since $\iota_8(G_8)$ is a maximal subgroup of G_9 (it is a complement in the extension $2^{24^2} . (\mathrm{Co}_1 \wr 2)$ that acts irreducibly on the subgroup 2^{24^2}) and $\iota_9(G_9)^l$ is clearly a core-free subgroup of the almost simple group $G_{10} = \mathrm{SL}(48, 2) . 2$, we deduce the existence of a Γ_{10}^*-complex with automorphism group isomorphic to $\mathrm{SL}(48, 2) . 2$.

However, as was pointed out to us by J.N. Bray, any intersection of two conjugates of $\mathrm{SL}(48, 2) . 2$ in $G_{11}^l = 2 \times \Omega^+(96, 2)$ that contains a subgroup $2^{24^2} . (\mathrm{Co}_1 \wr 2)$ must necessarily contain the larger group $2^{24^2} . (\mathrm{Sp}(24, 2) \wr 2)$. Hence the condition $H_9 = H_{10} \cap H_{10}^{\phi(u_{11})}$ does not hold, and so these methods cannot be applied to construct Γ_n^*-complexes for $n > 10$.

9 The presentations

We now briefly consider the finitely-presented groups U_n and U_n^*. Soicher [25] proved that $U_2 \cong L_3(2) : 2$ and $U_3 \cong (3 \times U_3(3)) : 2$, and commented that "$U_4$ may in fact be infinite".

We can confirm this. In 2005 Bernd Souvignier (private communication) exhibited a subgroup of U_4 having a free abelian quotient of dimension 4. He used the low-index subgroup algorithm [23, Section 5.6] to investigate subgroups of index 36 in the subgroup of U_4 of index 200 that maps onto $U_3(3)$. The kernel K of the map of one of these subgroups onto a 2-quotient of order 2^5 has such an abelianisation. Note that K has index 691200 in U_4; we have also found a subgroup of index 172800 in U_4 with infinite abelianisation.

It seems plausible that the homomorphism υ_n from U_n to U_{n+1} defined in Section 8 is an embedding for all n, which would imply that U_n is infinite for all $n \geq 4$. However we were not able to prove this. Neither did investigations of subgroups of low index in U_n for $n \geq 5$ yield a proof that they are infinite.

Of course U_n^* may also be infinite for sufficiently large $n \geq 9$, but again we failed to prove this. For $n \geq 5$, U_n and U_n^* have perfect derived group of index 2. For $n > 8$, the only finite homomorphic images of U_n^* of order greater than 2 that we could construct are those listed in Table 1; these do not provide subgroups of sufficiently low index to allow us to compute their abelianisation. We established that the derived groups of U_9^*, U_{10}^* and U_{11}^* have no simple homomorphic images of order up to 10^8.

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