The Alperin and Dade conjectures for the O'Nan and Rudvalis simple groups

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Abstract

We classify the radical subgroups and chains of the O'Nan and Rudvalis simple groups O$'$N and Ru, and then verify the Alperin weight conjecture and the Dade final conjecture for the two groups.

1 Introduction

In [2] and [3], we presented a (modified) local strategy to decide the Alperin and Dade conjectures for the finite simple groups and demonstrated its computational effectiveness by using it to verify these conjectures for the Conway simple group Co_2 and the Fischer simple group Fi_{23}. In [4], we verified the two conjectures for the Fischer simple group Fi_{22} using the local strategy. In this paper, we apply the strategy to verify the Alperin and Dade conjectures for the O'Nan and Rudvalis simple groups, O$'$N and Ru.

The ordinary conjecture for the Rudvalis simple group Ru has been verified by Dade (see [9], p. 99), so it suffices to verify the projective conjecture for the covering group 2.Ru. However, the calculations to verify the projective conjecture also allow us to establish the ordinary conjecture, and so for completeness, we include its verification.

Let G be a finite group, p a prime and B a p-block of G. Alperin [1] conjectured that the number of B-weights equals the number of irreducible Brauer characters of B. Dade [8] generalized the Knörr-Robinson version of the Alperin weight conjecture and presented his ordinary conjecture exhibiting the number of ordinary irreducible characters of a fixed defect in B in terms of an alternating sum of related values for p-blocks of some p-local subgroups of G. Dade [11] presented several other forms of his conjecture and announced that his final conjecture needs only to be verified for finite non-abelian simple groups; in addition, if a finite group has a cyclic outer automorphism group, then the projective invariant conjecture is equivalent to the final conjecture.

The outer automorphism groups of O$'$N and Ru have orders 2 and 1 respectively. Hence Dade's final conjecture for O$'$N (respectively for Ru) is equivalent to the invariant (ordinary) conjecture for O$'$N (Ru) and the projective invariant conjecture for 3.O$'$N (2.Ru).

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The paper is organized as follows. In Section 2, we fix notation and state Alperin's weight conjecture, and Dade's ordinary, invariant, projective, and projective invariant conjectures. In Section 3, we recall our modified local strategy and explain how we applied it to determine the radical subgroups of O′N and Ru. In Sections 4 and 5, we classify the radical subgroups of O′N and Ru, respectively, up to conjugacy and verify Alperin's weight conjecture. In Sections 6 and 7, we do some cancellations in the alternating sum of Dade's conjecture when $p = 2$ or 3, and then determine radical chains (up to conjugacy) and their local structures. In Sections 8 and 9, we verify Dade's invariant conjecture for O′N and ordinary conjecture for Ru. In Sections 10 and 11, we verify Dade's projective invariant conjecture for 3.O′N and 2.Ru, respectively. Four appendices provide details of the degrees of character tables.

2 The Alperin and Dade conjectures

Let R be a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and let $\mathrm{Blk}(G)$ be the set of p-blocks, $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a B*-weight* if φ has p-defect 0 and $B(\varphi)^G = B$, where the number $\log_p(\frac{|G|_p}{\varphi(1)_p})$ is the p-defect of φ, $B(\varphi)$ is the block of $N(R)$ containing φ and $B(\varphi)^G$ is the block of G corresponding to $B(\varphi)$ under the Brauer correspondence. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights, and $\ell(B)$ the number of irreducible Brauer characters of B. Alperin conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$.

Given a p-subgroup chain

$$C : P_0 < P_1 < \cdots < P_n \tag{2.1}$$

of G, define $|C| = n$, $C_k : P_0 < P_1 < \cdots < P_k$, $C(C) = C_G(P_n)$, and

$$N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \cdots \cap N(P_n). \tag{2.2}$$

The chain C is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$ for $1 \le k \le n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G.

Let Z be a cyclic group and $G^* = Z.G$ a central extension of Z by G, and $C \in \mathcal{R}(G)$. Denote by $N_{G^*}(C)$ the preimage $\eta^{-1}(N(C))$ of $N(C)$ in G^*, where η is the natural group homomorphism from G^* onto G with kernel Z. Let ρ be a faithful linear character of Z and B^* a block of G^* covering the block $B(\rho)$ of Z containing ρ. Denote by $\mathrm{Irr}(N_{G^*}(C), B^*, d, \rho)$ the irreducible characters ψ of $N_{G^*}(C)$ such that ψ lies over ρ, $\mathrm{d}(\psi) = d$ and $B(\psi)^{G^*} = B^*$ and set $\mathrm{k}(N_{G^*}(C), B^*, d, \rho) = |\mathrm{Irr}(N_{G^*}(C), B^*, d, \rho)|$.

Dade's Projective Conjecture [11]. If $O_p(G) = 1$ and B^* is a p-block of G^* covering $B(\rho)$ with defect group $D(B^*) \neq O_p(Z)$, then

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_{G^*}(C), B^*, d, \rho) = 0, \tag{2.3}$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

If, moreover, E^* is an extension of G^* centralizing Z and $N_{E^*}(C,\psi)$ is the stabilizer of $(N_{G^*}(C),\psi)$ in E^*, then $N_{E^*/G^*}(C,\psi) = N_{E^*}(C,\psi)/N_{G^*}(C)$ is a subgroup of E^*/G^*. For a subgroup $U^* \leq E^*/G^*$, denote by $\mathrm{k}(N_{G^*}(C), B^*, d, U^*, \rho)$ the number of characters ψ in $\mathrm{Irr}(N_{G^*}(C), B^*, d, \rho)$ such that $N_{E^*/G^*}(C,\psi) = U^*$. In the notation above, the Projective Invariant Conjecture is stated as follows.

Dade's Projective Invariant Conjecture [11]. If $O_p(G) = 1$ and B^* is a p-block of G^* covering $B(\rho)$ with $D(B^*) \neq O_p(Z)$, then

$$\sum_{C\in\mathcal{R}/G} (-1)^{|C|}\mathrm{k}(N_{G^*}(C), B^*, d, U^*, \rho) = 0. \tag{2.4}$$

In addition, if E^*/G^* is cyclic and $u = |U^*|$, then we set

$$\mathrm{k}(N_{G^*}(C), B^*, d, u, \rho) = \mathrm{k}(N_{G^*}(C), B^*, d, U^*, \rho).$$

In particular, if $Z = 1$, then ρ is the trivial character of Z, $G^* = G$ and B^* is a block B of G, we set $U = U^*$ and

$$\mathrm{k}(N(C), B, d, U) = \mathrm{k}(N_{G^*}(C), B^*, U^*, \rho).$$

Then the Projective Invariant Conjecture is reduced to the Invariant Conjecture.

Dade's Invariant Conjecture [11]. If $O_p(G) = 1$ and B is a p-block of G with defect $\mathrm{d}(B) > 0$, then

$$\sum_{C\in\mathcal{R}/G} (-1)^{|C|}\mathrm{k}(N(C), B, d, U) = 0. \tag{2.5}$$

Moreover, if $E^* = G^* = G$ and $Z = 1$, then $U^* = 1$ and ρ is the trivial character of Z. We set $B^* = B$, and so

$$\mathrm{k}(N_{G^*}(C), B^*, d, U^*, \rho) = \mathrm{k}(N_G(C), B, d).$$

Thus the Projective Invariant Conjecture is reduced to the Ordinary Conjecture.

Dade's Ordinary Conjecture [8]. If $O_p(G) = 1$ and B is a p-block of G with defect $\mathrm{d}(B) > 0$, then

$$\sum_{C\in\mathcal{R}/G} (-1)^{|C|}\mathrm{k}(N(C), B, d) = 0. \tag{2.6}$$

3 The modified local strategy

The maximal subgroups of O′N were classified by [14], [19] and [20], and those of Ru were classified by Wilson [18]. Using these results, we deduce that each radical 2- and 3-subgroup R of G is also radical in a maximal subgroups M of G and further that $N_G(R) = N_M(R)$, where $G = \mathrm{O'N}$ or Ru.

In [2] and [3], a modified local strategy was developed to classify the radical subgroups R. We review this method here.

Step (1). We first consider the case where M is a p-local subgroup. Let $Q = O_p(M)$, so that $Q \leq R$. Choose a subgroup X of M. Using MAGMA, we explicitly compute the coset action of M on the cosets of X in M; we obtain a group W representing this action, a group homomorphism f from M to W, and the kernel K of f. For a suitable X, we have $K = Q$ and the degree of the action of W on the cosets is much smaller than that of M. We can now directly classify the radical p-subgroup classes of W, and the preimages in M of the radical subgroup classes of W are the radical subgroup classes of M.

Step (2). Now consider the case where M is not p-local. We may be able to find its radical p-subgroup classes directly. Alternatively, we find a subgroup K of M such that $N_K(R) = N_M(R)$ for each radical subgroup R of M. If K is p-local, then we apply Step (1) to K. If K is not p-local, we can replace M by K and repeat Step (2).

Steps (1) and (2) constitute the *modified local strategy.* After applying the strategy, possible fusions among the resulting list of radical subgroups can be decided readily by testing whether the subgroups in the list are pairwise G-conjugate.

In the investigations of the conjecture for O′N, we used the minimal degree representation of O′N as a permutation group on 122760 points and a representation of 3.O′N as a permutation group on 368280 points. These representations were constructed by performing coset enumerations over particular subgroups of the finite presentations for O′N and 3.O′N given by Soicher [13]. In the investigations of the conjecture for Ru, we used the minimal degree representation of Ru as a permutation group on 4060 points and a representation of 2.O′N as a permutation group on 16240 points. The maximal subgroups of O′N and Ru were constructed using details supplied in [6] and the black-box algorithms of Wilson [17]. Note that algorithms for constructing these maximal subgroups are described on the Atlas website, http://www.mat.bham.ac.uk/atlas/. We also made extensive use of the algorithm described in [7] to construct random elements, and the procedures described in [2] and [3] for deciding the conjectures.

The computations reported in this paper were carried out using MAGMA V.2.6-2 [5] on a Sun UltraSPARC Enterprise 4000 server.

4 Radical subgroups and weights of O′N

Let $\Phi(G,p)$ be a set of representatives for conjugacy classes of radical p-subgroups of G. For $H, K \le G$, we write $H \le_G K$ if $x^{-1}Hx \le K$; and write $H \in_G \Phi(G,p)$ if $x^{-1}Hx \in \Phi(G,p)$ for some $x \in G$. We shall follow the notation of [6]. In particular, if p is odd, then $p_+^{1+2\gamma}$ is an extra-special group of order $p^{1+2\gamma}$ with exponent p; if δ is $+$ or $-$, then $2_\delta^{1+2\gamma}$ is an extra-special group of order $2^{1+2\gamma}$ with type δ. If X and Y are groups, we use $X.Y$ and $X : Y$ to denote an extension and a split extension of X by Y, respectively. Given a positive integer n, we use E_{p^n} or simply p^n to denote the elementary abelian group of order p^n, $\mathbb{Z}_n$ or simply n to denote the cyclic group of order n, and D_{2n} to denote the dihedral group of order $2n$.

Let G be the simple O'Nan group O′N and $E = Aut(G) = G.2$. Then

$$|G| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31,$$

and we may suppose $p \in \{2, 3, 7\}$, since both conjectures hold for a block with a cyclic defect group by [8] and [10].

We denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H and by $\mathrm{d}(H)$ the number $\log_p(|H|)$. Given $R \in \Phi(G,p)$, let $C(R) =$

$C_G(R)$ and $N = N_G(R)$. If $B_0 = B_0(G)$ is the principal p-block of G, then (c.f. (4.1) of [2])

$$\mathcal{W}(B_0) = \sum_R |\mathrm{Irr}^0(N/C(R)R)|, \tag{4.1}$$

where R runs over the set $\Phi(G,p)$ such that the p-part $\mathrm{d}(C(R)R/R) = 0$. The character table of $N/C(R)R$ can be calculated by MAGMA, and so we find $|\mathrm{Irr}^0(N/C(R)R)|$.

Lemma 4.1 *The non-trivial radical* 7*-subgroups R of* O′N (*up to conjugacy*) *are given in Table* 1*, where $(H)_a$ and $(H)_b$ denote subgroups of* O′N *such that $(H)_a \simeq H \simeq (H)_b$ and $(H)_a \neq_{\mathrm{O'N}} (H)_b$. In particular, τ permutes $(7^2)_a$ and $(7^2)_b$ for some $\tau \in E \backslash \mathrm{O'N}$.*

R	$C(R)$	$N_G(R)$	$N_E(R)$	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
$(7^2)_a$	7^2	$7^2{:}\,L_2(7){:}\,2$	$7^2{:}\,L_2(7){:}\,2$	2
$(7^2)_b$	7^2	$7^2{:}\,L_2(7){:}\,2$	$7^2{:}\,L_2(7){:}\,2$	2
7_+^{1+2}	7	$7_+^{1+2}{:}\,(3 \times D_8)$	$7_+^{1+2}{:}\,(3 \times D_8).2$	15

Table 1: Non-trivial radical 7-subgroups of O′N

PROOF: It follows by [20, Proposition 3.5]. □

Lemma 4.2 *The non-trivial radical* 3*-subgroups R of* O′N (*up to conjugacy*) *are given in Table* 2*.*

R	$C(R)$	N	$N_E(R)$	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
3^2	$3^2 \times A_6$	$(3^2{:}\,4 \times A_6).2$	$N.2$	
3^4	3^4	$3^4{:}\,2_+^{1+4}.D_{10}$	$N.2$	14

Table 2: Non-trivial radical 3-subgroups of O′N

PROOF: Suppose R is a radical 3-subgroup of $G = \mathrm{O'N}$ and $N(R) = N_{\mathrm{O'N}}(R)$ and let $K_1 = (3^2{:}\,4 \times A_6).2$ and $K_2 = 3^4{:}\,2_+^{1+4}.D_{10}$ be maximal subgroups of G. By [20, Proposition 3.4], any 3-local subgroup of O′N is conjugate to a subgroup of K_1 or K_2, so we may suppose $N(R) \leq K_i$ and $R \in \Phi(K_i, 3)$ with $N(R) = N_{K_i}(R)$ for some i. Since a Sylow 3-subgroup 3^4 of K_1 is the only radical subgroup of K_1 properly containing $O_3(K_1) = 3^2$ and $N_{K_1}(3^4) = 3^4.2^2.2^3$, it follows that $O_3(K_1)$ and $O_3(K_2)$ are the only non-trivial radical subgroups, up to conjugacy of O′N. The centralizers and the normalizers of R can be obtained by MAGMA. □

Lemma 4.3 *The non-trivial radical* 2*-subgroups R of* O′N (*up to conjugacy*) *are given in Table* 3*, where $S \in \mathrm{Syl}_2(\mathrm{O'N})$ is a Sylow* 2*-subgroup of* O′N*.*

PROOF: Let $M_1 \simeq 4.L_3(4){:}\,2$, $M_2 \simeq (3^2{:}\,4 \times A_6).2$ and $M_3 \simeq 4^3.L_3(2)$ be maximal subgroups of $G = \mathrm{O'N}$. Suppose R is a non-trivial radical 2-subgroup of G. By Yoshiara

R	$C(R)$	N	$N_E(R)$	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
2^2	$2^2 \times 3^2{:}\,4$	$(2^2 \times 3^2{:}\,4).S_3$	$N.2$	
4	$4.L_3(4)$	$4.L_3(4){:}\,2$	$N.2$	
D_8	$2 \times 3^2{:}\,4$	$(D_8 \times 3^2{:}\,4).2$	$N.2$	
4^3	4^3	$4^3.L_3(2)$	$N.2$	1
4.2^4	4	$4.2^4.A_5$	$N.2$	1
$(4 \times 2^2).2^4$	4	$(4 \times 2^2).2^4.S_3$	$N.2$	1
$(4^2 \times 2).2^3$	2^2	$(4^2 \times 2).2^3.S_3$	$N.2$	1
S	2	S	$N.2$	1

Table 3: Non-trivial radical 2-subgroups of O′N

[20, Proposition 3.3], we may suppose $R \in \Phi(M_i, 2)$ such that $N(R) = N_{M_i}(R)$ for some i, where $1 \leq i \leq 3$.

(1) Let $4 = O_2(M_1)$. Applying the local strategy of [3] to M_1, we have

$$\Phi(4.L_3(4).2, 2) = \{4, D_8, 4.2^4, (4 \times 2^2).2^4, S\},$$

where $S \in \mathrm{Syl}_2(G)$. In addition, $N_{M_1}(R) = N(R)$ for each $R \in \Phi(M_1, 2)$, so we may suppose $\Phi(M_1, 2) \subseteq \Phi(G, 2)$.

(2) Apply the local strategy to M_2. We may take

$$\Phi((3^2{:}\,4 \times A_4).2, 2) = \{4, 2^2, D_8, 4 \times 2^2, S'\},$$

where $S' \in \mathrm{Syl}_2(M_2)$. Moreover, $N_{M_2}(R) = N(R)$ for $R \in \{2^2, D_8\}$ and $N_{M_2}(R) \neq N(R)$ for $R \in \Phi(M_2, 2)\backslash\{2^2, D_8\}$.

(3) Let $4^3 = O_2(M_3)$ and apply the local strategy to $M_3 = 4^3.L_3(2)$. We may take

$$\Phi(4^3.L_3(2), 2) = \{4^3, (4^2 \times 2).2^3, (4 \times 2^2).2^4, S\},$$

and in addition, $N_{M_3}(R) = N(R)$ for each $R \in \Phi(M_3, 2)$. We may suppose $\Phi(M_3, 2) \subseteq \Phi(\mathrm{O'N}, 2)$.

This completes the classification of radical 2-subgroups of G. The centralizers and normalizers of $R \in \Phi(G, 2)$ are given by MAGMA. From the orders of the radical subgroups and their centres, we conclude that each radical subgroup is stabilized by some element of $E\backslash G$. □

Lemma 4.4 *Let $G = \mathrm{O'N}$, and let $\mathrm{Blk}^0(G, p)$ be the set of p-blocks with a non-trivial defect group and $\mathrm{Irr}^+(G)$ the characters of $\mathrm{Irr}(G)$ with positive p-defect.*

(a) *If $p = 7$, then $\mathrm{Irr}^0(G, 7) = \{B_0\}$, $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)$ and $\ell(B_0) = 19$.*

(b) *If $p = 3$, then $\mathrm{Blk}(G, 3) = \{B_0, B_1\}$ such that $D(B_1) \simeq 3^2$. In the notation of* [6, p. 133],

$$\mathrm{Irr}(B_1) = \{\chi_2, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{15}\},$$

$\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\backslash\mathrm{Irr}(B_1)$. *Moreover, $\ell(B_1) = 5$ and $\ell(B_0) = 14$.*

(c) *If* $p = 2$, *then* $\mathrm{Blk}(G, 2) = \{B_0, B_1\}$ *such that* $D(B_1) \simeq D_8$. *In the notation of* [6, p. 133], $\mathrm{Irr}(B_1) = \{\chi_2, \chi_3, \chi_4, \chi_7, \chi_{10}\}$ *and* $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash \mathrm{Irr}(B_1)$. *Moreover,* $\ell(B_1) = 3$ *and* $\ell(B_0) = 5$.

Proof: If $B \in \mathrm{Blk}(G, p)$ is non-principal with $D = D(B)$, then D is a radical p-subgroup and $C_G(D)$ is not a p-group. Thus $p \in \{3, 2\}$ and $D \in \{3^2, 2^2, 4, D_8\}$. Since $C_G(4)/4 = L_3(4)$ has a unique character, the Steinberg character θ, it follows that $N(\theta) = N(4) = 4.L_3(4){:}2$, where θ is regarded as a character of $C_G(4)$ and $N(\theta)$ is the stabilizer of θ in $N(4)$. In particular, $N(\theta)/C_G(4) = 2$ and 4 is not a defect group. Similarly, since $C_G(2^2) = 2^2 \times 3^2{:}4 \leq D_8 C_G(D_8) = D_8 \times 3^2{:}4$, it follows that D_8 normalizes any block of $C_G(2^2)$, so that 2^2 is also not a defect group. Thus $D \in_G \{3^2, D_8\}$, and G has exactly one block B_1 with $D(B_1) =_G D$ since $N(D)$ has exactly one orbit on non-trivial characters in $\mathrm{Irr}^0(C(D)D/D)$.

Using the method of central characters, $\mathrm{Irr}(B)$ is as above. If $D(B) = D_8$, then $\ell(B)$ is the number of B-weights (see [15]), so that $\ell(B_1) = 3$ when $p = 2$. Suppose $p = 3$ and $B = B_1$. Then $D(B) = 3^2$ and it contains a unique class of non-trivial elements x in $\mathrm{O'N}$, so that $\ell(B) = \mathrm{k}(B) - \ell(b)$, where $\mathrm{k}(B) = |\mathrm{Irr}(B)| = 6$ and b is the unique block of $C_{\mathrm{O'N}}(x) = 3^2 \times A_6$ inducing B. Thus $\ell(b) = 1$ and $\ell(B_1) = 5$.

If $\ell(G)$ is the number of p-regular G-conjugacy classes, then $\ell(B_0)$ can be calculated by the following equation due to Brauer:

$$\ell(G) = \bigcup_{B \in \mathrm{Blk}^0(G,p)} \ell(B) + |\mathrm{Irr}^0(G)|.$$

This completes the proof. $\square$

Theorem 4.5 *If* B *is a* p*-block of* $G = \mathrm{O'N}$, *then the number of* B*-weights is the number of irreducible Brauer characters of* B.

Proof: If $D(B)$ is cyclic, then the Alperin weight conjecture for B follows by [8]. If $D(B) = D_8$, the Alperin weight conjecture for B follows by [15]. If $D(B) = 3^2$, then $N(3^2)/C_G(3^2) \simeq Q_8$ is a quaternion group and B has exactly five B-weights.

If B is the principal block B_0, then the proof of Theorem 4.5 follows by Lemmas 4.1, 4.2, 4.3, 4.4 and (4.1). $\square$

5 Radical subgroups and weights of Ru

Let G be the simple Rudvalis group Ru. Then

$$|G| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29,$$

and by [8] and [10], we may suppose $p \in \{2, 3, 5\}$.

Lemma 5.1 *The non-trivial radical* 5*-subgroups* R *of* Ru *(up to conjugacy) are given in Table 4.*

R	$C(R)$	N	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
5	$5 \times A_5$	$5{:}\,4 \times A_5$	
5^2	5^2	$5^2{:}\,\mathrm{GL}_2(5)$	4
5_+^{1+2}	5	$5_+^{1+2}{:}\,4.D_8$	14

Table 4: Non-trivial radical 5-subgroups of Ru

Proof: By Magma, G has 3 radical 5-subgroup classes and by the structures of maximal subgroups [18], Theorem 2 (cf. [6], p. 126), each radical 5-subgroup R is the largest normal 5-subgroup of a maximal subgroup and $N(R)$ is maximal. □

Lemma 5.2 *The non-trivial radical* 3*-subgroups* R *of* Ru (*up to conjugacy*) *are given in Table* 5*.*

R	$C(R)$	N	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
3	$3.M_{10}$	$3.A_6.2^2$	
3^2	3^2	$3^2{:}\,\mathrm{GL}_2(3)$	2
3_+^{1+2}	3	$3_+^{1+2}:Q_8{:}\,2$	7

Table 5: Non-trivial radical 3-subgroups of Ru

Proof: As shown in the proof of [18], Section 2.6, G has 3 local subgroups, $N(3A) = 3.A_6.2^2$, $N(3^2) = 3^2{:}\,\mathrm{GL}_2(3)$ and $N(3_+^{1+2}) = 3_+^{1+2}{:}\,Q_8{:}\,2$. Thus $3 = O_3(N(3A))$, $3^2 = O_3(N(3^2))$ and $3_+^{1+2} \in \mathrm{Syl}_3(G)$ are radical 3-subgroups of G. By Magma, G has exactly three classes of radical 3-subgroups. This completes the proof. □

Lemma 5.3 *The non-trivial radical* 2*-subgroups* R *of* Ru (*up to conjugacy*) *are given in Table* 6*, where* $\mathrm{Sz}(8)$ *is the Suzuki group.*

R	$C(R)$	$N(R)$	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
2^2	$2^2 \times \mathrm{Sz}(8)$	$(2^2 \times \mathrm{Sz}(8)){:}\,3$	
2^6	2^6	$2^6{:}\,U_3(3){:}\,2$	1
2.2^{4+6}	2	$2.2^{4+6}.S_5$	0
2^{3+8}	2^3	$2^{3+8}{:}\,L_3(2)$	1
$2.2^{4+6}{:}\,2$	2	$2.2^{4+6}{:}\,2.S_3$	1
$2^{3+8}{:}\,2^2$	2^2	$2^{3+8}{:}\,2^2.S_3$	1
$2.2^{4+6}{:}\,2^2$	2	$2.2^{4+6}{:}\,2^2.S_3$	1
$2.2^{4+6}{:}\,D_8$	2	$2.2^{4+6}{:}\,D_8$	1

Table 6: Non-trivial radical 2-subgroups of Ru

Proof: Let $M_1 \simeq 2^6{:}U_3(3){:}2$, $M_2 \simeq (2^2 \times \mathrm{Sz}(8)){:}3$, $M_3 \simeq 2^{3+8}{:}L_3(2)$ and $M_4 \simeq 2.2^{4+6}{:}S_5$ be maximal subgroups of $G = \mathrm{Ru}$. As shown in the proof of [18], Sections 2.4-2.6, $M_4 = N(2A)$, $M_2 = N(O_2(C(2B)))$, and $M_3 = N(Z(O_2(N(K))))$, where $K \in \mathrm{Syl}_2(M_2)$. Moreover, $2^6 = O_2(M_1)$ is $2A$-pure and there is an involution $u \in M_3 \backslash O_3(M_3)$ such that $2^6 \leq C(u)$ with $|C(u)| = 2^8$. We use this information and random elements constructed using the algorithm of [7] to construct the maximal subgroups.

If R is a non-trivial radical 2-subgroup of $G = \mathrm{Ru}$, then $\Omega_1(Z(R))$ is elementary abelian. As shown in the proof of [18], Sections 2.4-2.6, we may suppose $N(\Omega_1(Z(R))) \leq M_i$ for some i where $1 \leq i \leq 4$. Thus $N_{M_i}(R) = N(R)$ and $R \in \Phi(M_i, 2)$.

(1) Apply the local strategy of [2] to $M_4 = 2.2^{4+6}{:}S_5$. Then we may take

$$\Phi(M_4, 2) = \{2.2^{4+6}, 2.2^{4+6}{:}2, 2.2^{4+6}{:}2^2, 2.2^{4+6}{:}D_8\}, \tag{5.1}$$

where $2.2^{4+6}{:}D_8 \in \mathrm{Syl}_2(M_4)$ is a Sylow 2-subgroup of M_4. In addition, $N_{M_4}(R) = N_G(R)$ for each $R \in \Phi(M_4, 2)$. We may suppose $\Phi(M_4, 2) \subseteq \Phi(G, 2)$.

(2) Similarly, applying the local strategy of [2] to $M_3 = 2^{3+8}{:}L_3(2)$, we may take

$$\Phi(M_3, 2) = \{2^{3+8}, 2^{3+8}{:}2^2, 2.2^{4+6}{:}2^2, 2.2^{4+6}{:}D_8\}, \tag{5.2}$$

and $N_{M_3}(R) = N_G(R)$ for each $R \in \Phi(M_3, 2)$. We may suppose $\Phi(M_3, 2) \subseteq \Phi(G, 2)$.

(3) Applying the local strategy of [2] to $M_2 = (2^2 \times \mathrm{Sz}(8)){:}3$, we may take

$$\Phi(M_2, 2) = \{2^2, 2^2 \times 2^{3+3}\},$$

where $2^2 \times 2^{3+3} \in \mathrm{Syl}_2(M_2)$. In addition, $C(2^2 \times 2^{3+3}) = 2^5$ and $N(2^2 \times 2^{3+3}) \neq N_{M_2}(2^2 \times 2^{3+3}) = (2^2 \times 2^{3+3}{:}7){:}3$.

(4) Again, applying the modified local strategy of [3] to $M_1 = 2^6{:}U_3(3){:}2$, we may take

$$\Phi(M_1, 2) = \{2^6, 2^6.2.2^4, 2^6.2^2.2^3, S'\},$$

where $S' \in \mathrm{Syl}_2(M_1)$. In addition, $N_{M_1}(R) \neq N(R)$ for $R \in \Phi(M_1, 2)\backslash\{2^6\}$, and moreover,

$$N_{M_1}(R) = \begin{cases} 2^6.2.2^4.S_3 & \text{if } R = 2^6.2.2^4, \\ 2^6.2^2.2^3.S_3 & \text{if } R = 2^6.2^2.2^3, \\ S' & \text{if } R = S'. \end{cases} \tag{5.3}$$

This completes the classification of radical 2-subgroups of G. The centralizers and normalizers of $R \in \Phi(G, 2)$ are given by Magma. $\square$

Lemma 5.4 *Let $G = \mathrm{Ru}$, and let $\mathrm{Blk}^0(G, p)$ be the set of p-blocks with a non-trivial defect group and $\mathrm{Irr}^+(G)$ consists of characters of $\mathrm{Irr}(G)$ with positive p-defect.*

(a) *If $p = 5$, then $\mathrm{Irr}^0(G, p) = \{B_0, B_1\}$, where $B_0 = B_0(G)$ is the principal block of G and $D(B_1) \simeq 5$. Lift each character of* Ru *to a character of the covering group $2.Ru$. Then each B_i is also a block of $2.Ru$ and $\mathrm{Irr}^0(2.Ru, p) = \{B_0, B_1, B_2\}$. In the notation of [6, p. 127], $\mathrm{Irr}(B_1) = \{\chi_8, \chi_{23}, \chi_{25}, \chi_{28}, \chi_{32}\}$, $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)\backslash\mathrm{Irr}(B_1)$ and $\mathrm{Irr}(B_2) = \mathrm{Irr}^+(2.Ru)\backslash\mathrm{Irr}(G)$. Moreover, $\ell(B_1) = 4$ and $\ell(B_0) = 18$.*

(b) *If* $p = 3$, *then* $\mathrm{Blk}(G, 2) = \{B_0, B_1, B_2\}$ *such that* $D(B_1) \simeq D(B_2) \simeq 3$. *Lift each character of* Ru *to a character of the covering group* $2.Ru$. *Then each* B_i *is also a block of* $2.Ru$ *and* $\mathrm{Irr}^0(2.Ru, p) = \{B_0, B_1, B_2, B_3, B_4\}$ *with* $D(B_3) \simeq 3^2$ *and* $D(B_4) \simeq D(B_0) \simeq 3_+^{1+2}$ *In the notation of* [6, p. 127],

$$\mathrm{Irr}(B_i) = \begin{cases} \{\chi_6, \chi_8, \chi_{10}\} & \text{if } i = 1, \\ \{\chi_7, \chi_{28}, \chi_{29}\} & \text{if } i = 2, \\ \{\chi_{41}, \chi_{42}, \chi_{43}, \chi_{44}, \chi_{47}, \chi_{48}\} & \text{if } i = 3, \end{cases}$$

and in addition, $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash (\mathrm{Irr}(B_1) \cup \mathrm{Irr}(B_2))$ *and*

$$\mathrm{Irr}(B_4) = \mathrm{Irr}^+(2.Ru) \backslash (\mathrm{Irr}(G) \cup \mathrm{Irr}(B_3)).$$

Moreover, $\ell(B_1) = \ell(B_2) = 2$ *and* $\ell(B_0) = 9$.

(c) *If* $p = 2$, *then* $\mathrm{Blk}(G, 2) = \{B_0, B_1\}$ *such that* $D(B_1) \simeq 2^2$. *Lift each character of* Ru *to a character of the covering group* $2.Ru$. *Then* $\mathrm{Irr}^0(2.Ru, p)$ *contains two blocks* $B_0(2.Ru)$ *and* $B_1(2.Ru)$. *In the notation of* [6, p. 127], $\mathrm{Irr}(B_1) = \{\chi_{32}, \chi_{34}, \chi_{35}, \chi_{36}\}$ *and* $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash \mathrm{Irr}(B_1)$, $\mathrm{Irr}(B_1(2.Ru)) = \mathrm{Irr}(B_1) \cup \{\chi_{49}, \chi_{50}, \chi_{60}\}$ *and*

$$\mathrm{Irr}(B_0(2.Ru)) = \mathrm{Irr}(B_0) \cup (\mathrm{Irr}^+(2.Ru) \backslash (\mathrm{Irr}(G) \cup \mathrm{Irr}(B_1(2.Ru)))).$$

Moreover, $\ell(B_1) = 3$ *and* $\ell(B_0) = 6$.

PROOF: If $B \in \mathrm{Blk}(G, p)$ is non-principal with $D = D(B)$, then $\mathrm{Irr}^0(C(D)D/D)$ has a non-trivial character, so by Lemmas 3.1, 3.2 and 3.4, $D \in_G \{5, 3, 2^2\}$, and moreover, if $Q \in \{5, 2^2\}$, then G has exactly one non-principal block B with $D(B) =_G Q$, since $|\mathrm{Irr}^0(C(Q)Q/Q)| = 1$. If $D = 3$, then $|\mathrm{Irr}^0(C(D)D/D)| = 2$, so G has exactly 2 blocks, B_i for $1 \leq i \leq 2$ with $D(B_i) =_G D$.

We use the method of central characters to prove that $\mathrm{Irr}(B)$ is as claimed. If $D(B)$ is cyclic or isomorphic to 2^2, then $\ell(B)$ is the number of B-weights (see [8] and [15]), so that

$$\ell(B_i) = \begin{cases} 4 & \text{if } p = 5 \text{ and } i = 1, \\ 2 & \text{if } p = 3 \text{ and } i \geq 1, \\ 3 & \text{if } p = 2 \text{ and } i = 1. \end{cases}$$

If $\ell(G)$ is the number of p-regular G-conjugacy classes, then $\ell(B_0)$ can be calculated using the following equation due to Brauer,

$$\ell(G) = \bigcup_{B \in \mathrm{Blk}^0(G,p)} \ell(B) + |\mathrm{Irr}^0(G)|.$$

This completes the proof. $\square$

Theorem 5.5 *Let* $G = \mathrm{Ru}$ *and let* B *be a* p*-block of* G. *Then the number of* B*-weights is the number of irreducible Brauer characters of* B.

PROOF: We may suppose B has a non-cyclic defect group and $D(B) \not\simeq 2^2$. Then the proof of Theorem 5.5 follows by (4.1) and Lemmas 5.1, 5.2, 5.3 and 5.4. $\square$

6 Radical chains of O′N

Let $G = \mathrm{O'N}$, $E = \mathrm{Aut}(G) = \mathrm{O'N}.2$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$.

Lemma 6.1 *In the notation of Lemma 4.1, the radical 7-chains C of G (up to conjugacy) are given in Table 7. In addition, τ permutes each pair $(C(2), C(4))$ and $(C(3), C(5))$ for some $\tau \in E \backslash G$.*

C		$N(C)$	$N_E(C)$
$C(1)$	1	$\mathrm{O'N}$	$\mathrm{O'N}.2$
$C(2)$	$1 < (7^2)_a$	$7^2{:}\,\mathrm{SL}_2(7){:}\,2$	$N(C)$
$C(3)$	$1 < (7^2)_a < 7_+^{1+2}$	$7_+^{1+2}{:}\,(3 \times 2^2)$	$N(C)$
$C(4)$	$1 < (7^2)_b$	$7^2{:}\,\mathrm{SL}_2(7){:}\,2$	$N(C)$
$C(5)$	$1 < (7^2)_b < 7_+^{1+2}$	$7_+^{1+2}{:}\,(3 \times 2^2)$	$N(C)$
$C(6)$	$1 < 7_+^{1+2}$	$7_+^{1+2}{:}\,(3 \times D_8)$	$N(C).2$

Table 7: Radical 7-chains of O′N

Proof: It follows by Lemma 4.1. □

Lemma 6.2 *In the notation of Lemma 4.2, the radical 3-chains C of G (up to conjugacy) are given in Table 8.*

C		$N(C)$	$N_E(C)$
$C(1)$	1	$\mathrm{O'N}$	$\mathrm{O'N}.2$
$C(2)$	$1 < 3^2$	$(3^2{:}\,4 \times A_6).2$	$N(C).2$
$C(3)$	$1 < 3^2 < 3^4$	$3^4.2^2.2^3$	$N(C).2$
$C(4)$	$1 < 3^4$	$3^4{:}\,2_-^{1+4}.D_{10}$	$N(C).2$

Table 8: Radical 3-chains of O′N

Proof: It follows by Lemma 4.2. □

Lemma 6.3 (a) *In the notation of Lemma 4.3 and its proof, the radical 2-chains $C(i)$ for $1 \le i \le 8$ and their normalizers are given in Table 9. Moreover, $N_E(C(i)) = N(C(i)).2$ for each i.*

(b) *Let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that $\mathcal{R}^0(G)/G = \{C(i) : 1 \le i \le 8\}$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B_0, d, u) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B_0, d, u)$$

for all integers $d, u \ge 0$.

C		$N(C)$
$C(1)$	1	O′N
$C(2)$	$1 < 4$	$4.L_3(4){:}\,2$
$C(3)$	$1 < 2^2 < D_8$	$D_8 \times 3^2{:}\,4$
$C(4)$	$1 < 2^2$	$(2^2 \times 3^2{:}\,4).S_3$
$C(5)$	$1 < 2^2 < 4 \times 2^2$	$(4 \times 2^2).S_3$
$C(6)$	$1 < 2^2 < D_8 < 4 \times D_8$	$4 \times D_8$
$C(7)$	$1 < 4^3 < (4 \times 2).2^4$	$(4 \times 2^2).2^4.S_3$
$C(8)$	$1 < 4^3$	$4^3.L_3(2)$

Table 9: Radical 2-chains of O′N

PROOF: (b) Let C' be a radical 2-chain such that

$$C' : 1 < P_1' < \dots P_m' \tag{6.1}$$

and let $C \in \mathcal{R}(G)$ be given by (2.1) with $P_1 \in \Phi(G,2)$.

Case (1). Let $R \in \Phi(4.L_3(4){:}\,2,2)\backslash\{4\}$ and define G-invariant subfamilies $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ of $\mathcal{R}(G)$, such that

$$\begin{aligned} \mathcal{M}^+(R)/G &= \{C' \in \mathcal{R}/G : P_1' = R\}, \\ \mathcal{M}^0(R)/G &= \{C' \in \mathcal{R}/G : P_1' = 4, P_2' = R\}. \end{aligned} \tag{6.2}$$

For $C' \in \mathcal{M}^+(R)$ given by (6.1), the chain

$$g(C') : 1 < 4 < P_1' = R < \dots < P_m' \tag{6.3}$$

is a chain of $\mathcal{M}^0(R)$, $N(C') = N(g(C'))$ and $N_E(C') = N_E(g(C')) = N(C').2$. Thus for any $B \in \mathrm{Blk}(G)$,

$$\mathrm{k}(N(C'), B, d, u) = \mathrm{k}(N(g(C')), B, d, u). \tag{6.4}$$

In addition, g induces a bijection between $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$, so we may suppose

$$C \notin \bigcup_{R \in \Phi(M_1,2)\backslash\{4\}} (\mathcal{M}^+(R) \cup \mathcal{M}^0(R)).$$

By Lemma 4.3, we may suppose $P_1 \in \{2^2, 4, 4^3, (4^2 \times 2).2^3\}$ and if $P_1 = 4$, then $C =_{\mathrm{O'N}} C(2)$.

Case (2). Let $(4^2 \times 2).2^3$ be a subgroup of $\Phi(4^3.L_3(2), 2)$, and let $\mathcal{M}^+((4^2 \times 2).2^3)$ and $\mathcal{M}^0((4^2 \times 2).2^3)$ be the G-invariant subfamilies $\mathcal{R}(G)$ defined by (6.2) with 4 replaced by 4^3 and R by $(4^2 \times 2).2^3$. Then $N_E((4^2 \times 2).2^3) = N((4^2 \times 2).2^3).2 = N_{N_E(4^3)}((4^2 \times 2).2^3)$ and so for $B \in \mathrm{Blk}(G)$ and integers $d, u \geq 0$, (6.4) holds for $C' \in \mathcal{M}^+((4^2 \times 2).2^3)$, where $g(C')$ is defined by (6.3) with 4 replaced by 4^3 and R by $(4^2 \times 2).2^3$. We may suppose

$$C \notin (\mathcal{M}^+((4^2 \times 2).2^3) \cup \mathcal{M}^0((4^2 \times 2).2^3)).$$

Let $C' : 1 < 4^3 < S$ and $g(C') : 1 < 4^3 < (4\times 2^2).2^4 < S$. Then $N(C') = N(g(C')) = S$ and $N_E(C') = N_E(g(C')) = S.2$, so that (6.4) still holds. It follows that we may suppose that $P_1 \neq_{\mathrm{O'N}} (4^2 \times 2).2^3$ and if $P_1 = 4^3$, then $C \in_{\mathrm{O'N}} \{C(7), C(8)\}$.

Case (3). Suppose $P_1 = 2^2$, so that $N(2^2) = (2^2 \times 3^2{:}\,4).S_3$. By MAGMA, we may take

$$\Phi((2^2 \times 3^2{:}\,4).S_3, 2) = \{2^2, D_8, 4\times 2^2, 4\times D_8\}$$

and $N_{N(2^2)}(D_8) = D_8 \times 3^2{:}\,4 = C_G(D_8)D_8$, $N_{N(2^2)}(4\times 2^2) = (4\times 2^2).S_3$, $N_{N(2^2)}(D_8\times 4) = D_8 \times 4$ and $N_{N_E(2^2)}(R) = N_{N(2^2)}(R).2$ for each $R \in \Phi((2^2 \times 3^2{:}\,4).S_3, 2)$.

Let $C' : 1 < 2^2 < 4\times D_8$ and $g(C') : 1 < 2^2 < 4\times 2^2 < 4\times D_8$. Then $N(C') = N(g(C')) = 4\times D_8$ and $N_E(C') = N_E(g(C')) = (4\times D_8).2$, so that (6.4) holds. It follows that if $P_1 = 2^2$, then $C \in_{\mathrm{O'N}} \{C(3), C(4), C(5), C(6)\}$.

The proof of (a) follows easily by that of (b) or Lemma 4.3. $\square$

Remark 6.4 *Let G^* be a covering group of $G = \mathrm{O'N}$, ρ a faithful linear character of $Z(G^*)$ and B^* a block of G^* covering the block $B(\rho)$ containing ρ. If $D(B^*) \neq O_p(Z(G^*))$ and $p = 2$, then*

$$\sum_{C\in\mathcal{R}(G)/G} (-1)^{|C|}\mathrm{k}(N_{G^*}(C), B^*, d, \rho) = \sum_{C\in\mathcal{R}^0(G)/G} (-1)^{|C|}\mathrm{k}(N_{G^*}(C), B^*, d, \rho)$$

for all integers $d \geq 0$.

The proof of the Remark is the same as that of Lemma 9, since $N(C') = N(g(C'))$ implies $N_{G^*}(C') = N_{G^*}(g(C'))$.

7 Radical chains of Ru

We use the notation and terminology of Sections 2 and 5. Let $G = \mathrm{Ru}$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$.

Lemma 7.1 *In the notation of Lemma 5.1, the radical 5-chains C of G (up to conjugacy) are given in Table 10.*

C		$N(C)$	C		$N(C)$
$C(1)$	1	Ru	$C(2)$	$1 < 5$	$5{:}\,4 \times A_5$
$C(3)$	$1 < 5 < 5^2$	$5{:}\,4\times 5{:}\,2$	$C(4)$	$1 < 5^2$	$5^2{:}\,\mathrm{GL}_2(5)$
$C(5)$	$1 < 5^2 < 5^{1+2}_+$	$5^{1+2}_+{:}\,4^2$	$C(6)$	$1 < 5^{1+2}_+$	$5^{1+2}_+{:}\,4.D_8$

Table 10: Radical 5-chains of Ru

PROOF: It follows by Lemma 5.1. $\square$

Lemma 7.2 (a) *In the notation of Lemma* 5.2*, the radical* 3*-chains* $C(i)$ *for* $1 \leq i \leq 4$ *and their normalizers are given in Table* 11.

C		$N(C)$	C		$N(C)$
$C(1)$	1	Ru	$C(2)$	$1 < 3$	$3.A_6.2^2$
$C(3)$	$1 < 3^2 < 3_+^{1+2}$	$3_+^{1+2}{:}\, 2^2$	$C(4)$	$1 < 3^2$	$3^2{:}\, \mathrm{GL}_2(3)$

Table 11: Radical 3-chains of Ru

(b) *Let* $\mathcal{R}^0(G)$ *be the* G*-invariant subfamily of* $\mathcal{R}(G)$ *such that* $\mathcal{R}^0(G)/G = \{C(i) : 1 \leq i \leq 4\}$*. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d)$$

for all integers $d \geq 0$ *and* $B \in \mathcal{R}(G)$*.*

Proof: If C is a chain given by (2.1), then we may suppose $P_1 \in \Phi(G, 3)$. Suppose $P_1 = 3$. By Magma, $\Phi(3.A_6.2^2, 3) = \{3, 3_+^{1+2}\}$ and $N(R) = N_{3.A_6.2^2}(R)$ for each $R \in \Phi(3.A_6.2^2, 3)$. Let $C' : 1 < 3 < 3_+^{1+2}$ and $g(C') : 1 < 3_+^{1+2}$. Since $N_{3.A_6.2^2}(3_+^{1+2}) = N_{\mathrm{Ru}}(3_+^{1+2})$, it follows that $N(C') = N(g(C'))$, so that

$$\mathrm{k}(N_{\mathrm{Ru}}(C'), B, d) = \mathrm{k}(N_{\mathrm{Ru}}(g(C')), B, d)$$

for any block B of Ru with $D(B) \neq 1$ and any integer d. Since $|C'| = |g(C')| + 1$, it follows that we may delete the two chains C' and $g(C')$ in the right hand side of (2.6). We may suppose $P_1 \neq_G 3_+^{1+2}$ and if $P_1 = 3$, then $C =_G C(2)$. If $P_1 = 3^2$, then either $C =_G C(4)$ or $|C| \geq 2$, so that P_2 is a radical subgroup of $N(3^2)$ and $P_2 > 3^2$. So $P_2 = 3_+^{1+2}$ is a Sylow subgroup of $N(3^2)$.

This proves (b) and the proof of (a) follows easily by Lemma 5.2. □

Lemma 7.3 (a) *In the notation of Lemma* 5.3 *and its proof, the radical* 2*-chains* $C(i)$ *for* $1 \leq i \leq 10$ *and their normalizers are given in Table* 12.

C		$N(C)$
$C(1)$	1	Ru
$C(2)$	$1 < 2^6$	$2^6{:}\, U_3(3){:}\, 2$
$C(3)$	$1 < 2^6 < 2^6{:}\, 2.2^4$	$2^6{:}\, 2.2^4.S_3$
$C(4)$	$1 < 2^6 < 2^6{:}\, 2.2^4 < S'$	S'
$C(5)$	$1 < 2^6 < 2^6{:}\, 2^2.2^3$	$2^6{:}\, 2^2.2^3.S_3$
$C(6)$	$1 < 2^2$	$(2^2 \times \mathrm{Sz}(8)){:}\, 3$
$C(7)$	$1 < 2^2 < 2^2 \times 2^{3+3}$	$(2^2 \times 2^{3+3}{:}\, 7){:}\, 3$
$C(8)$	$1 < 2^{3+8}$	$2^{3+8}{:}\, L_3(3)$
$C(9)$	$1 < 2.2^{4+6} < 2.2^{4+6}{:}\, 2^2$	$2.2^{4+6}{:}\, 2^2.S_3$
$C(10)$	$1 < 2.2^{4+6}$	$2.2^{4+6}{:}\, S_5$

Table 12: Radical 2-chains of Ru

(b) *Let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that $\mathcal{R}^0(G)/G = \{C(i) : 1 \leq i \leq 10\}$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d)$$

for all integers $d \geq 0$ and $B \in \mathrm{Blk}^0(G)$.

Proof: (b) Let $C \in \mathcal{R}(G)$ be given by (2.1) with $P_1 \in \Phi(G, 2)$.

Suppose $P_1 = O_2(M_1) = 2^6$. Then we may suppose $P_2 \in \Phi(M_1, 2)$ when $|C| \geq 2$. Let $C' : 1 < 2^6 < S'$ and $g(C') : 1 < 2^6 < 2^6{:}\, 2^2.2^3 < S'$, where $S' \in \mathrm{Syl}_2(M_1)$. Then $N(C') = N(g(C')) = N(S')$ and we may suppose $C \neq_G C'$ and $C \neq_G g(C')$. Thus $C \in_G \{C(2), C(3), C(4), C(5)\}$.

If $P_1 = O_2(M_2) = 2^2$, then $C \in_G \{C(6), C(7)\}$.

Suppose $P_1 = O_2(M_3) = 2^{3+8}$. Then we may suppose $\Phi(M_3, 2) \subseteq \Phi(G, 2)$. Let $R \in \Phi(M_3, 2) \backslash \{2^{3+8}\}$, so that $N(R) = N_{M_3}(R)$. Let $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ be subfamilies of $\mathcal{R}(G)$ given by (6.2) with 4 replaced by 2^{3+8}. Then for $C' \in \mathcal{M}^+(R)$ given by (6.1), the chain

$$g(C') : 1 < 2^{3+8} < P'_1 = R < P'_2 < \ldots < P'_m \tag{7.1}$$

is a chain in $\mathcal{M}^0(R)$ and $N(C') = N(g(C'))$. For any $B \in \mathrm{Blk}(G)$ and for any integer $d \geq 0$,

$$\mathrm{k}(N(C'), B, d) = \mathrm{k}(N(g(C')), B, d). \tag{7.2}$$

In addition, g is a bijection between $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$. So we may suppose

$$C \notin \bigcup_{R \in \Phi(M_3,2) \backslash \{2^{3+8}\}} (\mathcal{M}^+(R) \cup \mathcal{M}^0(R)).$$

Thus $P_1 \notin \{2^{3+8}.2^2, 2.2^{4+6}{:}\, 2^2, 2.2^{4+8}{:}\, D_8\}$ and if $P_1 = 2^{3+8}$, then $C =_G C(8)$.

¿From the proof above, we may suppose $P_1 \in \{2.2^{4+6}, 2.2^{4+6}{:}\, 2\}$. Let $\mathcal{M}^+(2.2^{4+6}{:}\, 2)$ and $\mathcal{M}^0(2.2^{4+6}{:}\, 2)$ be defined as (6.2) with 4 replaced by 2.2^{4+6} and R by $2.2^{4+6}{:}\, 2$. If $C' \in \mathcal{M}^+(2.2^{4+6}{:}\, 2)$ and $g(C')$ is defined by (7.1) with 2^{3+8} replaced by 2.2^{4+6}, then g is a bijection between $\mathcal{M}^+(2.2^{4+6}{:}\, 2)$ and $\mathcal{M}^0(2.2^{4+6}{:}\, 2)$, and $N(C') = N(g(C'))$. Thus (7.2) holds and we may suppose

$$C \notin (\mathcal{M}^+(2.2^{4+6}{:}\, 2) \cup \mathcal{M}^0(2.2^{4+6}{:}\, 2)).$$

In particular, we may suppose $P_1 \neq_G 2.2^{4+6}{:}\, 2$ and if $P_1 = 2.2^{4+6}$ and $|C| \geq 2$, then $P_2 \neq_G 2.2^{4+6}{:}\, 2$. Let $C' : 1 < 2.2^{4+6} < S$ and $g(C') : 1 < 2.2^{4+6} < 2.2^{4+6}{:}\, 2^2 < S$. Then $N(C') = N(g(C')) = N(S)$ and (7.2) holds. Thus $C \in_G \{C(9), C(10)\}$.

This completes the classification of the radical 2-chains; the normalizers of the chains are also given by the proof above or that of Lemma 5.3. □

Remark 7.4 *Let G^* be a covering group of $G = \mathrm{Ru}$, ρ a faithful linear character of $Z(G^*)$ and B^* a block of G^* covering the block $B(\rho)$ containing ρ. If $D(B^*) \neq O_p(Z(G^*))$ and $p = 2$ or 3, then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N_{G^*}(C), B^*, d, \rho) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N_{G^*}(C), B^*, d, \rho)$$

for all integers $d \geq 0$.

The proof of the Remark is the same as that of Lemma 12, since $N(C') = N(g(C'))$ implies $N_{G^*}(C') = N_{G^*}(g(C'))$.

8 Dade's invariant conjecture for O′N

Let $N(C)$ be the normalizer of a radical p-chain C. If $N(C)$ is a maximal subgroup of O′N, then the character table of $N(C)$ can be found in the library of character tables distributed with GAP [12]. If this is not the case, we construct a "useful" description of $N(C)$ and attempt to compute directly its character table using MAGMA.

If $N(C)$ is soluble, we construct a power-conjugate presentation for $N(C)$ and use this presentation to obtain the character table.

If $N(C)$ is insoluble, we construct faithful representations for $N(C)$ and use them as input to the character table construction function. We employ two strategies to obtain faithful representations of $N(C)$.

1. Construct the actions of $N(C)$ on the cosets of soluble subgroups of $N(C)$.
2. Construct the orbits of $N(C)$ on the underlying set of O′N; for the stabilizer of an orbit representative, construct the action of $N(C)$ on its cosets.

The tables listing degrees of irreducible characters referenced in the proof of Theorem 8.1 are in Appendix A.

Theorem 8.1 *Let B be a p-block of $G = \mathrm{O'N}$ with positive defect. Then B satisfies the invariant conjecture of Dade.*

PROOF: We may suppose B has a non-cyclic defect group and let $E = \mathrm{Aut}(G) = \mathrm{O'N}.2$.

(1). Suppose $p = 7$, so that by Lemma 4.4 (a), we may suppose $B = B_0$. Let $C = C(2)$, $C' = C(3)$. By Lemma 6.1, $N(C) \simeq 7^2{:}\,\mathrm{SL}_2(7){:}\,2 = N_E(C)$, $N(C') \simeq 7_+^{1+2}{:}\,(3 \times 2^2) = N_E(C')$, whose degrees are given in Tables A–18 and A–19. Thus

$$\mathrm{k}(N(C(2)), B_0, d, u) = \mathrm{k}(N(C(3)), B_0, d, u) = \begin{cases} 19 & \text{if } d = 3 \text{ and } u = 1, \\ 2 & \text{if } d = 2 \text{ and } u = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

Since $N_E(C(4)) = N(C(4)) \simeq N(C(2))$ and $N_E(C(5)) = N(C(5)) \simeq N(C(3))$, it follows that

$$\mathrm{k}(N(C(4)), B_0, d, u) = \mathrm{k}(N(C(5)), B_0, d, u),$$

which is also give by (8.1).

Similarly, $N(C(6)) \simeq 7_+^{1+2}.(3 \times D_8)$ and $N_E(C(6)) = N(C(6)).2$, whose degrees are given in Tables A–20 and A–21, respectively. By Lemma 4.4 (a) and [6, p. 133], we have

$$\mathrm{k}(G, B_0, d, u) = \mathrm{k}(N(C(4))), B_0, d, u) = \begin{cases} 10 & \text{if } d = 3 \text{ and } u = 2, \\ 10 & \text{if } d = 3 \text{ and } u = 1, \\ 4 & \text{if } d = 2 \text{ and } u = 2, \\ 0 & \text{otherwise.} \end{cases}$$

This implies the theorem when $p = 7$.

(2). Suppose $p = 3$, so that by Lemma 4.4 (b), $B = B_0$ or B_1. By Lemma 6.2, $N(C(2)) \simeq (3^2{:}\,4 \times A_6).2$ and $N_E(C(2)) = N(C(2)).2$. By MAGMA, $N(C(2))$ and $N_E(C(2))$ have 30 and 45 irreducible characters, respectively, whose degrees are given in Tables A–12 and A–13. In addition, $N(C(2))$ has two blocks and the principal block contains exactly 24 irreducible characters. By Lemma 4.4 (b) and [6, p. 133], we have

$$\mathrm{k}(G, B_1, d, u) = \mathrm{k}(N(C(2))), B_1, d, u) = \begin{cases} 4 & \text{if } d = 2 \text{ and } u = 2, \\ 2 & \text{if } d = 2 \text{ and } u = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The group $N(C(3)) \simeq 3^4{:}\,2^2.2^3$ and $N_E(C(3)) = N(C(3)).2$ have 24 and 36 irreducible characters, respectively, whose degrees are given in Tables A–14 and A–15. It follows that

$$\mathrm{k}(N(C(2)), B_0, d, u) = \mathrm{k}(N(C(3))), B_1, d, u) = \begin{cases} 16 & \text{if } d = 4 \text{ and } u = 2, \\ 8 & \text{if } d = 4 \text{ and } u = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, $N(C(4)) \simeq 3^4{:}\,2_-^{1+4}.D_{10}$ and $N_E(C(4)) = N(C(4)).2$ have 18 and 24 irreducible characters, respectively, whose degrees are given in Tables A–16 and A–17. Thus

$$\mathrm{k}(G, B_0, d, u) = \mathrm{k}(N(C(4))), B_1, d, u) = \begin{cases} 10 & \text{if } d = 4 \text{ and } u = 2, \\ 8 & \text{if } d = 4 \text{ and } u = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This implies the theorem when $p = 3$.

(3). Suppose $p = 2$, so that by Lemma 4.4 (c) and [15], we may suppose $B = B_0$.

The groups $N(C(3)) \simeq D_8 \times 3^2{:}\,4$ and $N_E(C(3)) = N(C(3)).2$ have 30 and 45 irreducible characters, respectively, whose degrees are given in Tables A–3 and A–4. In addition, $N(C(3))$ has two blocks and the principal block contains exactly 20 irreducible characters. Similarly, $N(C(4)) \simeq (2^2 \times 3^2{:}\,4).S_3$ and $N_E(C(4)) = N(C(4)).2$ have 30 and 45 irreducible characters, respectively, whose degrees are given in Tables A–5 and A–6. Moreover, the principal block of $N(C(4))$ also contains exactly 20 irreducible characters. It follows that

$$\mathrm{k}(N(C(3)), B_0, d, u) = \mathrm{k}(N(C(4))), B_0, d, u) = \begin{cases} 16 & \text{if } d = 5 \text{ and } u = 2, \\ 4 & \text{if } d = 4 \text{ and } u = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{8.2}$$

Similarly, $N(C(5)) \simeq (4 \times 2^2).S_3$ and $N(C(6)) \simeq 4 \times D_8$ both have 20 irreducible characters, and the degrees of $\mathrm{Irr}(N(C(5))$ are given in Table A–7. In addition, each element of $N_E(C(i)) = N(C(i)).2$ fixes each character of $N(C(i))$ for $i = 5, 6$. Thus

$$\mathrm{k}(N(C(5)), B_0, d, u) = \mathrm{k}(N(C(6)), B_0, d, u), \tag{8.3}$$

which is also given by (8.2).

Set $\mathrm{k}(i, d, u) = \mathrm{k}(N(C(i)), B_0, d, u)$. The group $N(C(7)) \simeq (4 \times 2^2).2^4.S_3$ and $N_E(C(7)) = N(C(7)).2$ have 24 and 33 irreducible characters, respectively, whose degrees are given in Tables A–8 and A–9. By Lemma 4.4 (c) and [6, p. 133], the values $\mathrm{k}(1, d, u)$ and $\mathrm{k}(7, d, u)$ are as in Table 13.

Defect d	9	9	8	8	7	7	6	4	otherwise
Number u	2	1	2	1	2	1	2	2	0
k(1, d, u)	4	4	4	2	1	2	1	2	0

Defect d	9	9	8	8	7	7	6	6	otherwise
Number u	2	1	2	1	2	1	2	1	0
k(7, d, u)	4	4	4	2	3	2	3	2	0

Table 13: Values of $\mathrm{k}(1, d, u)$ and $\mathrm{k}(7, d, u)$

If $\mathrm{k}(\mathrm{odd}, d, u) = \sum_{i \in \{1,7\}} \mathrm{k}(N(C(i)), B_0, d, u)$, the values are recorded in Table 14.

Defect d	9	9	8	8	7	7	6	6	4	otherwise
Number u	2	1	2	1	2	1	2	1	2	0
$\mathrm{k}(\mathrm{odd}, d, u)$	8	8	8	4	4	4	4	2	2	0

Table 14: Values of $\mathrm{k}(\mathrm{odd}, d, u)$

The groups $N(C(2)) \simeq 4.L_3(4){:}2$ and $N_E(C(2)) = N(C(2)).2$ have 31 and 44 irreducible characters, respectively, whose degrees are given in Tables A–1 and A–2. In addition, $N(C(2))$ has two blocks and the principal block contains exactly 26 irreducible characters. Similarly, the groups $N(C(8)) \simeq 4^3.L_3(2)$ and $N_E(C(8)) = N(C(8)).2$ have 18 and 24 irreducible characters, respectively, whose degrees are given in Tables A–10 and A–11. The values $\mathrm{k}(2, d, u)$ and $\mathrm{k}(8, d, u)$ are as in Table 15.

Defect d	9	9	8	8	7	7	6	6	4	otherwise
Number u	2	1	2	1	2	1	2	1	2	0
k(2, d, u)	4	4	4	2	3	2	3	2	2	0

Defect d	9	9	8	8	7	7	6	otherwise
Number u	2	1	2	1	2	1	2	0
k(8, d, u)	4	4	4	2	1	2	1	0

Table 15: Values of $\mathrm{k}(2, d, u)$ and $\mathrm{k}(8, d, u)$

If $\mathrm{k}(\mathrm{even}, d, u) = \sum_{i \in \{2,4\}} \mathrm{k}(N(C(i)), B_0, d, u)$, the values are recorded in Table 16.

Defect d	9	9	8	8	7	7	6	6	4	otherwise
Number u	2	1	2	1	2	1	2	1	2	0
$\mathrm{k}(\mathrm{even}, d, u)$	8	8	8	4	4	4	4	2	2	0

Table 16: Values of $\mathrm{k}(\mathrm{even}, d, u)$

This implies the theorem. $\square$

9 Dade's ordinary conjecture for Ru

If C is a radical p-chain of Ru, then the character table of $N_{\mathrm{Ru}}(C)$ can either be found in the library of character tables distributed with GAP or computed directly as in Section 8. The tables listing degrees of characters in $\mathrm{Irr}(N_{\mathrm{Ru}}(C))$ referenced in the proofs of Theorem 9.1 are in Appendix B.

Theorem 9.1 *Let B be a p-block of the simple Rudvalis group $G = \mathrm{Ru}$ with a positive defect. Then B satisfies the ordinary conjecture of Dade.*

PROOF: We may suppose B has a non-cyclic defect group, so that $p \in \{2, 3, 5\}$.

(1) Suppose $p = 5$, so that by Lemma 5.4 (a), we may suppose $B = B_0$. If C is a radical chain of G, then denote by $b_i(C)$ the block of $N_{\mathrm{Ru}}(C)$ inducing B_i for $i = 0, 1$.

By Lemma 7.1, $N_{\mathrm{Ru}}(C(2)) \simeq 5{:}\,4 \times A_5$ and $N_{\mathrm{Ru}}(C(3)) \simeq 5{:}\,4 \times 5{:}\,2$, have 25 and 20 irreducible characters, respectively, whose degrees are given in Tables B–13 and B–14. In addition, the principal block of $N_{\mathrm{Ru}}(C(2))$ contains exactly 20 characters of defect 2. It follows that

$$\mathrm{k}(N_{\mathrm{Ru}}(C(2)), B_0, d) = \mathrm{k}(N_{\mathrm{Ru}}(C(3)), B_0, d) = \begin{cases} 20 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{9.1}$$

The groups $N_{\mathrm{Ru}}(C(4)) \simeq 5{:}\,\mathrm{GL}_2(5)$ and $N_{\mathrm{Ru}}(C(5)) \simeq 5_+^{1+2}{:}\,4^2$ both have 29 irreducible characters, whose degrees are given in Tables B–15 and B–16, respectively. It follows that

$$\mathrm{k}(N_{\mathrm{Ru}}(C(4)), B_0, d) = \mathrm{k}(N_{\mathrm{Ru}}(C(5)), B_0, d) = \begin{cases} 25 & \text{if } d = 3, \\ 4 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{9.2}$$

The group $N_{\mathrm{Ru}}(C(6)) \simeq 5_+^{1+2}{:}\,4.D_8$ has 25 irreducible characters, whose degrees are given by Table B–17. It follows by Lemma 5.4 (a) that

$$\mathrm{k}(G, B_0, d) = \mathrm{k}(N_{\mathrm{Ru}}(C(6)), B_0, d) = \begin{cases} 20 & \text{if } d = 3, \\ 5 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{9.3}$$

Thus Theorem 9.1 follows when $p = 5$.

(2) Suppose $p = 3$, so that by Lemma 5.4 (b), $B = B_0$. The groups $N_{\mathrm{Ru}}(C(3)) \simeq 3_+^{1+2}{:}\,2^2$ and $N_{\mathrm{Ru}}(C(4)) \simeq 3^2{:}\,\mathrm{GL}_2(3)$ both have 11 irreducible characters, whose degrees are given in Tables B–11 and B–12, respectively. It follows that

$$\mathrm{k}(N_{\mathrm{Ru}}(C(3)), B_0, d) = \mathrm{k}(N_{\mathrm{Ru}}(C(4)), B_0, d) = \begin{cases} 9 & \text{if } d = 3, \\ 2 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The group $N_{\mathrm{Ru}}(C(2)) \simeq 3.A_6.2^2$ has 20 irreducible characters, whose degrees are given in Table B–10. In addition, the principal block of $N_{\mathrm{Ru}}(C(2))$ contains exactly 14 irreducible characters of defect 3 and 2. It follows by Lemma 5.4 (b) that

$$\mathrm{k}(G, B_0, d) = \mathrm{k}(N_{\mathrm{Ru}}(C(2)), B_0, d) = \begin{cases} 9 & \text{if } d = 3, \\ 5 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus Theorem 9.1 follows when $p = 3$.

(3) Suppose $p = 2$, so that by Lemma 5.4 (c), we may suppose $B = B_0$ or B_1. Since $D(B_1) \simeq 2^2$, it follows by [15] that Dade's ordinary conjecture holds for B_1. We may suppose $B = B_0$.

First, we consider the chains C such that the defect $\mathrm{d}(N_{\mathrm{Ru}}(C)) = 8$, so that $C \in \{C(6), C(7)\}$.

The subgroups $N_{\mathrm{Ru}}(C(6)) \simeq (2^2 \times \mathrm{Sz}(8)){:}\,3$ and $N_{\mathrm{Ru}}(C(7)) \simeq (2^2 \times 3^{3+3}{:}\,7){:}\,3$ have 28 and 24 irreducible characters, respectively, whose degrees are given in Tables B–5 and B–6. It follows that

$$\mathrm{k}(N_{\mathrm{Ru}}(C(6)), B_0, d) = \mathrm{k}(N_{\mathrm{Ru}}(C(7)), B_0, d) = \begin{cases} 16 & \text{if } d = 8, \\ 8 & \text{if } d = 7, \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the chains C such that $\mathrm{d}(N_{\mathrm{Ru}}(C)) = 12$, so that

$$C \in \{C(2), C(3), C(4), C(5)\}.$$

Set $\mathrm{k}(i, d) = \mathrm{k}(N(C(i)), B_0, d)$. The groups $N_{\mathrm{Ru}}(C(3)) \simeq 2^6{:}\,2.2^4.S_3$ and $N_{\mathrm{Ru}}(C(5)) \simeq 2^6{:}\,2^2.2^3.S_3$ have 47 and 56 irreducible characters, respectively whose degrees are given in Tables B–2 and B–4. The values $\mathrm{k}(3, d)$ and $\mathrm{k}(5, d)$ are as in Table 17.

Defect d	12	11	10	9	8	otherwise
k(3, d)	16	12	12	3	4	0

Defect d	12	11	9	7	6	otherwise
k(5, d)	16	28	9	2	1	0

Table 17: Values of $\mathrm{k}(3, d)$ and $\mathrm{k}(5, d)$

If $\mathrm{k}(\mathrm{odd}_1, d) = \sum_{i \in \{3,5\}} \mathrm{k}(N(C(i)), B_0, d)$, the values are recorded in Table 18.

Defect d	12	11	10	9	8	7	6	otherwise
$\mathrm{k}(\mathrm{odd}_1, d)$	32	40	12	12	4	2	1	0

Table 18: Values of $\mathrm{k}(\mathrm{odd}_1, d)$

The groups $N_{\mathrm{Ru}}(C(2)) \simeq 2^6{:}\,U_3(3){:}\,2$ and $N_{\mathrm{Ru}}(C(4)) \simeq S'$ have 30 and 73 irreducible characters, respectively whose degrees are given in Tables B–1 and B–3. The values $\mathrm{k}(2, d)$ and $\mathrm{k}(4, d)$ are as in Table 19.

Defect d	12	11	9	6	otherwise
k(2, d)	16	12	1	1	0

Defect d	12	11	10	9	8	7	otherwise
k(4, d)	16	28	12	11	4	2	0

Table 19: Values of $\mathrm{k}(2, d)$ and $\mathrm{k}(4, d)$

If $\mathrm{k}(\mathrm{even}_1, d) = \sum_{i \in \{2,4\}} \mathrm{k}(N(C(i)), B_0, d)$, the values are recorded in Table 20.

Defect d	12	11	10	9	8	7	6	otherwise
$\mathrm{k}(\mathrm{even}_1, d)$	32	40	12	12	4	2	1	0

Table 20: Values of $\mathrm{k}(\mathrm{even}_1, d)$

It follows that

$$\sum_{i=2}^{5}(-1)^{|C(i)|}\mathrm{k}(N_{\mathrm{Ru}}(C(i)), B_0, d) = 0.$$

Finally, we consider the chains $C \in \{C(1), C(8), C(9), C(10)\}$. Then $N_{\mathrm{Ru}}(C(9)) \simeq 2.2^{4+6}{:}\, 2^2.S_3$ has 52 irreducible characters, whose degrees are given in Table B–8. From Lemma 5.4 (c), the values $\mathrm{k}(1, d)$ and $\mathrm{k}(9, d)$ are as in Table 21.

Defect d	14	13	12	11	7	otherwise
k(1, d)	8	6	4	12	2	0

Defect d	14	13	12	11	10	9	8	7	otherwise
k(9, d)	8	10	16	10	4	1	2	1	0

Table 21: Values of $\mathrm{k}(1, d)$ and $\mathrm{k}(9, d)$

If $\mathrm{k}(\mathrm{odd}_2, d) = \sum_{i \in \{1,9\}} \mathrm{k}(N(C(i)), B_0, d)$, the values are recorded in Table 22.

Defect d	14	13	12	11	10	9	8	7	otherwise
$\mathrm{k}(\mathrm{odd}_2, d)$	16	16	20	22	4	1	2	3	0

Table 22: Values of $\mathrm{k}(\mathrm{odd}_2, d)$

The groups $N_{\mathrm{Ru}}(C(8)) \simeq 2^{3+8}{:}\, L_3(3)$ and $N_{\mathrm{Ru}}(C(10)) \simeq 2.2^{4+6}{:}\, S_5$ have 35 and 49 irreducible characters, respectively whose degrees are given in Tables B–7 and B–9. The values $\mathrm{k}(8, d)$ and $\mathrm{k}(10, d)$ are as in Table 23.

Defect d	14	13	12	11	10	9	otherwise
k(8, d)	8	6	4	12	4	1	0

Defect d	14	13	12	11	8	7	otherwise
k(10, d)	8	10	16	10	2	3	0

Table 23: Values of $\mathrm{k}(8, d)$ and $\mathrm{k}(10, d)$

If $\mathrm{k}(\mathrm{even}_2, d) = \sum_{i \in \{8,10\}} \mathrm{k}(N(C(i)), B_0, d)$, the values are recorded in Table 24.

Defect d	14	13	12	11	10	9	8	7	otherwise
$\mathrm{k}(\mathrm{even}_2, d)$	16	16	20	22	4	1	2	3	0

Table 24: Values of $\mathrm{k}(\mathrm{even}_2, d)$

It follows that

$$\sum_{i\{1,8,9,10\}} (-1)^{|C(i)|}\mathrm{k}(N_{\mathrm{Ru}}(C(i)), B_0, d) = 0$$

and so Dade's ordinary conjecture holds. □

10 Dade's projective conjecture for $3.\mathrm{O'N}$

Let C be a radical p-chain of $\mathrm{O'N}$ and $N_{3.\mathrm{O'N}}(C) = 3.N_{\mathrm{O'N}}(C)$. Then the character table of $N_{3.\mathrm{O'N}}(C)$ can either be found in the library of character tables distributed with GAP or computed directly using MAGMA as in Section 8.

Let $H = N_{3.\mathrm{O'N}}(C)$ and let ζ_1 and ζ_2 be the faithful linear characters of $Z(3.\mathrm{O'N})$. Denote by $\mathrm{Irr}(H \mid \zeta_i)$ the subset of $\mathrm{Irr}(H)$ consisting of characters covering ζ_i. The tables listing degrees of characters in $\mathrm{Irr}(H \mid \zeta_i)$ referenced in the proofs of Theorem 10.1 are in Appendix C.

Theorem 10.1 *Let B be a p-block of $G = 3.\mathrm{O'N}$ with $D(B) \neq O_p(G)$. Then B satisfies the projective conjecture of Dade.*

PROOF: We may suppose B has a non-cyclic defect group and let $N(C) = N_G(C)$ for each $C \in \mathcal{R}(\mathrm{O'N})$.

(1). Suppose $p = 7$, $S = 7_+^{1+2} \in \mathrm{Syl}_7(G)$ and $C = C(i)$ is a radical chain of $\mathrm{O'N}$. Since $N_G(S) = 3 \times S{:}(3 \times D_8)$, it follows that G has exactly three full defect blocks B_0, B_1, B_2. By MAGMA,

$$\mathrm{k}(N(C(i)), B_\ell, d, \zeta_\ell) = \mathrm{k}(N_{\mathrm{O'N}}(C(i)), B_0(\mathrm{O'N}), d), \qquad (10.1)$$

for integers $\ell \in \{1, 2\}$, $d \geq 0$ and $i \geq 2$. In addition, by [6, p. 133], (10.1) still holds when $i = 1$ and $\mathrm{k}(G, \zeta_\ell) \cap \mathrm{Irr}(B_0) = \emptyset$, so we may suppose $B \in \{B_1, B_2\}$. Since

$\mathrm{k}(N_{\mathrm{O'N}}(C(i)), B_0(\mathrm{O'N}), d) = \sum_{u \geq 0} \mathrm{k}(N_{\mathrm{O'N}}(C(i)), B_0(\mathrm{O'N}), d, u)$ for each i, Theorem 10.1 follows by the proof (1) of Theorem 8.1.

(2). Suppose $p = 3$. Then G has eight blocks, $\mathrm{Blk}(G, 3) = \{B_i : 0 \leq i \leq 7\}$ such that $D(B_i) \simeq 3$ for $i = 2, \ldots 8$ and $D(B_1) \simeq 3^2$. In the notation of [6, pp. 156-159],

$$\begin{cases} \mathrm{Irr}(B_1) = \mathrm{Irr}(B_1(\mathrm{O'N})) \cup \{\chi_j : j \in \{39, 40, 41, 42, 43, 44, 53, 54, 55, 56\}\} \\ \cup_{i=2}^{7} \mathrm{Irr}(B_i) = \{\chi_j : j \in \{21, 22, 25-28, 61-64, 69, 79, 71-76\}\} \\ \mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash (\cup_{i=1}^{7} \mathrm{Irr}(B_i)) \end{cases}$$

We may suppose $B = B_0$ or B_1. It follows by Tables C–1, C–5, C–6 and C–7 that

$$\mathrm{k}(N_G(C(i)), B_0, d, \zeta_\ell) = \begin{cases} 14 & \text{if } d = 3, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, 4$ and in addition,

$$\mathrm{k}(G, B_1, d, \zeta_\ell) = \mathrm{k}(N_G(C(2)), B_1, d, \zeta_\ell) = \begin{cases} 5 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

This proves the theorem when $p = 3$.

(3) Suppose $p = 2$. Then G has three blocks B_0, B_1, B_2 with full defect and one block with defect group D_8 and in addition, $\mathrm{Irr}(G \mid \zeta_\ell) \cap \mathrm{Irr}(B_0) = \emptyset$. We may suppose $B = B_1$ or B_2, and suppose B_ℓ covers the block of $Z(G)$ containing ζ_ℓ.

The groups $N(C(3)) \simeq D_8 \times 3.3^2{:}\,4$ and $N(C(4)) \simeq 3.(2^2 \times 3^2{:}\,4)$ both have exactly 20 irreducible characters covering ζ_ℓ, and the degrees of $\mathrm{Irr}(D_8{:}\,3.3^2{:}\,4 \mid \zeta_\ell)$ and $\mathrm{Irr}(3.(2^2 \times 3^2{:}\,4) \mid \zeta_\ell)$ are given in Tables C–3 and C–4. It follows that

$$\mathrm{k}(N(C(3)), B_\ell, d, \zeta_\ell) = \mathrm{k}(N(C(4))), B_\ell, d, \zeta_\ell) = \begin{cases} 16 & \text{if } d = 5, \\ 4 & \text{if } d = 4, \\ 0 & \text{otherwise.} \end{cases}$$

By Magma, the degrees in $\mathrm{Irr}(N(C(i)) \mid \zeta_\ell)$ is the same as that in $\mathrm{Irr}(B_0(N_{\mathrm{O'N}}(C(i))))$ for $i = 5, 6, 7, 8$. It follows by (8.3) that

$$\mathrm{k}(N(C(5)), B_\ell, d, \zeta_\ell) = \mathrm{k}(N(C(6))), B_\ell, d, \zeta_\ell).$$

Set $\mathrm{k}(i, d) = \mathrm{k}(N(C(i)), B_\ell, d, \zeta_\ell)$. The degrees of $\mathrm{Irr}(G \mid \zeta_\ell)$ are given by Table C–1 (cf. [6, p. 133]). It follows by Tables C–1 and 13 that the values $\mathrm{k}(1, d)$ and $\mathrm{k}(7, d)$ are as in Table 25.

Defect d	9	8	7	6	4	otherwise
k(1, d)	8	6	3	1	2	0

Defect d	9	8	7	6	otherwise
k(7, d)	8	6	5	5	0

Table 25: Values of $\mathrm{k}(1, d)$ and $\mathrm{k}(7, d)$

If $\mathrm{k}(\mathrm{odd}, d) = \sum_{i\in\{1,7\}} \mathrm{k}(N(C(i)), B_\ell, d, \zeta_\ell)$, the values are recorded in Table 26.

Defect d	9	8	7	6	4	otherwise
$\mathrm{k}(\mathrm{odd}, d)$	16	12	8	6	2	0

Table 26: Values of $\mathrm{k}(\mathrm{odd}, d)$

The group $N(C(2)) \simeq 3.4.L_3(4){:}2$ and the degrees of $\mathrm{Irr}(N(C(2)) \mid \zeta_\ell)$ are given in Table C–2. In addition, the values $\mathrm{k}(8, d)$ can be determined by Table 15. Thus the values $\mathrm{k}(2, d)$ and $\mathrm{k}(8, d)$ are as in Table 27.

Defect d	9	8	7	6	4	otherwise
k(2, d)	8	6	5	5	2	0

Defect d	9	8	7	6	otherwise
k(8, d)	8	6	3	1	0

Table 27: Values of $\mathrm{k}(2, d)$ and $\mathrm{k}(8, d)$

If $\mathrm{k}(\mathrm{even}, d) = \sum_{i\in\{2,4\}} \mathrm{k}(N(C(i)), B_\ell, d, \zeta_\ell)$, the values are recorded in Table 28.

Defect d	9	8	7	6	4	otherwise
$\mathrm{k}(\mathrm{even}, d)$	16	12	8	6	2	0

Table 28: Values of $\mathrm{k}(\mathrm{even}, d)$

This implies the theorem. □

11 Dade's projective conjecture for 2.Ru

Let C be a radical p-chain of Ru and $N_{2.\mathrm{Ru}}(C) = 2.N_{\mathrm{Ru}}(C)$. Then the character table of $N_{2.\mathrm{Ru}}(C)$ can either be found in the library of character tables distributed with GAP or computed directly using MAGMA as in Section 8.

Let $H = N_{2.\mathrm{Ru}}(C)$ and let ξ be the faithful linear characters of $Z(2.\mathrm{Ru})$. Denote by $\mathrm{Irr}(H \mid \xi)$ the subset of $\mathrm{Irr}(H)$ consisting of characters covering ξ. The tables listing degrees of characters in $\mathrm{Irr}(H \mid \xi)$ referenced in the proofs of Theorem 11.1 are in Appendix D.

Theorem 11.1 *Let B be a p-block of $G = 2.\mathrm{Ru}$ with $D(B) \neq O_p(G)$. Then B satisfies the projective conjecture of Dade.*

PROOF: We may suppose B has a non-cyclic defect group and let $N(C) = N_G(C)$ for each $C \in \mathcal{R}(\mathrm{Ru})$.

(1) Suppose $p=5$, so that by Lemma 5.4 (a), $B=B_2$.

Given $3 \leq i \leq 6$ and $C(i) \in \mathcal{R}(\mathrm{Ru})$, it follows by MAGMA that the degrees in $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(i)) \mid \xi)$ are the same as that in $\mathrm{Irr}(B_0(N_{\mathrm{Ru}}(C(i))))$, so that

$$\mathrm{k}(N_{2.\mathrm{Ru}}(C(i)), B, d, \xi) = \mathrm{k}(N_{\mathrm{Ru}}(C(i)), B_0(\mathrm{Ru}), d),$$

which are given by (9.1), (9.2) and (9.3). The degrees of characters in $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(2)) \mid \xi)$ are given in Table D–10. It follows by Lemma 5.4 (a) that the equations (9.1), (9.2) and (9.3) still hold if we replace $\mathrm{k}(N_{\mathrm{Ru}}(C(i)), B_0(\mathrm{Ru}), d)$ by $\mathrm{k}(N_{2.\mathrm{Ru}}(C(i)), B, d, \xi)$. This implies Theorem 11.1 when $p=5$.

(2) Suppose $p=3$, so that by Lemma 5.4 (b), we may suppose $B=B_3$ or B_4. By MAGMA, the degrees in $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(i)) \mid \xi) \cap \mathrm{Irr}(B_4)$ is the same as that in $\mathrm{Irr}(B_0(N_{\mathrm{Ru}}(C(i))))$ for $i>2$, so that

$$\mathrm{k}(N_{2.\mathrm{Ru}}(C(i)), B_4, d, \xi) = \mathrm{k}(N_{\mathrm{Ru}}(C(i)), B_0(\mathrm{Ru}), d).$$

Again by Lemma 5.4 (b) and [6, p. 127] the equation above also holds when $i=1$. In addition, $N_{2.\mathrm{Ru}}(C(2))$ has exactly one block $b_3(C(2))$ inducing B_3 and $\mathrm{Irr}(b_3(C(2)))$ consists of 4 irreducible characters of degree 9 and 2 of degree 18, so that by Lemma 5.4 (b),

$$\mathrm{k}(N_{2.\mathrm{Ru}}(C(1)), B_3, d, \xi) = \mathrm{k}(N_{2.\mathrm{Ru}}(C(2)), B_3, d, \xi) = \begin{cases} 6 & \text{if } d=2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus Theorem 11.1 follows by Proof (2) of Theorem 9.1.

(3) Suppose $p=2$, so that by Lemma 5.4 (c), we may suppose $B=B_0(2.\mathrm{Ru})$.

By Lemma 7.3, $N_{2.\mathrm{Ru}}(C(6)) \simeq 2.(2^2 \times \mathrm{Sz}(8)){:}\,3$, $N_{2.\mathrm{Ru}}(C(7)) \simeq 2.(2^2 \times 3^{3+3}{:}\,7){:}\,3$, and the degrees of $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(6)) \mid \xi)$ and $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(7)) \mid \xi)$ are given in Tables D–5 and D–6, respectively. It follows that

$$\mathrm{k}(N_{2.\mathrm{Ru}}(C(6)), B_0, d, \xi) = \mathrm{k}(N_{2.\mathrm{Ru}}(C(7)), B_0, d, \xi) = \begin{cases} 8 & \text{if } d=8, \\ 6 & \text{if } d=7, \\ 0 & \text{otherwise.} \end{cases}$$

Set $\mathrm{k}(i,d) = \mathrm{k}(N(C(i)), B, d, \xi)$. By Lemma 7.3, $N_{2.\mathrm{Ru}}(C(3)) \simeq 2.2^6{:}\,2.2^4.S_3$, $N_{2.\mathrm{Ru}}(C(5)) \simeq 2.2^6{:}\,2^2.2^3.S_3$, and the degrees of $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(3)) \mid \xi)$ and $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(6)) \mid \xi)$ are given in Tables D–2 and D–4, respectively. The values $\mathrm{k}(3,d)$ and $\mathrm{k}(5,d)$ are as in Table 29.

Defect d	11	10	9	otherwise
k(3, d)	8	18	3	0

Defect d	11	10	9	8	otherwise
k(5, d)	8	10	9	4	0

Table 29: Values of $\mathrm{k}(3,d)$ and $\mathrm{k}(5,d)$

If $\mathrm{k}(\mathrm{odd}_1, d) = \sum_{i\in\{3,5\}} \mathrm{k}(N(C(i)), B, d, \xi)$, the values are recorded in Table 30.

Defect d	11	10	9	8	otherwise
$\mathrm{k}(\mathrm{odd}_1, d)$	16	28	12	4	0

Table 30: Values of $\mathrm{k}(\mathrm{odd}_1, d)$

By Lemma 7.3, $N_{2.\mathrm{Ru}}(C(2)) \simeq 2.2^6{:}U_3(3){:}2$, $N_{2.\mathrm{Ru}}(C(4)) \simeq 2.S'$, and the degrees of $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(2)) \mid \xi)$ and $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(4)) \mid \xi)$ are given in Tables D–1 and D–3, respectively. The values $\mathrm{k}(2, d)$ and $\mathrm{k}(4, d)$ are as in Table 31.

Defect d	11	10	9	8	otherwise
k(2, d)	8	10	1	4	0

Defect d	11	10	9	otherwise
k(4, d)	8	18	11	0

Table 31: Values of $\mathrm{k}(2, d)$ and $\mathrm{k}(4, d)$

If $\mathrm{k}(\mathrm{even}_1, d) = \sum_{i\in\{2,4\}} \mathrm{k}(N(C(i)), B, d, \xi)$, the values are recorded in Table 32.

Defect d	11	10	9	8	otherwise
$\mathrm{k}(\mathrm{even}_1, d)$	16	28	12	4	0

Table 32: Values of $\mathrm{k}(\mathrm{even}_1, d)$

It follows that

$$\sum_{i=2}^{5} (-1)^{|C(i)|}\mathrm{k}(N_{\mathrm{Ru}}(C(i)), B, d, \xi) = 0.$$

By Lemma 7.3, $N_{2.\mathrm{Ru}}(C(9)) \simeq 2.2.2^{4+6}{:}\, 2^2.S_3$ and the degrees of $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(9)) \mid \xi)$ are given in Table D–8. From Lemma 5.4 (c), the values $\mathrm{k}(1, d)$ and $\mathrm{k}(9, d)$ are as in Table 33.

Defect d	13	12	11	10	9	8	otherwise
k(1, d)	8	2	3	2	5	2	0

Defect d	13	12	11	10	9	otherwise
k(9, d)	8	2	11	9	1	0

Table 33: Values of $\mathrm{k}(1, d)$ and $\mathrm{k}(9, d)$

If $\mathrm{k}(\mathrm{odd}_2, d) = \sum_{i\in\{1,9\}} \mathrm{k}(N(C(i)), B, d, \xi)$, the values are recorded in Table 34.

Defect d	13	12	11	10	9	8	otherwise
$\mathrm{k}(\mathrm{odd}_2, d)$	16	4	14	11	6	2	0

Table 34: Values of $\mathrm{k}(\mathrm{odd}_2, d)$

By Lemma 7.3, $N_{2.\mathrm{Ru}}(C(8)) \simeq 2.2^{3+8}{:}\,L_3(3)$, $N_{2.\mathrm{Ru}}(C(10)) \simeq 2.2.2^{4+6}{:}\,S_5$, and the degrees of characters in $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(8)) \mid \xi)$ and $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C(10)) \mid \xi)$ are given in Tables D–7 and D–9. The values $\mathrm{k}(8, d)$ and $\mathrm{k}(10, d)$ are as in Table 35.

Defect d	13	12	11	10	9	8	otherwise
k(8, d)	8	2	5	2	2	1	0

Defect d	13	12	11	10	9	8	otherwise
k(10, d)	8	2	9	9	4	1	0

Table 35: Values of $\mathrm{k}(8, d)$ and $\mathrm{k}(10, d)$

If $\mathrm{k}(\mathrm{even}_2, d) = \sum_{i\in\{8,10\}} \mathrm{k}(N(C(i)), B, d, \xi)$, the values are recorded in Table 36.

Defect d	13	12	11	10	9	8	otherwise
$\mathrm{k}(\mathrm{even}_2, d)$	16	4	14	11	6	2	0

Table 36: Values of $\mathrm{k}(\mathrm{even}_2, d)$

It follows that

$$\sum_{i\{1,8,9,10\}} (-1)^{|C(i)|}\mathrm{k}(N_{2.\mathrm{Ru}}(C(i)), B, d, \xi) = 0$$

and so Dade's projective conjecture holds. □

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A The degrees of irreducible characters of chain normalizers of O′N

Degree	1	20	35	36	40	56	64	70	72	90	126	128	160
Number	2	3	6	2	1	3	4	2	1	3	1	1	2

Table A–1: The degrees of characters in $\mathrm{Irr}(4.L_3(4){:}\,2)$

Degree	1	20	35	40	56	64	70	72	90	112	126	128	160	180
Number	4	6	4	2	2	4	6	3	2	1	2	3	4	1

Table A–2: The degrees of characters in $\mathrm{Irr}((4.L_3(4){:}\,2).2)$

Degree	1	2	4	8
Number	16	4	8	2

Table A–3: The degrees of characters in $\mathrm{Irr}(D_8 \times 3^2{:}\,4)$

Degree	1	2	8	16
Number	32	8	4	1

Table A–4: The degrees of characters in $\mathrm{Irr}((D_8 \times 3^2{:}\,4).2)$

Degree	1	2	3	4	8	12
Number	8	4	8	4	2	4

Table A–5: The degrees of characters in $\mathrm{Irr}((2^2 \times 3^2{:}\,4).S_3)$

Degree	1	2	3	8	16	24
Number	16	8	16	2	1	2

Table A–6: The degrees of characters in $\mathrm{Irr}((2^2 \times 3^2{:}\,4).S_3.2)$

Degree	1	2	3
Number	8	4	8

Table A–7: The degrees of characters in $\mathrm{Irr}((4 \times 2^2).S_3)$

Degree	1	2	3	4	6	8	12	24
Number	2	1	6	2	5	4	3	1

Table A–8: The degrees of characters in $\mathrm{Irr}((4 \times 2^2).2^4.S_3)$

Degree	1	2	3	6	8	12	16	24
Number	4	2	4	8	5	7	1	2

Table A–9: The degrees of characters in $\mathrm{Irr}((4 \times 2^2).2^4.S_3.2)$

Degree	1	3	6	7	8	14	21	28	42
Number	1	2	1	3	1	1	2	3	4

Table A–10: The degrees of characters in $\mathrm{Irr}(4^3.L_3(2))$

Degree	1	3	6	7	8	14	28	42	56	84
Number	2	4	2	2	2	3	2	5	1	1

Table A–11: The degrees of characters in $\mathrm{Irr}(4^3.L_3(2).2)$

Degree	1	2	8	9	10	16	18	20	40	64	72	80
Number	4	1	1	4	8	4	1	1	2	2	1	1

Table A–12: The degrees of characters in $\mathrm{Irr}((3^2{:}\,4 \times A_6).2)$

Degree	1	2	8	9	10	16	18	20	32	64	72	80
Number	4	3	2	4	12	4	3	3	1	4	2	3

Table A–13: The degrees of characters in $\mathrm{Irr}((3^2{:}\,4 \times A_6).2.2)$

Degree	1	2	8	32
Number	8	6	8	2

Table A–14: The degrees of characters in $\mathrm{Irr}(3^4.2^2.2^3)$

Degree	1	2	4	8	64
Number	8	10	1	16	1

Table A–15: The degrees of characters in $\mathrm{Irr}(3^4.2^2.2^3.2)$

Degree	1	2	4	5	8	80
Number	2	2	2	6	2	4

Table A–16: The degrees of characters in $\mathrm{Irr}(3^4{:}\,2_-^{1+4}.D_{10})$

Degree	1	4	5	10	16	80
Number	4	5	4	2	1	8

Table A–17: The degrees of characters in $\mathrm{Irr}(3^4{:}\,2_-^{1+4}.D_{10}.2)$

Degree	1	6	7	8	48	96
Number	2	7	2	5	2	3

Table A–18: The degrees of characters in $\mathrm{Irr}(7^2{:}\,\mathrm{SL}_2(7){:}\,2)$

Degree	1	6	12	42
Number	12	4	3	2

Table A–19: The degrees of characters in $\mathrm{Irr}(7_+^{1+2}{:}\,(3 \times 2^2))$

Degree	1	2	12	24	42
Number	12	3	4	1	4

Table A–20: The degrees of characters in $\mathrm{Irr}(7_+^{1+2}.(3 \times D_8))$

Degree	1	2	24	42
Number	12	9	4	8

Table A–21: The degrees of characters in $\mathrm{Irr}(7_+^{1+2}.(3 \times D_8).2)$

B The degrees of irreducible characters of chain normalizers of Ru

Degree	1	6	7	14	21	27	42	56	63	64	126	189	378
Number	2	2	2	3	2	2	1	1	4	1	2	4	4

Table B–1: The degrees of characters in $\mathrm{Irr}(2^6{:}\,U_3(3){:}\,2)$

Degree	1	2	3	4	6	8	12	24	48
Number	4	5	12	2	7	1	10	2	4

Table B–2: The degrees of characters in $\mathrm{Irr}(2^6{:}\,2.2^4.S_3)$

Degree	1	2	4	8	16	32
Number	16	28	12	11	4	2

Table B–3: The degrees of characters in $\mathrm{Irr}(S')$

Degree	1	2	3	6	24	32	64
Number	8	4	8	24	9	2	1

Table B–4: The degrees of characters in $\mathrm{Irr}(2^6{:}\,2^2.2^3.S_3)$

Degree	1	3	14	42	64	91	105	192	195	273
Number	3	1	6	2	3	3	4	1	4	1

Table B–5: The degrees of characters in $\mathrm{Irr}((2^2 \times \mathrm{Sz}(8)){:}\,2)$

Degree	1	3	7	14	21	42
Number	3	9	3	6	1	2

Table B–6: The degrees of characters in $\mathrm{Irr}((2^2 \times 2^{3+3}\colon 7)\colon 3)$

Degree	1	3	6	7	8	21	24	28	42	56	84	112	224	336
Number	1	2	1	1	1	4	7	2	5	4	2	2	1	2

Table B–7: The degrees of characters in $\mathrm{Irr}(2^{3+8}\colon L_3(3))$

Degree	1	2	3	4	6	8	12	24	48	64	96	128
Number	2	1	6	2	9	4	14	6	4	2	1	1

Table B–8: The degrees of characters in $\mathrm{Irr}(2.2^{4+6}\colon 2^2.S_3)$

Degree	1	4	5	6	10	12	15	20
Number	2	2	2	5	4	2	4	4
Degree	24	30	40	60	120	128	192	
Number	3	1	1	8	6	3	2	

Table B–9: The degrees of characters in $\mathrm{Irr}(2.2^{4+6}\colon S_5)$

Degree	1	9	10	12	16	18	20	30
Number	4	4	2	3	2	2	1	2

Table B–10: The degrees of characters in $\mathrm{Irr}(3.A_6.2^2)$

Degree	1	2	4	6
Number	4	4	1	2

Table B–11: The degrees of characters in $\mathrm{Irr}(3_+^{1+2}\colon 2^2)$

Degree	1	2	3	4	8	16
Number	2	3	2	1	2	1

Table B–12: The degrees of characters in $\mathrm{Irr}(3^2\colon \mathrm{GL}_2(3))$

Degree	1	3	4	5	12	16	20
Number	4	8	5	4	2	1	1

Table B–13: The degrees of characters in $\mathrm{Irr}(5{:}\,4 \times A_5)$

Degree	1	2	4	8
Number	8	8	2	2

Table B–14: The degrees of characters in $\mathrm{Irr}(5{:}\,4 \times 5{:}\,2)$

Degree	1	4	5	6	24	96
Number	4	10	4	6	4	1

Table B–15: The degrees of characters in $\mathrm{Irr}(5^2{:}\,\mathrm{GL}_2(5))$

Degree	1	4	16	20
Number	16	8	1	4

Table B–16: The degrees of characters in $\mathrm{Irr}(5_+^{1+2}{:}\,4^2)$

Degree	1	2	8	16	20	40
Number	8	6	4	2	4	1

Table B–17: The degrees of characters in $\mathrm{Irr}(5^2{:}\,4.D_8)$

C The degrees of irreducible characters of chain normalizers of $3.\mathrm{O'N}$

Let ζ_1 and ζ_2 be the faithful irreducible characters of the cyclic group $3 = Z(3.\mathrm{O'N})$. For a radical p-chain $C \in \mathcal{R}(\mathrm{O'N})$, denote by $\mathrm{Irr}(N_{3.\mathrm{O'N}}(C) \mid \zeta_i)$ the irreducible characters of the stabilizer $N_{3.\mathrm{O'N}}(C)$ covering the character ζ_i.

Degree	342	495	5643	52668	58311	58653	63612	111321
Number	2	2	3	2	1	1	1	1
Degree	116622	122760	169290	169632	175770	207360	253440	
Number	1	1	2	2	1	3	1	

Table C–1: The degrees of characters in $\mathrm{Irr}(3.\mathrm{O'N} \mid \zeta_i)$

Degree	6	15	21	36	72	84	90	96	120	126	168
Number	2	6	2	2	1	3	3	2	3	1	1

Table C–2: The degrees of characters in $\mathrm{Irr}(3.(4.L_3(4){:}\,2) \mid \zeta_i)$

Degree	3	6
Number	16	4

Table C–3: The degrees of characters in $\mathrm{Irr}(D_8 \times 3.3^2{:}\,4 \mid \zeta_i)$

Degree	3	6	9
Number	8	4	8

Table C–4: The degrees of characters in $\mathrm{Irr}(3.(2^2 \times 3^2{:}\,4) \mid \zeta_i)$

Degree	18	27	36	45	54	90
Number	8	4	1	4	1	1

Table C–5: The degrees of characters in $\mathrm{Irr}(3.(3^2{:}\,4 \times A_6).2 \mid \zeta_i)$

Degree	9	18
Number	8	6

Table C–6: The degrees of characters in $\mathrm{Irr}(3.(3^4.2^2.2^3) \mid \zeta_i)$

Degree	9	18	36	45	72
Number	2	2	2	6	2

Table C–7: The degrees of characters in $\mathrm{Irr}(3.(3^4{:}\,2_-^{1+4}.D_{10}) \mid \zeta_i)$

D The degrees of irreducible characters of chain normalizers of 2.Ru

Let ξ be the faithful irreducible character of the cyclic group $2 = Z(2.\mathrm{Ru})$. For a radical p-chain $C \in \mathcal{R}(\mathrm{Ru})$, denote by $\mathrm{Irr}(N_{2.\mathrm{Ru}}(C) \mid \xi)$ the irreducible characters of the stabilizer $N_{2.\mathrm{Ru}}(C)$ covering the character ξ.

Degree	28	36	56	168	216	224	252	288	336
Number	4	2	3	4	3	2	2	2	1

Table D–1: The degrees of characters in $\mathrm{Irr}(2.(2^6.U_3(3){:}\,2) \mid \xi)$

Degree	4	8	12	16	24	48
Number	4	7	4	1	11	2

Table D–2: The degrees of characters in $\mathrm{Irr}(2.2^6.2.2^4.S_3 \mid \xi)$

Degree	4	8	16
Number	8	18	11

Table D–3: The degrees of characters in $\mathrm{Irr}(2.S' \mid \xi)$

Degree	8	12	16	24	32
Number	2	8	9	8	4

Table D–4: The degrees of characters in $\mathrm{Irr}(2.2^6.2^2.2^3.S_3 \mid \xi)$

Degree	2	28	128	182	210	390
Number	3	6	3	3	1	1

Table D–5: The degrees of characters in $\mathrm{Irr}(2.(2^2 \times \mathrm{Sz}(8)){:}\,3 \mid \xi)$

Degree	2	6	14	28
Number	3	2	3	6

Table D–6: The degrees of characters in $\mathrm{Irr}(2.(2^2 \times 2^{3+3}{:}\,7){:}\,3 \mid \xi)$

Degree	28	56	64	84	96	112	128	336
Number	4	2	2	4	2	3	1	2

Table D–7: The degrees of characters in $\mathrm{Irr}(2.(2^{3+8}{:}\,L_3(3)) \mid \xi)$

Degree	12	16	24	32	48	64	96
Number	8	4	2	7	7	1	2

Table D–8: The degrees of characters in $\mathrm{Irr}(2.(2.2^{4+6}\colon S_4) \mid \xi)$

Degree	12	16	32	48	60	64	80	96	120	128	160
Number	4	4	1	1	4	4	4	4	2	1	4

Table D–9: The degrees of characters in $\mathrm{Irr}(2.(2.2^{4+6}\colon S_5) \mid \xi)$

Degree	2	4	6	8	16	24
Number	8	4	4	2	1	1

Table D–10: The degrees of characters in $\mathrm{Irr}(2.(5\colon 4 \times A_5) \mid \xi)$