A local strategy to decide the Alperin and Dade Conjectures

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ABSTRACT

We present a new strategy which exploits both the maximal and p-local subgroup structure of a given finite simple group in order to decide the Alperin and Dade conjectures for this group. We demonstrate the computational effectiveness of this approach by using it to verify these conjectures for the Conway simple group Co_2.

INTRODUCTION

In this paper, we present a new strategy to decide the Alperin and Dade conjectures for the finite simple groups and demonstrate its computational effectiveness by using it to verify these conjectures for the Conway simple group Co_2. We also outline the contents of a software library which may be used to decide these conjectures for an arbitrary finite group.

Already, these conjectures have been verified for a substantial number of finite simple groups; see [8] for an indication of the current standing of the problem. The major challenge in deciding the conjectures for a given group is to determine the radical subgroups of this group and hence to obtain the radical chains. In practice, some of the radical chains of the outstanding finite simple groups cannot be determined explicitly using existing computational approaches. Our new *local strategy* uses knowledge of both the maximal and p-local subgroup structure of the finite simple group to determine the radical subgroups of the group.

1991 *Mathematics Subject Classification*. Primary 20C20, 20C34, 20C08

Let G be a finite group, p a prime and B a p-block of G. Alperin [1] conjectured that the number of B-weights equals the number of irreducible Brauer characters of B. Dade [7] generalized the Knörr-Robinson version of the Alperin weight conjecture and presented his ordinary conjecture exhibiting the number of ordinary irreducible characters of a fixed defect in B in terms of an alternating sum of related values for p-blocks of some p-local subgroups of G. Dade [8] announced that his final conjecture needs only to be verified for finite non-abelian simple groups; in addition, if a finite group has both trivial Schur multiplier and outer automorphism group, then the ordinary conjecture is equivalent to the final conjecture.

We demonstrate the computational effectiveness of our local strategy by using it to verify the Alperin weight conjecture and the Dade ordinary conjecture, and so the final one, for the Conway simple group Co_2.

The outline of the paper is as follows. In Section 1, we fix our notation and state the two conjectures in detail. In Section 2, we discuss the computational tools used in deciding the conjectures. In Section 3, we introduce our local strategy and discuss how we employed it to determine the radical subgroups of Co_2. In Section 4, we classify the radical subgroups of Co_2 up to conjugacy and verify the Alperin weight conjecture. In Section 5, we do some cancellations in the alternating sum of Dade's conjecture when $p = 2$ or 3, and then determine radical chains (up to conjugacy) and their local structures. In the last section, we verify Dade's conjecture.

1. THE ALPERIN AND DADE CONJECTURES

Let R be a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and let $\mathrm{Blk}(G)$ be the set of p-blocks, let $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a B-*weight* if φ has p-defect 0 (see [7, (5.5)] for the definition) and $B(\varphi)^G = B$ (in the sense of Brauer), where $B(\varphi)$ is the block of $N(R)$ containing φ. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights, and $\ell(B)$ the number of irreducible Brauer characters of B. Alperin [1] conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$.

Given a p-subgroup chain

$$C : P_0 < P_1 < \cdots < P_n \tag{1.1}$$

of a finite group G, define $|C| = n$, $C_k : P_0 < P_1 < \cdots < P_k$, $C(C) = C_G(P_n)$, and

$$N(C) = N_G(C) = N_G(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_n). \tag{1.2}$$

The chain C is *radical* if it satisfies the following conditions:

$$\text{(a)} \;\; P_0 = O_p(G) \quad \text{and} \quad \text{(b)} \;\; P_k = O_p(N(C_k)) \text{ for } 1 \le k \le n.$$

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G. For $B \in \mathrm{Blk}(G)$ and integer $d \ge 0$, let $\mathrm{k}(N(C), B, d)$ be the number of characters in the set

$$\mathrm{Irr}(N(C), B, d) = \{\psi \in \mathrm{Irr}(N(C)) : B(\psi)^G = B, \; \mathrm{d}(\psi) = d\},$$

where $\mathrm{d}(\psi)$ is the defect of ψ.

Dade's Ordinary Conjecture [7]. If $O_p(G) = 1$ and B is a p-block of G with positive defect, then for any integer $d \ge 0$,

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \, \mathrm{k}(N(C), B, d) = 0, \tag{1.3}$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

2. COMPUTATIONAL TOOLS

As part of this study, we have developed and implemented a library of procedures which can be used to (partially or completely) decide the Alperin weight conjecture and the Dade ordinary conjecture for an arbitrary finite group. The group can be described by a matrix or permutation representation.

These procedures are written in the language of the computational algebra system MAGMA (see [3] for details). They perform the following tasks:

[1.] Determine the G-conjugacy classes of radical p-subgroups for a given prime p.

[2.] Determine the blocks of the normaliser of each radical subgroup.

[3.] Determine the weights for each block of a radical subgroup.

[4.] Identify up to isomorphism type the defect groups.

[5.] Construct the p-radical chains, up to conjugacy, and eliminate redundant chains.

[6.] For each non-trivial chain, determine its local structure and evaluate the corresponding term of the alternating sum.

These procedures can be executed in sequence and hence, within the limits of computational resources, allow a user to decide both conjectures for an arbitrary finite group. Details of the algorithms used will be presented elsewhere. We plan to extend our algorithms to deal with other forms of Dade's conjecture.

The computations reported in this paper were carried out using these procedures running MAGMA V.2.20-7 on a Sun UltraSPARC Enterprise 4000 server.

In our investigation, we used the minimal degree representation of Co_2 as a permutation group on 2300 points. In constructing maximal subgroups of Co_2, we made extensive use of the algorithm described in [4] to construct random elements.

3. DETERMINING THE RADICAL SUBGROUPS OF Co_2

The major computational challenge in deciding the conjectures for Co_2 is to determine the radical subgroups of Co_2.

In summary, our *standard algorithm* to determine the radical p-subgroups of a group G for a given prime p is the following: compute the subgroup classes of the Sylow p-subgroup of G; for each p-subgroup R, compute the largest normal p-subgroup $O_p(N(R))$ of the normaliser $N(R)$ in G of R; if $O_p(N(R))$ equals R then R is radical.

This algorithm suffices to compute both the radical 3- and 5-subgroups of Co_2. However, the Sylow 2-subgroup of Co_2 has order 2^{18} and, using available computing resources, we could not determine the conjugacy classes of subgroups of this 2-group. Instead we use the following local strategy to obtain the radical 2-subgroups of Co_2.

Wilson [10] classifies the maximal subgroups of Co_2. In (4C), we use his classification to deduce that each radical 2-subgroup R of Co_2 is radical in one of seven maximal subgroups M and further that $N_G(R) = N_M(R)$.

(1). We first consider the case where M is a 2-local subgroup. Let $Q = O_2(M)$, so that $Q \leq R$. We find all the subgroup classes of a Sylow 2-subgroup D of M containing Q. Using MAGMA, we explicitly compute the quotient M/Q and the natural homomorphism $\eta : M \longrightarrow M/Q$. This approach provides a regular representation for M/Q, whose (potentially large) degree is usually computationally limiting. Hence, we construct a power-conjugate presentation for the quotient group $\eta(D) = D/Q$ since such presentations are computationally very effective. We now compute all subgroup classes in D/Q. The preimages in D of the subgroup classes of D/Q are the subgroup classes of D containing Q.

We select those class representatives R which are radical by deciding whether $R = O_2(N_M(R))$. Since computing the normalizer in M of R is potentially very expensive, we also seek to limit the time taken by this step. In some cases, the quotient M/Q is a well-known group. If a small degree permutation representation of M/Q is available, we explore this representation independently to find the radical 2-subgroup classes of M/Q and then use this information to guide our investigations and to provide termination conditions for our computations.

For example, if $M = 2_+^{1+8} : S_6(2)$, then D has 3200 subgroup classes containing $Q = 2_+^{1+8}$. By studying a permutation representation of degree 28 of $M/Q = S_6(2)$, we learn that $S_6(2)$ has 7 non-trivial radical 2-subgroups: one each of order 2^5, 2^6, 2^7 and 2^9, and three of order 2^8. Hence, we now know that the radical 2-subgroups of M have orders 2^k for $14 \leq k \leq 18$. We partition the 3200 classes according to their orders and search in each partition only until we find the required number of radical subgroups of this order.

(2). Now consider the case where M is not 2-local. We may be able to find its radical 2-subgroup classes directly. Alternatively, we find a subgroup K of M such that $N_K(R) = N_M(R)$ for each radical subgroup R of M. If K is 2-local, then we apply Step (1) to K. If K is not 2-local, we can replace M by K and repeat Step (2).

After applying the local strategy, possible fusions among the resulting list of radical subgroups can be decided readily by testing whether the subgroups in the list are pairwise G-conjugate.

Although it was not necessary, we used the local strategy to construct the radical 3-subgroups of Co_2 since it was significantly more efficient than the standard algorithm.

4. RADICAL SUBGROUPS AND WEIGHTS

Let $\Phi(G, p)$ be a set of representatives for conjugacy classes of radical p-subgroups of G. For $H, K \leq G$, we write $H \leq_G K$ if $x^{-1}Hx \leq K$; and write $H \in_G \Phi(G, p)$ if $x^{-1}Hx \in \Phi(G, p)$ for some $x \in G$. We use the notation of [6]. In particular, if p is odd, then $p_+^{1+2\gamma}$ is an extra-special group of order $p^{1+2\gamma}$ with exponent p; if δ is $+$ or $-$, then $2_\delta^{1+2\gamma}$ is an extra-special group of order $2^{1+2\gamma}$ with type δ. If X and Y are groups, we use $X.Y$ and $X : Y$ to denote an extension and a split extension of X by Y, respectively. Given $n \in \mathbb{N}$, we use E_{p^n} or simply p^n to denote the elementary abelian group of order p^n, $\mathbb{Z}_n$ or simply n to denote the cyclic group of order n, and D_{2n} to denote the dihedral group of order $2n$.

Let G be Co_2. Then $|G| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$, and we may suppose $p \in \{2, 3, 5\}$, since both conjectures hold for a block with a cyclic defect group by [7, Theorem 9.1].

We denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H and by $\mathrm{d}(H)$ the number $\log_p(|H|)$. Given $R \in \Phi(G, p)$, let $C(R) = C_G(R)$ and $N = N_G(R)$. If $B_0 = B_0(G)$ is the principal p-block of G, then by [2, (1.3)],

$$\mathcal{W}(B_0) = \sum_R |\,\mathrm{Irr}^0(N/C(R)R)|, \tag{4.1}$$

where R runs over the set $\Phi(G,p)$ such that the p-part $\mathrm{d}(C(R)R/R)=0$. The character table of $N/C(R)R$ can be constructed using MAGMA, hence we can find $|\operatorname{Irr}^0(N/C(R)R)|$.

(4A). *The non-trivial radical* 5*-subgroups* R *of* Co_2 *(up to conjugacy) are*

R	$C(R)$	N	$\lvert\operatorname{Irr}^0(N/C(R)R)\rvert$
5	$5\times S_5$	$F_5^4\times S_5$	
5_+^{1+2}	5	$5_+^{1+2}{:}\,4S_4$	$16,$

where F_n^m *is a Frobenius group with kernel* $\mathbb{Z}_n$ *and complement* $\mathbb{Z}_m$*.*

Proof. If $G=\mathrm{Co}_2$ and x is an element of class $5B$, then $N_G(\langle x\rangle)=5.4\times S_5$ (cf. [10, p. 111]), so that $5=\langle x\rangle$ is radical and $C_G(x)=5\times S_5$. In addition, if $5_+^{1+2}\in\operatorname{Syl}_5(G)$ is a Sylow 5-subgroup of G, then $N_G(5_+^{1+2})=5_+^{1+2}.4S_4$. By MAGMA, $\Phi(G,5)=\{5,5_+^{1+2}\}$. $\square$

(4B). *The non-trivial radical* 3*-subgroups* R *of* Co_2 *(up to conjugacy) are*

R	$C(R)$	N	$\lvert\operatorname{Irr}^0(N/C(R)R)\rvert$
3	$3\times U_4(2).2$	$S_3\times U_4(2).2$	
3^4	3^4	$3^4.A_6.D_8$	5
3_+^{1+4}	3	$3_+^{1+4}:2_-^{1+4}.S_5$	4
S	3	$S.(SD_{2^4}\times 2)$	$14,$

where $S\in\operatorname{Syl}_3(\mathrm{Co}_2)$ *and* SD_{2^4} *is the semidihedral group of order* 2^4*.*

Proof. Let M_1, M_2 and M_3 be subgroups of $G=\mathrm{Co}_2$ such that $M_1\simeq 3_+^{1+4}{:}\,2_-^{1+4}.S_5$, $M_2\simeq S_3\times U_4(2).2$ and $M_3\simeq 3^4.A_6.D_6$. Then M_1 and M_2 are normalizers of a $3A$ and $3B$ element, respectively. Suppose $1\neq R\in\Phi(G,3)$. Then $X=\Omega_1(Z(R))$ is an elementary abelian subgroup of G.

If $|X|=3$, then we may suppose $N_G(X)=M\in\{M_1,M_2\}$, so that $N_G(R)\le M$, $R\in\Phi(M,3)$ and $N_M(R)=N_G(R)$. Assume $M=M_1$ and $3_+^{1+4}=O_3(M)$. Then

$$\Phi(M_1,3)=\{3_+^{1+4},S\},$$

where $S\in\operatorname{Syl}_3(G)$. Assume $M=M_2$ and $3=O_3(M_2)$. It follows by [6, p. 26] and MAGMA that

$$\Phi(M_2,3)=\{3,3\times 3_+^{1+2},3^4,S'\}, \tag{4.2}$$

where $S'\in\operatorname{Syl}_3(M_2)$. Moreover, $N_G(R)\neq N_M(R)$ for each $R\in\Phi(M_2,3)\backslash\{3\}$. In addition, $C_G(3\times 3_+^{1+2})=C_G(S')=3^2$, $C_G(3^4)=3^4$ and (see [6, p. 26])

$$N_{M_2}(R)=\begin{cases} S_3\times 3_+^{1+2}{:}\,2S_4 & \text{if } R=3\times 3_+^{1+2},\\ S_3\times 3^3{:}\,(S_4\times 2) & \text{if } R=3^4,\\ S_3\times 3^3{:}\,(S_3\times 2) & \text{if } R=S'.\end{cases} \tag{4.3}$$

Suppose $|X| \geq 9$, so that X is noncyclic. By [10, p. 112], X contains an element x of class $3B$. Thus $X \leq C_G(x) = \langle x \rangle \times U_4(2).2$. Moreover, either $N_G(X) \leq N(3A)$ or $C_G(X)$ contains a normal subgroup shape 3^4. In the latter case, $C_G(X) \leq 3 \times 3^3 : D_8$ or $C_G(X) \leq 3 \times 3^3 : 2^2$, so that $C_G(X)$ has exactly one Sylow 3-subgroup of order 3^4. Since $N_G(R) \leq N_G(X)$ and $R \leq C_G(X)$, it follows by [9, Lemma 2.1] that R is a radical subgroup of $C_G(X)$. In particular, $3^4 \leq O_3(C_G(X)) \leq R$. Therefore $R = 3^4$. Hence $M = M_3$, and by MAGMA,

$$\Phi(M_3, 3) = \{3^4, S\}; \tag{4.4}$$

moreover, $N_G(R) = N_M(R)$ for each $R \in \Phi(M_3, 3)$. □

(4C). *Given integer $1 \leq i \leq 7$, let M_i be the maximal subgroups of $G = \mathrm{Co}_2$ such that $M_1 \simeq 2_+^{1+8} : S_6(2)$, $M_2 \simeq (2_+^{1+6} \times 2^4).A_8$, $M_3 \simeq 2^{4+10}.(S_5 \times S_3)$, $M_4 \simeq 2^{10} : M_{22} : 2$, $M_5 \simeq M^cL$, $M_6 \simeq M_{23}$ and $M_7 \simeq U_6(2) : 2$. Suppose R is a non-trivial radical 2-subgroup of G. Then $N_G(R) \leq_G M_i$ for some i. In particular, if $N_G(R) \leq M_i$, then $N_G(R) = N_{M_i}(R)$ and $R \in_G \Phi(M_i, 2)$.*

Proof. By [10, Theorem], each M_i is a maximal subgroup of G. If $1 \neq R \in \Phi(G, 2)$, then $X = \Omega_2(Z(R))$ is an elementary abelian subgroup of G, and $N_G(R) \leq N_G(X)$. By [10, Proposition 4] and the proof given in [10, pp. 113-114], $N_G(X) \leq M_i$ for some i and so $N_G(R) \leq M_i$. □

How do we construct these maximal subgroups of Co_2? From [6, 10], we learn that $M_1 = N(2A)$, $M_2 = N(2B)$ and $M_4 = N(O_2(N(2C)))$. The subgroup $2^4 = Z(O_2(M_3))$ contains 5 elements of class $2A$ and 10 of class $2B$, and it is also a subgroup of $2^{10} = O_2(M_4)$. Now 2^{10} has 77 elements of class $2A$ and 330 of class $2B$. Clearly we can assume that 2^4 contains a central involution z of a Sylow 2-subgroup of M_4. Thus a necessary condition for an involution $x \in 2^{10}$ to be an element of 2^4 is that x and xz are of class $2A$ or $2B$. This insight and repeated random element selection using the algorithm of [4] allowed us to construct 2^4 and so M_3. The remaining three maximal subgroups were constructed using the black-box algorithms of Wilson [11].

(4D). *The non-trivial radical* 2*-subgroups* R *of* Co_2 *(up to conjugacy) are*

R	$C(R)$	$N/C(R)R$	$\vert\mathrm{Irr}^0(N/C(R)R)\vert$
2_+^{1+8}	2	$S_6(2)$	1
2^{10}	2^{10}	$M_{22}{:}\,2$	0
$2_+^{1+6}\times 2^4$	2^5	A_8	1
$2_+^{1+8}.2^5$	2	$S_4(2)$	1
2^{4+10}	2^4	$S_5\times S_3$	0
$2^{10}.2^4$	2^4	$L_3(2)$	1
$2_+^{1+8}.2^6$	2	$L_3(2)$	1
$2^{4+10}.2$	2^3	$S_3\times S_3$	1
$2^{10}.2^5$	2^4	S_5	0
$2_+^{1+8}.2^3.2^4$	2	$S_3\times S_3$	1
$2^{10}.2^3.2^3$	2^3	S_3	1
$2_+^{1+8}.2^2.2^2.2^4$	2	S_3	1
$2_+^{1+8}.2^3.2^2.2^3$	2	S_3	1
$2_+^{1+8}.2^3.2^5$	2	S_3	1
S	2	1	1,

where $S\in\mathrm{Syl}_2(\mathrm{Co}_2)$ *is a Sylow* 2*-subgroup of* Co_2*.*

Proof. Suppose $1\neq R\in\Phi(G,2)$. Then we may assume that $R\in\Phi(M_i,2)$ for some $i=1,2,\ldots,7$.

We first consider those maximal subgroups – namely, M_5, M_6, and M_7 – which are not 2-local.

(1) Let M be either $M_5\simeq M^cL$ or $M_6\simeq M_{23}$. Then $\Phi(M,2)$ can be calculated directly using MAGMA, and M has no radical subgroups R such that $N_G(R)=N_M(R)$.

Suppose $M=M_7\simeq U_6(2):2$. A Sylow 2-subgroup S of M has order 2^{16}; hence we could not use the standard algorithm to classify the radical 2-subgroups of M. Instead we use Step (2) of the local strategy.

Suppose $1\neq D\in\Phi(M,2)$. If $H=U_6(2)$ is a subgroup of M of index 2, then by [9, Lemma 2.1], $D\cap H$ is a radical 2-subgroup of H. Moreover, if $D\cap H=1$, then $|D|=2$ and D is generated by an involution x. Thus $N_G(D)=C_G(x)$ and so $O_2(C_G(x))\leq D$. But $|O_2(C_G(x))|\geq 2^7$ (cf. [10, Table II]), so $|D|\neq 2$ and $D\cap H\neq 1$. By the Borel-Tits Theorem [5], $N_H(D\cap H)$ is a parabolic subgroup of $U_6(2)$ and $D\cap H=O_2(N_H(D\cap H))$. Thus $N_H(D\cap H)$ is a subgroup of a maximal parabolic subgroup L of H. Since $N_M(D)\leq$

$N_M(D \cap H) \leq L.2 \leq M$, it follows that $D \in_M \Phi(L.2, 2)$ and $N_{L.2}(D) = N_M(D)$. From [6, p. 115] we may suppose

$$L.2 \in \{2_+^{1+8}{:}\,U_4(2){:}\,2, (2^{4+8}{:}\,(3 \times A_5){:}\,2).2, 2^9{:}\,L_3(4){:}\,2\}.$$

The parabolic subgroup $2_+^{1+8}{:}\,U_4(2)$ is a centralizer W of an involution of class $2A$ and $N_M(W) = 2_+^{1+8}{:}\,U_4(2){:}\,2$. If W is the centralizer of an involution of class $2B$, then $O_2(W) = 2^{4+8}$; also $2^{4+8}{:}\,(3 \times A_5){:}\,2 = N_H(O_2(W))$ and $(2^{4+8}{:}\,(3 \times A_5){:}\,2).2 = N_M(O_2(W))$. Moreover, $N_G(O_2(W))$ is conjugate to $M_3 \simeq 2^{4+10}.(S_5 \times S_3)$ in G. If W is the centralizer of an involution of class $2C$ and $Q = O_2(W)$, then $Q \simeq 2^9$; further, $N_H(Q) = 2^9{:}\,L_3(4)$ and $N_M(Q) = 2^9{:}\,L_3(4){:}\,2$.

Applying the local strategy to each maximal subgroup $L.2$ of M, we obtained the radical subgroups D of M and none satisfies $N_M(D) = N_G(D)$.

We now consider the case where $R \in \Phi(M_i, 2)$ and $i \in \{1, 2, 3, 4\}$. Since each M_i is a 2-local subgroup of G, we can apply Step (1) of the local strategy to each M_i.

(2) Let $2_+^{1+8} = O_2(M_1)$ and apply the local strategy to $M_1 \simeq 2_+^{1+8} : S_6(2)$. Then

$$\Phi(M_1, 2) = \{2_+^{1+8}, 2_+^{1+8}.2^5, 2_+^{1+8}.2^6, 2_+^{1+8}.2^3.2^4, 2_+^{1+8}.2^2.2^2.2^4, 2_+^{1+8}.2^3.2^2.2^3, 2_+^{1+8}.2^3.2^5, S\}$$

and $N_{M_1}(R) = N_G(R)$ for each $R \in \Phi(M_1, 2)$. We may suppose $\Phi(M_1, 2) \subseteq \Phi(G, 2)$.

(3) Let $2_+^{1+6} \times 2^4 = O_2(M_2)$ and $S' \in \mathrm{Syl}_2(M_2)$. Then

$$\begin{aligned} \Phi(M_2, 2) = \{&2_+^{1+6} \times 2^4, (2_+^{1+6} \times 2^4).2^3, 2^{10}.2^4, \\ &2^{4+10}.2, 2^{10}.2^3.2^3, (2_+^{1+6} \times 2^4).2^2.2^3, (2_+^{1+6} \times 2^4).2.2^4, S'\}, \end{aligned} \tag{4.5}$$

and moreover, $N_{M_2}(R) = N(R)$ for each $R \in \{2_+^{1+6} \times 2^4, 2^{10}.2^4, 2^{4+10}.2, 2^{10}.2^3.2^3\}$. In addition, for $R \in \{(2_+^{1+6} \times 2^4).2^3, (2_+^{1+6} \times 2^4).2^2.2^3, (2_+^{1+6} \times 2^4).2.2^4, S'\}$,

$$N_{M_2}(R)/R = \begin{cases} L_3(2) & \text{if } R = (2_+^{1+6} \times 2^4).2^3, \\ S_3 & \text{if } R = (2_+^{1+6} \times 2^4).2^2.2^3, \\ S_3 & \text{if } R = (2_+^{1+6} \times 2^4).2.2^4, \\ S' & \text{if } R = S'. \end{cases} \tag{4.6}$$

and $C_G(R) = 2^2$.

(4) Let $2^{4+10} = O_2(M_3)$. Then

$$\Phi(M_3, 2) = \{2^{4+10}, 2^{10}.2^5, 2^{4+10}.2, 2_+^{1+8}.2^3.2^4, 2^{10}.2^3.2^3, 2_+^{1+8}.2^3.2^5, 2_+^{1+8}.2^3.2^2.2^3, S\}.$$

Also $N_{M_3}(R) = N_G(R)$ for $R \in \Phi(M_3, 2)$. Hence we may suppose $\Phi(M_3, 2) \subseteq \Phi(G, 2)$.

(5) Let $2^{10} = O_2(M_4)$. Then

$$\Phi(M_4, 2) = \{2^{10}, 2_+^{1+8}.2^5, 2^{10}.2^4, 2^{10}.2^5, 2^{10}.2^3.2^3, 2_+^{1+8}.2^3.2^2.2^3, 2_+^{1+8}.2^3.2^5, S\}.$$

Also $N_{M_4}(R) = N(R)$ for $R \in \Phi(M_4, 2)$. Hence we may suppose $\Phi(M_4, 2) \subseteq \Phi(G, 2)$. $\square$

In all cases, the normalizers and centralizers of each radical subgroup of G can be computed using MAGMA.

Denote by $D(B)$ a defect group of a block B, $\mathrm{Irr}(B)$ the set of irreducible ordinary characters of B.

(4E). *Let $G = \mathrm{Co}_2$ and let $\mathrm{Blk}^0(G, p)$ be the set of p-blocks with a non-trivial defect group.*

(a) *If $p \in \{5, 3\}$, then $\mathrm{Irr}^0(G, p) = \{B_0, B_1, B_2\}$ such that $D(B_1) \simeq D(B_2) \simeq \mathbb{Z}_p$, where $B_0 = B_0(G)$ is the principal block of G. In the notation of* [6, pp. 154–155],

$$\mathrm{Irr}(B_1) = \begin{cases} \{\chi_4, \chi_{20}, \chi_{24}, \chi_{38}, \chi_{43}\} & \text{if } p = 5, \\ \{\chi_{19}, \chi_{40}, \chi_{43}\} & \text{if } p = 3, \end{cases}$$

and

$$\mathrm{Irr}(B_2) = \begin{cases} \{\chi_8, \chi_{14}, \chi_{26}, \chi_{39}, \chi_{44}\} & \text{if } p = 5, \\ \{\chi_{33}, \chi_{36}, \chi_{44}\} & \text{if } p = 3. \end{cases}$$

In addition, $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash (\mathrm{Irr}(B_1) \cup \mathrm{Irr}(B_2))$, where $\mathrm{Irr}^+(G)$ consists of characters of $\mathrm{Irr}(G)$ with positive defects.

(b) *If $p = 2$, then $\mathrm{Blk}(G, 2) = \{B_0\}$ and $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G)$. Moreover,*

$$\ell(B_1) = \ell(B_2) = \begin{cases} 4 & \text{if } p = 5, \\ 2 & \text{if } p = 3, \end{cases} \qquad \ell(B_0) = \begin{cases} 16 & \text{if } p = 5, \\ 23 & \text{if } p = 3, \\ 12 & \text{if } p = 2. \end{cases}$$

Proof. If $B \in \mathrm{Blk}(G, p)$ is non-principal with $D = D(B)$, then $\mathrm{Irr}^0(C(D)D/D)$ has a non-trivial character, so by (4A), (4B) and (4D), $p = 5, 3$ and $D \in_G \{5, 3\}$. Moreover, for each such D, $|\mathrm{Irr}^0(C(D)D/D)| = 2$, so G has exactly two blocks, B_1 and B_2 with a defect group D. Using the method of central characters, we deduce that $\mathrm{Irr}(B)$ is described by (a).

If $D(B)$ is cyclic, then $\ell(B)$ is the number of B-weights, so that $\ell(B_1) = \ell(B_2)$ is 4 or 2 according as $p = 5$ or 3. If $\ell(G)$ is the number of p-regular G-conjugacy classes, then $\ell(B_0)$ can be calculated using the following equation due to Brauer:

$$\ell(G) = \bigcup_{B \in \mathrm{Blk}^0(G,p)} \ell(B) + |\mathrm{Irr}^0(G)| \ . \ \square$$

(4F). *Let B be a p-block of* Co_2 *with a non-cyclic defect group. Then the number of B-weights is the number of irreducible Brauer characters of B.*

Proof. Follows by (4.1) and (4A), (4B), (4D) and (4E). □

5. RADICAL CHAINS

Let $G = \mathrm{Co}_2$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$.

(5A). *In the notation of* (4A), *the radical* 5*-chains C of G (up to conjugacy) are:*

C	$N(C)$	C	$N(C)$
$C(1): 1$	G	$C(2): 1 < 5$	$F_5^4 \times S_5$
$C(3): 1 < 5 < 5^2$	$F_5^4 \times F_5^4$	$C(4): 1 < 5_+^{1+2}$	$5_+^{1+2}{:}\,4S_4$,

where $5^2 \in \mathrm{Syl}_5(F_5^4 \times S_5)$.

Proof. Straightforward. □

(5B). (a) *In the notation of* (4B) *and* (4.2), *the radical* 3*-chains $C(i)$ for $1 \le i \le 8$ and their normalizers $N(C)$ are:*

C	$N(C)$	C	$N(C)$
$C(1): 1$	Co_2	$C(2): 1 < 3$	$S_3 \times U_4(2).2$
$C(3): 1 < 3 < 3^4$	$S_3 \times 3^3{:}\,(S_4 \times 2)$	$C(4): 1 < 3^4$	$3^4.A_6.D_8$
$C(5): 1 < 3 < 3 \times 3_+^{1+2}$	$S_3 \times 3_+^{1+2}{:}\,2S_4$	$C(6): 1 < 3_+^{1+4}$	$3_+^{1+4} : (2_-^{1+4}.S_5)$
$C(7): 1 < 3_+^{1+4} < S$	$S.(SD_{2^4} \times 2)$	$C(8): 1 < 3 < K < S'$	$S_3 \times 3^3{:}\,(S_3 \times 2)$,

where $K = 3 \times 3_+^{1+2}$ *and* $S' \in \mathrm{Syl}_3(3 \times U_4(2).2)$.

(b) *Let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that $\mathcal{R}^0(G)/G = \{C(i) : 1 \le i \le 8\}$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|}\,\mathrm{k}(N(C), B_0, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|}\,\mathrm{k}(N(C), B_0, d)$$

for all integers $d \ge 0$.

Proof. If $C : 1 < S$ and $C' : 1 < 3^4 < S$, then $N(C) = N(C') = N(S)$, and we can delete C and C' in the sum (1.3). Similarly, if $C : 1 < 3 < S'$ and $C' : 1 < 3 < 3^4 < S'$, then $N(C) = N(C') = N_{M_2}(S')$. The rest follows from the proof of (4B). □

(5C). (a) *In the notation of* (4D) *and* (4.5), *the radical* 2*-chains* $C(i)$ *for* $1 \le i \le 16$ *and their normalizers* $N(C)$ *are:*

C	$N(C)$
$C(1): 1$	Co_2
$C(2): 1 < 2_+^{1+8}$	$2_+^{1+8}: S_6(2)$
$C(3): 1 < 2^{10} < 2_+^{1+8}.2^5$	$2_+^{1+8}.2^5.S_6$
$C(4): 1 < 2^{10}$	$2^{10}: M_{22}: 2$
$C(5): 1 < 2^{10} < 2^{10}.2^5$	$2^{10}.2^5.S_5$
$C(6): 1 < 2^{4+10}$	$2^{4+10}.(S_5 \times S_3)$
$C(7): 1 < 2^{4+10} < 2_+^{1+8}.2^3.2^4$	$2_+^{1+8}.2^3.2^4.(S_3 \times S_3)$
$C(8): 1 < 2^{10} < 2^{10}.2^5 < 2_+^{1+8}.2^3.2^5$	$2_+^{1+8}.2^3.2^5.S_3$
$C(9): 1 < 2_+^{1+6} \times 2^4 < (2_+^{1+6} \times 2^4).2^3$	$(2_+^{1+6} \times 2^4).2^3.L_3(2)$
$C(10): 1 < 2_+^{1+6} \times 2^4$	$(2_+^{1+6} \times 2^4).A_8$
$C(11): 1 < 2^{10} < 2^{10}.2^4$	$2^{10}.2^4.L_3(2)$
$C(12): 1 < 2^{10} < 2^{10}.2^4 < (2_+^{1+6} \times 2^4).2.2^4$	$(2_+^{1+6} \times 2^4).2.2^4.S_3$
$C(13): 1 < 2^{4+10} < 2^{4+10}.2$	$(2^{4+10}.2).(S_3 \times S_3)$
$C(14): 1 < 2^{10} < 2^{10}.2^5 < 2^{10}.2^3.2^3$	$2^{10}.2^3.2^3.S_3$
$C(15): 1 < 2^{10} < 2^{10}.2^5 < 2^{10}.2^3.2^3 < S'$	S'
$C(16): 1 < 2^{4+10} < 2^{4+10}.2 < (2_+^{1+6} \times 2^4).2^2.2^3$	$(2_+^{1+6} \times 2^4).2^2.2^3.S_3.$

(b) *Let* $\mathcal{R}^0(G)$ *be the* G*-invariant subfamily of* $\mathcal{R}(G)$ *such that* $\mathcal{R}^0(G)/G = \{C(i) : i = 1, 2, \ldots, 16\}$. *Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|}\, \mathrm{k}(N(C), B, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|}\, \mathrm{k}(N(C), B, d)$$

for all integers $d \ge 0$ *and for each block* B *with a non-cyclic defect group.*

Proof. (b) Suppose C' is a radical 2-chain such that

$$C' : 1 < P_1' < \ldots < P_m'. \tag{5.1}$$

Let $C \in \mathcal{R}(G)$ be given by (1.1) with $P_1 \in \Phi(G, 2)$.

Case (1). Let $R \in \Phi(M_1, 2)\backslash\{2_+^{1+8}\}$. Define G-invariant subfamilies $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ of $\mathcal{R}(G)$, such that

$$\begin{aligned} \mathcal{M}^+(R)/G &= \{C' \in \mathcal{R}/G : P_1' = R\}, \quad \text{and} \\ \mathcal{M}^0(R)/G &= \{C' \in \mathcal{R}/G : P_1' = 2_+^{1+8}, P_2' = R\}. \end{aligned} \tag{5.2}$$

For $C' \in \mathcal{M}^+(R)$ given by (5.1), the chain

$$g(C') : 1 < 2_+^{1+8} < P_1' = R < P_2' < \ldots < P_m' \tag{5.3}$$

is a chain in $\mathcal{M}^0(R)$ and $N(C') = N(g(C'))$. For any $B \in \mathrm{Blk}(G)$ and for any integer $d \geq 0$,

$$\mathrm{k}(N(C'), B, d) = \mathrm{k}(N(g(C')), B, d). \tag{5.4}$$

In addition, g is a bijection between $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$. So we may assume

$$C \notin \bigcup_{R \in \Phi(M_1,2)\backslash\{2_+^{1+8}\}} (\mathcal{M}^+(R) \cup \mathcal{M}^0(R)) \,.$$

Thus $P_1 \notin \{2_+^{1+8}.2^5, 2_+^{1+8}.2^6, 2_+^{1+8}.2^3.2^4, 2_+^{1+8}.2^2.2^2.2^4, 2_+^{1+8}.2^3.2^2.2^3, 2_+^{1+8}.2^3.2^5, S\}$, and if $P_1 = 2_+^{1+8}$, then $C =_G C(2)$. We may suppose

$$P_1 \in \Phi_1(G,2) = \{2^{10}, 2_+^{1+6} \times 2^4, 2^{4+10}, 2^{10}.2^4, 2^{4+10}.2, 2^{10}.2^5, 2^{10}.2^3.2^3\} \subseteq \Phi(G,2).$$

Case (2). Let $\Phi_2(G,2) = \{2^{10}, 2^{4+10}, 2^{10}.2^5\} \subseteq \Phi_1(G,2)$ and assume $R \in \Omega = \{2^{10}.2^4, 2^{4+10}.2, 2^{10}.2^3.2^3\} \subseteq \Phi(M_2,2)$. Repeat the proof above with 2_+^{1+8} replaced by $2_+^{1+6} \times 2^4$. Then we may suppose $P_1 \in \Phi_2(G,2) \cup \{2_+^{1+6} \times 2^4\}$, and if $P_1 = 2_+^{1+6} \times 2^4$, then $P_2 \in \Phi(M_2,2)\backslash\Omega$. Now $N_{M_2}((2_+^{1+6} \times 2^4).2^3) = N(C(9)) \simeq (2_+^{1+6} \times 2^4).2^3.L_3(2)$ and by MAGMA

$$\Phi((2_+^{1+6} \times 2^4).2^3.L_3(2), 2) = \{(2_+^{1+6} \times 2^4).2^3, (2_+^{1+6} \times 2^4).2^2.2^3, (2_+^{1+6} \times 2^4).2.2^4, S'\},$$

which is a subset of $\Phi(M_2,2)$. In addition, $N_{N(C(9))}(R) = N_{M_2}(R)$ for each radical subgroup $R \in \Phi((2_+^{1+6} \times 2^4).2^3.L_3(2), 2)$.

Given $Q \in \Phi((2_+^{1+6} \times 2^4).2^3.L_3(2), 2)\backslash\{(2_+^{1+6} \times 2^4).2^3\}$, define G-invariant subfamilies $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ of $\mathcal{R}(G)$, such that

$$\begin{aligned} \mathcal{L}^+(Q)/G &= \{C' \in \mathcal{R}/G : P_1' = 2_+^{1+6} \times 2^4, P_2' = Q\}, \text{ and} \\ \mathcal{L}^0(Q)/G &= \{C' \in \mathcal{R}/G : P_1' = 2_+^{1+6} \times 2^4, P_2' = (2_+^{1+6} \times 2^4).2^3, P_3' = Q\}. \end{aligned} \tag{5.5}$$

A similar proof to above shows that we may suppose

$$C \notin \bigcup_{Q \in I} (\mathcal{L}^+(Q) \cup \mathcal{L}^0(Q)), \tag{5.6}$$

where $I = \Phi((2_+^{1+6} \times 2^4).2^3.L_3(2), 2) \backslash \{2_+^{1+6} \times 2^4\}$. It follows that if $P_1 = 2_+^{1+6} \times 2^4$, then $C \in_G \{C(9), C(10)\}$, and we may suppose

$$P_1 \in \Phi_2(G, 2) = \{2^{10}, 2^{4+10}, 2^{10}.2^5\}.$$

Case (3). Let $\mathcal{M}^+(2^{10}.2^5)$ and $\mathcal{M}^0(2^{10}.2^5)$ be given by (5.2) with R replaced by $2^{10}.2^5$ and 2_+^{1+8} by 2^{4+10}. Then (5.4) holds for $C' \in \mathcal{M}^+(2^{10}.2^5)$ and we may suppose $P_1 \neq_G 2^{10}.2^5$ and if $P_1 = 2^{4+10}$, then $P_2 \neq_G 2^{10}.2^5$. Since $N(2_+^{1+8}.2^3.2^4) \simeq (2_+^{1+8}.2^3.2^4).(S_3 \times S_3) \leq M_3$, it follows that

$$\Phi((2_+^{1+8}.2^3.2^4).(S_3 \times S_3), 2) = \{2_+^{1+8}.2^3.2^4, 2_+^{1+8}.2^3.2^5, 2_+^{1+8}.2^3.2^2.2^3, S\} \subseteq \Phi(G, 2),$$

and moreover, $N_{N(2_+^{1+8}.2^3.2^4)}(R) = N_{M_3}(R) = N(R)$ for all $R \in \Phi(N(2_+^{1+8}.2^3.2^4), 2)$. Let $\Omega' = \{2_+^{1+8}.2^3.2^5, 2_+^{1+8}.2^3.2^2.2^3, S\} \subseteq \Phi(N(2_+^{1+8}.2^3.2^4), 2)$, and $W \in \Omega'$. Replace Q by W, $2_+^{1+6} \times 2^4$ by 2^{4+10} and $(2_+^{1+6} \times 2^4).2^3$ by $2_+^{1+8}.2^3.2^4$ in the definition of (5.5). A similar proof to above shows that we may suppose

$$C \notin \bigcup_{W \in \Omega'} (\mathcal{L}^+(W) \cup \mathcal{L}^0(W)).$$

Thus if $P_1 = 2^{4+10}$, then we may suppose $P_2 \in \{2^{4+10}.2, 2_+^{1+8}.2^3.2^4, 2^{10}.2^3.2^3\}$, and moreover, if $P_2 =_G 2_+^{1+8}.2^3.2^4$, then $C =_G C(7)$.

Similarly, $N_{M_3}(2^{4+10}.2) = N(2^{4+10}.2) \simeq 2^{4+10}.2.(S_3 \times S_3)$, and

$$\Phi(2^{4+10}.2.(S_3 \times S_3), 2) = \{2^{4+10}.2, 2^{10}.2^3.2^3, (2_+^{1+6} \times 2^4).2^2.2^3, S'\} \subseteq \Phi(M_2, 2),$$

and moreover, $N_{N(2^{4+10}.2)}(R) = N_{M_2}(R)$ for each $R \in \Phi(N(2^{4+10}.2), 2)$. Replace Q by $2^{10}.2^3.2^3$, $2_+^{1+6} \times 2^4$ by 2^{4+10} and $(2_+^{1+6} \times 2^4).2^3$ by $2^{4+10}.2$ in the definition of (5.5). We may suppose $P_2 \neq_G 2^{10}.2^3.2^3$, and if $P_1 = 2^{4+10}$ and $P_2 =_G 2^{4+10}.2$, then $P_3 \neq_G 2^{10}.2^3.2^3$.

Let C' be the chain $1 < 2^{4+10} < 2^{4+10}.2 < (2_+^{1+6} \times 2^4).2^2.2^3 < S'$, and $g(C') : 1 < 2^{4+10} < 2^{4+10}.2 < S'$. Then $N(C') = N(g(C')) = S'$ and (5.4) holds. It follows that if $P_1 = 2^{4+10}$, then $C \in_G \{C(6), C(7), C(13), C(16)\}$.

Case (4). Suppose $P_1 = 2^{10}$. By (4D), $N_{M_4}(2_+^{1+8}.2^5) = N(2_+^{1+8}.2^5) = 2_+^{1+8}.2^5.S_6$ and by MAGMA,

$$\Phi(2_+^{1+8}.2^5.S_6, 2) = \{2_+^{1+8}.2^5, 2_+^{1+8}.2^3.2^2.2^3, 2_+^{1+8}.2^3.2^5, S\} \subseteq \Phi(G, 2),$$

and moreover, $N_{N(2_+^{1+8}.2^5)}(R) = N(R)$ for each $R \in \Phi(N(2_+^{1+8}.2^5), 2)$. Suppose $Q \in \Phi(N(2_+^{1+8}.2^5), 2)\backslash\{2_+^{1+8}.2^5\}$. Replace $2_+^{1+6} \times 2^4$ by 2^{10} and $(2_+^{1+6} \times 2^4).2^3$ by $2_+^{1+8}.2^5$ in the definition of (5.5). The same proof shows that we may suppose

$$C \notin \bigcup_{Q \in \Phi(N(2_+^{1+8}.2^5),2)\backslash\{2_+^{1+8}.2^5\}} (\mathcal{L}^+(Q) \cup \mathcal{L}^0(Q)).$$

Thus we may suppose $P_2 \in_G \{2_+^{1+8}.2^5, 2^{10}.2^4, 2^{10}.2^5, 2^{10}.2^3.2^3\}$, and if $P_2 = 2_+^{1+8}.2^5$, then $C =_G C(3)$. Since $N_{M_4}(2^{10}.2^4) = N(2^{10}.2^4) = 2^{10}.2^4.L_3(2)$, it follows by MAGMA that

$$\Phi(2^{10}.2^4.L_3(2), 2) = \{2^{10}.2^4, 2^{10}.2^3.2^3, (2_+^{1+6} \times 2^4).2.2^4, S'\} \subseteq \Phi(M_2, 2)$$

and moreover, $N_{N(2^{10}.2^4)}(R) = N_{M_2}(R)$ for each $R \in \Phi(N(2^{10}.2^4), 2)$.

Let $\mathcal{L}^+(2^{10}.2^3.2^3)$ and $\mathcal{L}^0(2^{10}.2^3.2^3)$ be defined by (5.5) with Q replaced by $2^{10}.2^3.2^3$, $2_+^{1+6} \times 2^4$ by 2^{10} and $(2_+^{1+6} \times 2^4).2^3$ by $2^{10}.2^4$. A similar proof shows that we may suppose $P_2 \neq_G 2^{10}.2^3.2^3$ and if $P_2 = 2^{10}.2^4$, then $P_3 \neq_G 2^{10}.2^3.2^3$.

Let $C' : 1 < 2^{10} < 2^{10}.2^4 < S'$ and $g(C') : 1 < 2^{10} < 2^{10}.2^4 < (2_+^{1+6} \times 2^4).2.2^4 < S'$. Then $N(C') = N(g(C')) = S'$ and (5.4) holds. Thus if $P_1 = 2^{10}$ and $P_2 = 2^{10}.2^4$, then $C \in_G \{C(11), C(12)\}$.

Similarly, $N(2^{10}.2^5) = N_{M_4}(2^{10}.2^5) = 2^{10}.2^5.S_5$ and

$$\Phi(2^{10}.2^5.S_5, 2) = \{2^{10}.2^5, 2^{10}.2^3.2^3, 2_+^{1+8}.2^3.2^5, S\} \subseteq \Phi(G, 2)$$

and moreover, $N_{N(2^{10}.2^5)}(R) = N(R)$ for all $R \in \Phi(N(2^{10}.2^5), 2)$. Let $C' : 1 < 2^{10} < 2^{10}.2^5 < S$ and $g(C') : 1 < 2^{10} < 2^{10}.2^5 < 2_+^{1+8}.2^3.2^5 < S$. Then $N(C') = N(g(C')) = S$ and (5.4) holds.

Finally, $\Phi(N(2^{10}.2^3.2^3), 2) = \{2^{10}.2^3.2^3, S'\} \subseteq \Phi(M_2, 2)$ and for each radical subgroup $R \in \Phi(N(2^{10}.2^3.2^3), 2)$, $N_{N(2^{10}.2^3.2^3)}(R) = N_{M_2}(R)$. Thus if $P_1 = 2^{10}$ and $P_2 = 2^{10}.2^5$, then we may suppose $C \in_G \{C(5), C(8), C(14), C(15)\}$. This completes the proof of (b).

(a). The proof follows easily by that of (b) or (4D). □

6. THE PROOF OF DADE'S CONJECTURE

(6A). *Let B be a p-block of $G = \mathrm{Co}_2$ with positive defect. If p is odd, then B satisfies the ordinary conjecture of Dade.*

Proof. We may suppose $p = 5$ or 3, and $B = B_0$.

Suppose $p = 5$ and let $C = C(2)$, $C' = C(3)$. Then $N(C) \simeq F_5^4 \times S_5$ and $N(C') \simeq F_5^4 \times F_5^4$. The principal blocks of $N(C)$ and $N(C')$ both have exactly 25 irreducible characters of height 0, so that

$$\mathrm{k}(N(C), B_0, d) = \mathrm{k}(N(C'), B_0, d).$$

for all integers $d \geq 0$. The subgroup $N(C(4)) \simeq 5_+^{1+2}.2S_4$ has 27 irreducible characters.

The degrees of characters of $\mathrm{Irr}(5_+^{1+2}.2S_4)$

Degree	1	2	3	4	20	24	40	60
Number	4	6	4	2	3	4	3	1

It follows by [6, p. 154] and (4E) that

$$\mathrm{k}(G, B_0, d) = \mathrm{k}(N(C(4))), B_0, d) = \begin{cases} 20 & \text{if } d = 3, \\ 7 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus (6A) holds when $p = 5$.

Suppose $p = 3$. Then $N(C(2)) \simeq S_3 \times U_4(2).2$ and $N(C(3)) \simeq S_3 \times 3^3{:}\,(S_4 \times 2)$ have 75 and 66 irreducible characters, respectively.

The degrees of characters of $\mathrm{Irr}(S_3 \times U_4(2).2)$

Degree	1	2	6	10	12	15	20	24	30	40	48	60	64	80	81	90	120	128	160	162	180
Number	4	2	4	2	2	8	7	4	8	3	2	8	4	2	4	2	3	2	1	2	1

The degrees of characters of $\mathrm{Irr}(S_3 \times 3^3{:}\,(S_4 \times 2))$

Degree	1	2	3	4	6	8	12	16	24	32
Number	8	8	8	2	12	4	14	4	5	1

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 27 & \text{if } d = 5, \\ 39 & \text{if } d = 4, \\ \alpha & \text{if } d = 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(2), C(3)\}$ and $\alpha = 3$ or 0 according as $C = C(2)$ or $C(3)$.

The subgroups $N(C(5)) \simeq S_3 \times 3_+^{1+2}{:}\,2S_4$ and $N(C(8)) \simeq S_3 \times 3^3{:}\,(S_3 \times 2)$ have 54 and 51 irreducible characters, respectively.

The degrees of characters of $\mathrm{Irr}(S_3 \times 3_+^{1+2}{:}\,2S_4)$

Degree	1	2	3	4	6	8	12	16	18	24	32	36
Number	4	8	4	5	8	5	9	4	2	3	1	1

The degrees of characters of $\mathrm{Irr}(S_3 \times 3^3{:}\,(S_3 \times 2))$

Degree	1	2	4	6	8	12
Number	8	12	6	16	1	8

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 27 & \text{if } d = 5, \\ 24 & \text{if } d = 4, \\ \beta & \text{if } d = 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(5), C(8)\}$ and $\alpha = 3$ or 0 according as $C = C(5)$ or $C(8)$. Thus

$$\mathrm{k}(N(C(2)), B_0, d) + k(N(C(8)), B_0, d) = \mathrm{k}(N(C(3)), B_0, d) + k(N(C(5)), B_0, d).$$

The subgroups $N(C(4)) \simeq 3^4.A_6.D_8$ and $N(C(7)) \simeq S.(SD_{2^4} \times 2)$ have 42 and 45 irreducible characters, respectively.

The degrees of characters of $\mathrm{Irr}(3^4.A_6.D_8)$

Degree	1	2	9	10	16	18	20	40	60	120	160	180
Number	4	1	4	8	4	1	5	3	4	2	2	4

The degrees of characters of $\mathrm{Irr}(S.(SD_{2^4} \times 2))$

Degree	1	2	4	8	16	18	24	36	48	72
Number	8	10	3	4	2	4	4	7	2	1

It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 27 & \text{if } d = 6, \\ 6 & \text{if } d = 5, \\ \gamma & \text{if } d = 4, \\ 0 & \text{otherwise,} \end{cases} \tag{6.1}$$

where $C \in \{C(4), C(7)\}$ and $\gamma = 9$ or 12 according as $C = C(4)$ or $C(7)$.

Finally, the subgroup $N(C(6)) \simeq 3_+^{1+4}{:}\,(2_-^{1+4}.S_5)$ has 50 irreducible characters.

The degrees of characters of $\mathrm{Irr}(3_+^{1+4}\colon(2_-^{1+4}.S_5))$

Degree	1	4	5	6	10	15	16	18	20	24	54
Number	2	4	4	1	5	2	2	1	4	1	2
Degree	72	80	90	160	180	216	240	270	288	320	360
Number	2	2	4	3	2	2	2	1	1	1	2

It follows by [6, p. 154] that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 27 & \text{if } d = 6, \\ 6 & \text{if } d = 5, \\ \delta & \text{if } d = 4, \\ 5 & \text{if } d = 3, \\ 0 & \text{otherwise}, \end{cases} \tag{6.2}$$

where $C \in \{C(1), C(6)\}$ and $\delta = 9$ or 12 according as $C = C(1)$ or $C(6)$. Thus Dade's conjecture follows by (6.1) and (6.2). $\square$

(6B). *Let B be a 2-block of $G = \mathrm{Co}_2$ with positive defect. Then B satisfies the ordinary conjecture of Dade.*

Proof. We may suppose $B = B_0 = B_0(G)$. Since $C(C)$ is a 2-subgroup for each chain $C \neq C(1)$, it follows that $\mathrm{Irr}(B_0(N(C))) = \mathrm{Irr}(N(C))$. We first consider the chains $C(j)$ such that $\mathrm{d}(N(C(j))) = 17$. So $9 \leq j \leq 16$.

The subgroup $N(C(10)) \simeq (2_+^{1+6} \times 2^4).A_8$ has 111 irreducible characters.

The degrees of characters of $\mathrm{Irr}((2_+^{1+6} \times 2^4).A_8)$

Degree	1	7	8	14	15	20	21	28	35	45	56	64	70
Number	1	1	1	1	1	1	3	3	5	4	2	1	5
Degree	90	105	112	120	140	160	168	210	224	252	280	315	360
Number	1	7	1	2	5	1	3	5	1	2	3	10	4
Degree	420	448	512	560	630	720	840	960	1260	1680	2520		
Number	9	2	1	1	4	1	7	1	8	1	2		

Thus $\mathrm{k}(10, d) = \mathrm{k}(N(C(10)), B_0, d)$ are as follows:

Defect d	17	16	15	14	13	12	11	8	otherwise
$k(10, d)$	32	16	28	24	4	2	4	1	0

The subgroup $N(C(12)) \simeq (2_+^{1+6} \times 2^4).2.2^4.S_3$ has 345 irreducible characters.

The degrees of characters of $\mathrm{Irr}((2_+^{1+6} \times 2^4).2.2^4.S_3)$

Degree	1	2	3	4	6	8	12	16	24	32	48	64	96	128	192
Number	4	2	28	28	30	22	64	4	110	8	20	6	16	1	2

Thus $\mathrm{k}(12, d) = \mathrm{k}(N(C(12)), B_0, d)$ are as follows:

Defect d	17	16	15	14	13	12	11	10	otherwise
$k(12, d)$	32	32	92	132	24	24	8	1	0

The subgroup $N(C(14)) \simeq (2^{10}.2^3.2^3).S_3$ has 354 irreducible characters.

The degrees of characters of $\mathrm{Irr}((2^{10}.2^3.2^3).S_3)$

Degree	1	2	3	4	6	8	12	16	24	32	48	64	96	192
Number	16	12	16	2	60	8	90	22	56	13	38	2	17	2

Thus $\mathrm{k}(14, d) = \mathrm{k}(N(C(14)), B_0, d)$ are as follows:

Defect d	17	16	15	14	13	12	11	otherwise
$k(14, d)$	32	72	92	64	60	30	4	0

The subgroup $N(C(16)) \simeq (2_+^{1+6} \times 2^4).2^2.2^3.S_3$ has 333 irreducible characters.

The degrees of characters of $\mathrm{Irr}((2_+^{1+6} \times 2^4).2^2.2^3.S_3)$

Degree	1	2	3	4	6	8	12	16	24	32	48	64	96	128	192
Number	8	4	24	8	52	12	68	20	56	16	44	6	12	1	2

Thus $\mathrm{k}(16, d) = \mathrm{k}(N(C(16)), B_0, d)$ are as follows:

Defect d	17	16	15	14	13	12	11	10	otherwise
$k(16, d)$	32	56	76	68	64	28	8	1	0

If $\mathrm{k}_e = \sum_{j=5}^{8} \mathrm{k}(N(C(2j)), B_0, d)$, then k_e are as follows:

Defect d	17	16	15	14	13	12	11	10	8	otherwise
k_e	128	176	288	288	152	84	24	2	1	0

The subgroup $N(C(9)) \simeq (2_+^{1+6} \times 2^4).2^3.L_3(2)$ has 174 irreducible characters.

The degrees of characters of $\mathrm{Irr}((2_+^{1+6} \times 2^4).2^3.L_3(2))$

Degree	1	3	6	7	8	14	21	24	28	42	48
Number	1	2	1	9	3	4	20	4	22	11	2
Degree	56	64	84	112	168	192	224	336	384	448	512
Number	21	3	22	4	32	2	4	2	1	3	1

Thus $\mathrm{k}(9,d) = \mathrm{k}(N(C(9)), B_0, d)$ are as follows:

Defect d	17	16	15	14	13	12	11	10	8	otherwise
$k(9,d)$	32	16	44	60	8	4	8	1	1	0

The subgroup $N(C(11)) \simeq (2^{10}.2^4).L_3(2)$ has 186 irreducible characters:

The degrees of characters of $\mathrm{Irr}((2^{10}.2^4).L_3(2))$

Degree	1	3	6	7	8	14	21	24	28	42	48	56	64	84	112	168	224	336	448	672
Number	2	4	2	14	4	10	12	4	18	20	2	14	2	42	4	18	5	6	2	1

Thus $\mathrm{k}(11,d) = \mathrm{k}(N(C(11)), B_0, d)$ are as follows:

Defect d	17	16	15	14	13	12	11	otherwise
$k(11,d)$	32	32	60	40	12	6	4	0

The subgroup $N(C(13)) = 2^{4+10}.2.(S_3 \times S_3)$ has 262 irreducible characters.

The degrees of characters of $\mathrm{Irr}(2^{4+10}.2.(S_3 \times S_3))$

Degree	1	2	3	4	6	8	9	12	16	18	24	32	36	48	64	72	96	144	192	288
Number	8	8	8	2	8	4	16	14	12	40	10	9	44	20	2	18	13	20	2	4

Thus $\mathrm{k}(13,d) = \mathrm{k}(N(C(13)), B_0, d)$ are as follows:

Defect d	17	16	15	14	13	12	11	otherwise
$k(13,d)$	32	56	60	32	52	26	4	0

The subgroup $N(C(15)) = S' = (2_+^{1+6} \times 2^4).2.2^4.2 \in \mathrm{Syl}_2((2_+^{1+6} \times 2^4).A_8)$ has 521 irreducible characters.

The degrees of characters of $\mathrm{Irr}((2_+^{1+6} \times 2^4).2.2^4.2)$

Degree	1	2	4	8	16	32	64	128
Number	32	72	124	156	80	48	8	1

Thus $\mathrm{k}(15, d) = \mathrm{k}(N(C(15)), B_0, d)$ are as follows:

Defect d	17	16	15	14	13	12	11	10	otherwise
$k(15, d)$	32	72	124	156	80	48	8	1	0

It follows that

$$\sum_{j=5}^{8} \mathrm{k}(N(C(2j)), B_0, d) = \sum_{j=5}^{8} \mathrm{k}(N(C(2j-1)), B_0, d) = \begin{cases} 128 & \text{if } d = 17, \\ 176 & \text{if } d = 16, \\ 288 & \text{if } d = 15, \\ 288 & \text{if } d = 14, \\ 152 & \text{if } d = 13, \\ 84 & \text{if } d = 12, \\ 24 & \text{if } d = 11, \\ 2 & \text{if } d = 10, \\ 1 & \text{if } d = 8, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we consider the 2-chains $C(j)$ such that $\mathrm{d}(N(C(j))) = 18$, so that $1 \leq j \leq 8$. The subgroup $N(C(2)) \simeq 2_+^{1+8} : S_6(2)$ has 100 irreducible characters:

The degrees of characters of $\mathrm{Irr}(2_+^{1+8} : S_6(2))$

Degree	1	7	15	16	21	27	35	56	70	84	105	112	120	135	168
Number	1	1	1	1	2	1	2	1	1	1	3	1	3	1	1
Degree	189	210	216	240	280	315	336	378	405	420	432	512	560	720	810
Number	3	2	1	1	2	1	3	1	3	1	1	1	2	2	1
Degree	840	896	945	1080	1120	1344	1680	1890	1920	2520	2688	2835	3024	3240	3360
Number	2	1	5	1	1	1	6	2	1	2	1	8	3	2	2
Degree	3456	3780	4480	5040	5376	5670	6048	6480	6720	7560	7680	8192			
Number	1	2	2	2	1	1	1	1	2	1	1	1			

Thus $\mathrm{k}(2, d) = \mathrm{k}(N(C(2)), B_0, d)$ are as follows:

Defect d	18	17	16	15	14	13	12	11	10	9	5	otherwise
$k(2, d)$	32	8	4	16	23	4	3	6	1	2	1	0

The subgroup $N(C(4)) \simeq 2^{10}: M_{22}: 2$ has 79 irreducible characters.

The degrees of characters of $\mathrm{Irr}(2^{10}: M_{22}: 2)$

Degree	1	21	22	45	55	99	154	210	231	385	440	560	770
Number	2	2	2	4	2	2	2	2	6	2	2	1	6
Degree	924	990	1155	1386	1408	1540	2772	3080	3465	4620	6930	9240	13860
Number	4	4	4	2	2	2	1	4	8	4	6	2	1

Thus $\mathrm{k}(4, d) = \mathrm{k}(N(C(4)), B_0, d)$ are as follows:

Defect d	18	17	16	15	14	11	otherwise
$k(4, d)$	32	24	12	8	1	2	0

The subgroup $N(C(6)) \simeq 2^{4+10}.(S_5 \times S_3)$ has 156 irreducible characters.

The degrees of characters of $\mathrm{Irr}(2^{4+10}.(S_5 \times S_3))$

Degree	1	2	3	4	5	6	8	10	12	15	18	20	30	40
Number	4	2	4	4	4	2	2	2	5	4	2	4	2	4
Degree	45	60	80	90	120	160	180	240	320	360	480	640	720	960
Number	16	13	1	14	8	8	10	1	8	14	10	2	1	5

Thus $\mathrm{k}(6, d) = \mathrm{k}(N(C(6)), B_0, d)$ are as follows:

Defect d	18	17	16	15	14	13	12	11	otherwise
$k(6, d)$	32	24	36	28	3	18	13	2	0

The subgroup $N(C(8)) \simeq (2_+^{1+8}.2^3.2^5).S_3$ has 264 irreducible characters.

The degrees of characters of $\mathrm{Irr}((2_+^{1+8}.2^3.2^5).S_3)$

Degree	1	2	3	4	6	8	12	16	24	32	48	64	96	128	192	256
Number	8	6	24	9	34	6	35	9	46	6	28	9	32	6	5	1

Thus $\mathrm{k}(8, d) = \mathrm{k}(N(C(8)), B_0, d)$ are as follows:

Defect d	18	17	16	15	14	13	12	11	10	otherwise
$k(8, d)$	32	40	44	52	37	38	14	6	1	0

If $\mathrm{k}_e = \sum_{j=1}^{4} \mathrm{k}(N(C(2j)), B_0, d)$, then k_e are as follows:

Defect d	18	17	16	15	14	13	12	11	10	9	5	otherwise
k_e	128	96	96	104	64	60	30	16	2	2	1	0

The subgroup $N(C(3)) \simeq (2_+^{1+8}.2^5).S_6$ has 148 irreducible characters.

The degrees of characters of $\mathrm{Irr}((2_+^{1+8}.2^5).S_6)$

Degree	1	5	6	9	10	15	16	20	24	30	36	40	45	60	80
Number	2	4	2	2	6	8	3	1	2	6	1	4	16	4	4
Degree	90	96	120	144	160	180	240	256	320	360	384	480	576	640	720
Number	10	2	10	2	6	6	8	1	1	16	2	4	1	4	10

Thus $\mathrm{k}(3, d) = \mathrm{k}(N(C(3)), B_0, d)$ are as follows:

Defect d	18	17	16	15	14	13	12	11	10	otherwise
$k(3, d)$	32	24	12	32	27	12	2	6	1	0

The subgroup $N(C(5)) \simeq (2^{10}.2^5).S_5$ has 187 irreducible characters.

The degrees of characters of $\mathrm{Irr}((2^{10}.2^5).S_5)$

Degree	1	2	4	5	6	8	10	12	15	20	30	40	60	80	120	160	240	320	480	640
Number	8	2	8	8	4	2	2	1	16	8	32	6	27	1	20	16	10	12	2	2

Thus $\mathrm{k}(5, d) = \mathrm{k}(N(C(5)), B_0, d)$ are as follows:

Defect d	18	17	16	15	14	13	12	11	otherwise
$k(5, d)$	32	40	44	28	11	18	12	2	0

The subgroup $N(C(7)) \simeq (2_+^{1+8}.2^3.2^4).(S_3 \times S_3)$ has 205 irreducible characters.

The degrees of characters of $\mathrm{Irr}((2_+^{1+8}.2^3.2^4).(S_3 \times S_3))$

Degree	1	2	3	4	6	8	9	12	16	18	24	32
Number	4	4	8	5	6	4	20	9	5	14	8	4
Degree	36	48	64	72	96	128	144	192	256	288	384	
Number	22	9	5	24	14	4	11	10	1	12	2	

Thus $\mathrm{k}(7, d) = \mathrm{k}(N(C(7)), B_0, d)$ are as follows:

Defect d	18	17	16	15	14	13	12	11	10	otherwise
$k(7,d)$	32	24	36	36	25	30	15	6	1	0

It follows by [6, p. 154] that $\mathrm{k}(1,d) = \mathrm{k}(G, B_0, d)$ are as follows:

Defect d	18	17	16	15	14	12	11	9	5	otherwise
$k(1,d)$	32	8	4	8	1	1	2	2	1	0

It follows that

$$\sum_{j=1}^{4} \mathrm{k}(N(C(2j)), B_0, d) = \sum_{j=1}^{4} \mathrm{k}(N(C(2j-1)), B_0, d) = \begin{cases} 128 & \text{if } d = 18, \\ 96 & \text{if } d = 17, \\ 96 & \text{if } d = 16, \\ 104 & \text{if } d = 15, \\ 64 & \text{if } d = 14, \\ 60 & \text{if } d = 13, \\ 30 & \text{if } d = 12, \\ 16 & \text{if } d = 11, \\ 2 & \text{if } d = 10, \\ 2 & \text{if } d = 9, \\ 1 & \text{if } d = 5, \\ 0 & \text{otherwise}, \end{cases}$$

which implies (6B). □

Acknowledgement. We thank R.A. Wilson for his advice on constructing the maximal subgroups of Co_2.

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