On some questions about a family of cyclically presented groups

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Abstract

We study various questions about the generalised Fibonacci groups, a family of cyclically presented groups, which includes as special cases the Fibonacci, Sieradski, and Gilbert-Howie groups.

1 Introduction

Consider the class of groups with cyclic presentation:

\[ G_n(w) = \langle x_1, \ldots, x_n : w = 1, \theta(w) = 1, \ldots, \theta^{n-1}(w) = 1 \rangle \]

where \( w \) is a reduced word in the alphabet \( X = \{ x_i^\pm 1, \ldots, x_n^\pm 1 \} \) and \( \theta \) is the automorphism of the free group of rank \( n \) defined by setting \( \theta(x_i) = x_{i+1} \mod n \).

One of the motivations for the study of these groups is their connection with the topology of closed connected orientable 3-manifolds; see, for example, [5, 12].

If \( w = x_i x_{i+m} x_{i+k}^{-1} \), then we obtain the generalised Fibonacci groups introduced in [4]:

\[ G_n(m, k) = \langle x_1, \ldots, x_n : x_i x_{i+m} = x_{i+k} \ (i = 1, \ldots, n) \rangle \]

where the subscripts are taken modulo \( n \).

For particular choices of parameters, these groups are well-known: \( G_n(1, 2) \) are the Fibonacci groups \( F(2, n) \) (see [7, 17]); \( G_n(2, 1) \) are the Sieradski groups \( S(n) \) (see [16, 18]); \( G_n(m, 1) \) are the Gilbert-Howie groups \( H(n, m) \) (see [9]).

We can immediately restrict our attention to those groups \( G_n(m, k) \) whose parameters satisfy the conditions \( 0 < m < k < n \) and \( (n, m, k) = 1 \). Such groups are irreducible. Bardakov & Vesnin [2] prove:

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• if $G_n(m, k)$ is not irreducible, then it is either trivial, cyclic, or a free product of $G_{n'}(m', k')$ for smaller values of $n', m', k'$;

• if $G_n(m, k)$ is irreducible and either $(n, k) = 1$ or $(n, k - m) = 1$, then $G_n(m, k)$ is isomorphic to $G_n(t, 1) = H(n, t)$, where $tk \equiv m \mod n$ or $t(k - m) \equiv (n - m) \mod n$ respectively.

This motivates the following definition in [2]: $G_n(m, k)$ is strongly irreducible if it is irreducible and $(n, k) > 1$ and $(n, k - m) > 1$.

Bardakov & Vesnin [2] pose, and study, a number of questions about these groups. These include:

• Under what conditions is $G_n(m, k)$ aspherical? Finite and non-trivial?

• Determine the number of isomorphism types among $G_n(m, k)$.

• Determine the structure of the largest abelian quotient, $A_n(m, k)$, of $G_n(m, k)$.

• Under what conditions is $G_n(m, k)$ the fundamental group of a 3-orbifold (in particular, a hyperbolic closed 3-manifold) of finite volume?

We summarise recent progress in answering these questions.

With a few exceptions, Gilbert & Howie [9] identify those $H(n, m)$ which are aspherical or finite. Williams [19] proves that a strongly irreducible group $G_n(m, k)$ is not aspherical if and only if $(m, k) = 1$ and either $n = 2k$, or $n = 2(k - m)$. He determines sufficient conditions for an irreducible group to be perfect. If, as he conjectures, these are also necessary, then every strongly irreducible group is not perfect; and he describes the structure of those which are finite and non-trivial. We show that $H(9, 3)$ is infinite, thus reducing the undecided cases among irreducible (but not strongly irreducible) groups to 2.

Let $f(n)$ denote the number of isomorphism types among the irreducible groups $G_n(m, k)$. We obtain some new isomorphisms, and demonstrate that the known isomorphisms suffice to obtain $f(n)$ for all but four values of $n \leq 27$. We formulate a sharp conjecture for $f(p^\ell)$ where $p$ is a prime.

Under the hypothesis of irreducibility, Corollary 5.8 of [5] shows that $A_n(m, k)$ is infinite if and only if $n \equiv 0 \mod 6$, $m + k \equiv 3 \mod 6$, and $m$ is even. An equivalent result appears in [19, Theorem 4]. If $2k \equiv m \mod n$, then we obtain a complete description of $A_n(m, k)$.

Corollary 3.5 of [5] is a slight improvement on [2, Theorem 3.1]: if $n$ is odd and $(2k - m, n) = 1$, then $G_n(m, k)$ cannot be the fundamental group of a hyperbolic closed 3-orbifold of finite volume. If $G_n(m, k)$ is irreducible and $2k \equiv m \mod n$, then we show that $G_n(m, k) \cong S(n)$, the fundamental group of a closed connected orientable 3-manifold. Finally, we prove that the split extension of an irreducible $G_n(m, k)$ by a cyclic group of order $n$ has a homomorphism onto a particular triangle group if both $(n, k) = 1$ and $2(2k - m) \equiv 0 \mod n$.2
2 The isomorphism problem

The most general result on isomorphism is the following [2, Theorem 1.1].

**Theorem 1.** Let $G_n(m, k)$ and $G_n(m', k')$ be irreducible groups. Assume that $k'$ is divisible by $r = (n, k - m)$, $(n, k') = 1$, and there exist integers $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, n/r\}$ such that

\[
\begin{align*}
    i + j(k - m) &\equiv (1 - m) \mod n \\
    m' + 1 &\equiv (i + jk') \mod n.
\end{align*}
\]

Then $G_n(m, k) \cong G_n(m', k')$.

Observe that the extra condition, $(n, k') = 1$, omitted from the original statement is both necessary and a consequence of the proof: for example, $\mathbb{Z}_7 \cong G_6(1, 3) \not\cong G_6(3, 4) \cong \mathbb{Z}_3 \times \mathbb{Z}_7$.

Theorem 1 assumes both that $k'$ is divisible by $(n, k - m)$ and $(n, k') = 1$, so $r = 1$. Hence, as was pointed out by the referee, we obtain an equivalent and simpler formulation.

**Theorem 2.** Let $G_n(m, k)$ and $G_n(m', k')$ be irreducible groups and assume $(n, k') = 1$. If $m'(m - k) \equiv mk' \mod n$, then $G_n(m, k)$ is isomorphic to $G_n(m', k')$.

We record some obvious consequences.

**Corollary 3.**

1. If $n \geq 5$ is odd, then $G_n(n - 3, n - 1) \cong G_n(n - 3, n - 2)$.

2. $G_{2h+1}(h, h + 1) \cong G_{2h+1}(h, 2h) \cong G_{2h+1}(1, 2) = F(2, 2h + 1)$.

3. If $(2h + 1, k - 1) = 1$, then $G_{2h+1}(1, k) \cong G_{2h+1}(1, 2h + 2 - k)$.

**Proof.** We illustrate the method by proving (3). By hypothesis, $(2h+1, k-1) = 1$ and so $(2h + 1, 2h + 2 - k) = 1$. Since $(1 - k) \equiv (2h + 2 - k) \mod (2h + 1)$, the result follows.

**Corollary 4.** If there exists $\beta$ such that $\beta s \equiv 1 \mod n$ and $\beta(1 - t) \equiv 1 \mod n$, then $G_n(1, t) \cong G_n(1, s)$.

**Proof.** Since $\beta s \equiv 1 \mod n$, we conclude that $(n, s) = 1$.

**Proposition 5.** If $(n, m) = 1$, then $G_n(m, k)$ is isomorphic to $G_n(1, t)$, where $tm \equiv k \mod n$. 

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Proposition 7. If \( p \) is an odd prime, then there are at most \((p-1)/2\) isomorphism types among the irreducible groups \( G_p(m, k) \).
Proof. If \( p \) is prime, then \((p, m) = 1\). Proposition 5 implies that \( G_p(n, k) \cong G_p(1, t) \) for some \( t \in \{2, \ldots, p - 1\} \), where \( t m \equiv k \mod p \). Since \((p, t - 1) = 1\), there exists \( \beta \) such that \( \beta(1 - t) \equiv 1 \mod p \).

If \( 2 \leq t \leq (p + 1)/2 \), then \( s = p + 1 - t \) satisfies \((p + 1)/2 \leq s \leq p - 1\). Corollary 4 now implies that \( G_p(1, t) \cong G_p(1, s) \) since \( \beta s = \beta(p + 1 - t) \equiv 1 \mod p \).

Hence the isomorphism types arise by choosing \( t \in \{2, \ldots, (p + 1)/2\} \), and so \( f(p) \leq (p - 1)/2 \).

Our investigations, reported in Section 5, suggest the following stronger result.

Conjecture 8. If \( n = p^\ell \) for an odd prime \( p \) and positive integer \( \ell \), then \( f(n) = p^\ell - \frac{p^\ell - 1}{p-1} p^{(\ell-1)} - 1 \). If \( \ell > 2 \), then \( f(2^\ell) = 3(2^{\ell-2}) \).

3 The abelianisation of \( G_n(m, k) \)

We obtain a complete description of the abelianisation of \( G_n(m, k) \) when \( 2k \equiv m \mod n \), and so extend [19, Lemma 5].

Lemma 9. Assume \( 2k \equiv m \mod n \).

- \( G_n(m, k) \) is perfect if and only if \( n/(n, k) \equiv \pm 1 \mod 6 \).

- The abelianisation of \( G_n(m, k) \) is isomorphic to \( \mathbb{Z}_{2(n,k)}^2 \), \( \mathbb{Z}_3^{(n,k)} \), or \( \mathbb{Z}_2^{2(n,k)} \) if and only if \( n/(n, k) \equiv 0 \), \( n/(n, k) \equiv \pm 2 \), or \( n/(n, k) \equiv \pm 3 \mod 6 \) respectively.

Proof. If \( G_n(m, k) \) is irreducible, then \( 2k \equiv m \mod n \) implies that \( n, k = 1 \). Proposition 6(2) implies that \( G_n(m, k) \cong G_n(2k, k) \cong G_n(2, 1) = S(n) \). Recall from [6, Theorem 2.1] that \( S(n) \cong \pi_1(M_n) \), where \( M_n \) is the \( n \)-fold cyclic cover of the 3-sphere, branched over the trefoil knot. Thus \( M_n \) is the Brieskorn manifold of [13] and its abelianisation is well-known – see, for example, [15, p. 304].

If \( G_n(2k, k) \) is not irreducible, then [2, Lemma 1.2] shows that \( G_n(2k, k) \) is isomorphic to a free product of \( (n, k) \) copies of \( G_{n/(n,k)}(2k/(n, k), k/(n, k)) \), which is irreducible. Hence the abelianisation of \( G_n(2k, k) \) is trivial, \( \mathbb{Z}_{2(n,k)}^2 \), \( \mathbb{Z}_3^{(n,k)} \), or \( \mathbb{Z}_2^{2(n,k)} \) according to the stated congruence conditions.

In summary, Proposition 6(2), [6, Corollary 2.2] and [18, Theorems B-C] imply the following: if \( G_n(m, k) \) is irreducible and \( 2k \equiv m \mod n \), then \( G_n(m, k) \) is infinite if and only if \( n \geq 6 \), and it has a free subgroup of rank 2 when \( n \geq 7 \).

We now briefly discuss \( A_n(1, t) \) for arbitrary \( n \). It is well-known that \( A_n(1, 2) \) is finite, of order \( L_n - 1 - (-1)^n \), where \( L_n \) is the \( n \)-th Lucas number (see, for example, [11, Chapter 6]).
Proposition 10. The structure of \( A_n(1,t) \), where \( t \in \{2, \ldots, n-1\} \), is determined by the diagonal form of the integral \( t \times t \) matrix

\[
\begin{pmatrix}
{a_{n+1}} - 1 & a_{n+1} & a_{n+2} & \cdots & a_{n+3} & a_{n+2} \\
a_{n+2} & a_{n+1} + 1 & a_{n+2} & \cdots & a_{n+3} & a_{n+2} \\
a_{n+3} & a_{n+1} + 1 & a_{n+2} & \cdots & a_{n+3} & a_{n+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n+t-1} & a_{n+t+1} & a_{n+t+2} & \cdots & a_{n+t+3} & a_{n+t+2} \\
a_{n+t} & a_{n+t+1} & a_{n+t+2} & \cdots & a_{n+t+3} & a_{n+t+2} \\
a_{n+t+1} & a_{n+t+2} & a_{n+t+3} & \cdots & a_{n+t+4} & a_{n+t+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{n+1} & a_{n+1} \\
a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{n+1} & a_{n+1} \\
\end{pmatrix}
\]

where \( a_i + a_{i+1} = a_{i+t} (i \geq 1) \) and \( a_1 = 1 \), \( a_i = 0 \) (\( 2 \leq i \leq t \)).

Proof. We sketch a proof. The generators, \( x_1, \ldots, x_n \), of \( A_n(1,t) \) commute and satisfy the following relations:

\[
\begin{align*}
    x_1x_2 &= x_{t+1} \\
x_2x_3 &= x_{t+2} \\
x_3x_4 &= x_{t+3} \\
& \vdots \\
x_{t-1}x_t &= x_{2t-1} \\
x_t x_{t-1} &= x_{2t} \\
x_{t+1} x_{t+2} &= x_{2t+1} \\
x_{t+2} x_{t+3} &= x_{2t+2} \\
& \vdots \\
x_{n-t} x_{n-t+1} &= x_n
\end{align*}
\]

Hence \( \{x_1, \ldots, x_t\} \) generate \( A_n(1,t) \), and \( x_i = x_1^{a_i} x_2^{b_i^1} \cdots x_t^{b_i^t} \) for \( i > t \).

We use the relations \( x_i = x_{i-1} x_{i-1+1} \) and those implied by commutativity to deduce that

\[
\begin{align*}
a_i &= a_{i-t} + a_{i-t+1} & i > t \\
b_i^j &= b_{i-t}^j + b_{i-t+1}^j & 2 \leq j \leq t
\end{align*}
\]

where \( a_1 = 1, a_i = 0, b_i^j = \delta_{i,j} \) for \( 2 \leq i, j \leq t \). Thus \( b_i^j = a_{i-t+j+1} \) for all \( i \geq 1 \).

Hence the structure of \( A_n(1,t) \) can be deduced from the diagonal form of the \( t \times t \) matrix:

\[
\begin{pmatrix}
{a_{n-t+1}} + a_{n-t+2} - 1 & b_{n-t+1}^2 + b_{n-t+2}^2 & b_{n-t+1}^3 + b_{n-t+2}^3 & \cdots & b_{n-t+1}^t + b_{n-t+2}^t \\
a_{n-t+2} + a_{n-t+3} & b_{n-t+2}^2 + b_{n-t+3}^2 - 1 & b_{n-t+2}^3 + b_{n-t+3}^3 & \cdots & b_{n-t+2}^t + b_{n-t+3}^t \\
a_{n-t+3} + a_{n-t+4} & b_{n-t+3}^2 + b_{n-t+4}^2 & b_{n-t+3}^3 + b_{n-t+4}^3 - 1 & \cdots & b_{n-t+3}^t + b_{n-t+4}^t \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-1} + a_n & b_{n-1}^2 + b_n^2 & b_{n-1}^3 + b_n^3 & \cdots & b_{n-1}^t + b_n^t \\
a_{n} + 1 & b_n^2 & b_n^3 & \cdots & b_n^t - 1
\end{pmatrix}
\]

The result now follows readily. \( \square \)
4 Split extensions

Let \( E_n(m, k) \) denote the split extension of \( G_n(m, k) \) by \( \mathbb{Z}_n = \langle \theta : \theta^n = 1 \rangle \), where \( \theta \) is the automorphism sending each generator \( x_i \) to \( x_{i+1} \) (subscripts taken modulo \( n \)). The relations \( x_ix_{i+m} = x_{i+k} \) of \( G_n(m, k) \) imply

\[ x\theta^{-m}x\theta^m = \theta^{-k}x\theta^k \]

where \( x := x_n \), and \( x_i = \theta^{-i}x^\theta \). Setting \( y = \theta^m x^{-1} \) (and eliminating \( x = y^{-1}\theta^m \)) yields

\[ E_n(m, k) = \langle \theta, y : \theta^n = 1, \theta^{k-1}y^2 = y\theta^k \rangle. \]

Assume \( 2k \equiv m \mod n \). As we observed in Lemma 9, if \( G_n(m, k) \) is irreducible, then it is isomorphic to \( G_n(2, 1) = S(n) \). Further \( E_n(m, k) \) is isomorphic to the fundamental group of the 3-dimensional orbifold whose underlying space is the 3-sphere and whose singular set is the trefoil knot with branching index \( n \) (see for example [6, Theorem 2.1]).

Lemma 11. If \( n \geq 3 \) is odd, then \( G_n(1, (n + 1)/2) \cong S(n) \) is isomorphic to the derived group of the centrally extended triangle group

\[ \Gamma = \langle \gamma_1, \gamma_2, \gamma_3 : \gamma_1^n = \gamma_2^2 = \gamma_3^3 = \gamma_1\gamma_2\gamma_3 \rangle. \]

If \( n \geq 7 \) is odd, then the centre of \( G_n(1, (n + 1)/2) \) is \( \mathbb{Z} \), otherwise it is \( \mathbb{Z}_2 \). If \( p \geq 5 \) is a prime, there is a homomorphism from \( G_p(1, (p + 1)/2) \) onto \( \text{SL}(2, p) \). Furthermore, \( G_5(1, 3) \cong \text{SL}(2, 5) \) and \( G_3(1, 2) \cong Q_8 \).

Proof. The first two assertions follow from [13, Section 3] since \( G_n(1, (n + 1)/2) \) is isomorphic to the fundamental group of the Brieskorn manifold \( M(n, 2, 3) \). To prove the third, we define a map from \( E_p(1, (p + 1)/2) \) to \( \text{SL}(2, p) \):

\[ \theta \rightarrow A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad y \rightarrow B = \begin{pmatrix} 0 & -2 \\ (p + 1)/2 & 1 \end{pmatrix}. \]

One can easily verify that this is an epimorphism which induces an epimorphism from \( G_p(1, (p + 1)/2) \) to \( \text{SL}(2, p) \), sending \( x_i \mapsto A^iB^{-1}A^{i+1} \). The last follows from [9]. \( \square \)

Theorem 12. Let \( G_n(m, k) \) be irreducible.

(a) If \( 2(2k - m) \equiv 0 \mod n \), then \( E_n(m, k) \) has a homomorphism onto the subgroup of \( \text{SL}(2, \mathbb{C}) \) having presentation

\[ \{ A, B : A^n = B^3 = 1, A^{2k-m} = (BA^k)^2 \}. \]
(b) If \((n, k) = 1\), then \(E_n(m, k)\) has a homomorphism onto the group defined
by the presentation \(\{u, v : v^n = 1, (uv)^3 = 1, v^{-\eta(2k-m)} = u^2\}\) where \(\eta k \equiv 1 \mod n\), for some integer \(\eta\).

(c) If \((n, k) = 1\) and \(2(2k - m) \equiv 0 \mod n\), then \(E_n(m, k)\) covers the triangle
of type \((n, 2, 3)\) if \(n\) is odd, and of type \(((n, 2k-m), 2, 3)\) if \(n\) is even.
If \((n, k) = 1\), \(2(2k - m) \equiv 0 \mod n\) and \((n, 2k-m) \geq 6\), then \(G_n(m, k)\) is
infinite.

Proof. Recall \(E_n(m, k) = \langle \theta, y : \theta^n = 1, \theta^{k-m}y^2 = y\theta^k\rangle\).

(a) We will exhibit a homomorphism \(E_n(m, k) \to \text{SL}(2, \mathbb{C})\) which both satisfies
the relations of \(E_n(m, k)\) and sends
\[
\theta \to A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad y \to B = \begin{pmatrix} \alpha & \beta \\ 1 & \gamma \end{pmatrix},
\]
where \(\lambda^n = 1, \beta \neq 0\), and \(\alpha \gamma - \beta = 1\). Such a homomorphism implies that
\[
\left( \begin{array}{cc} \alpha & \beta \\ 1 & \gamma \end{array} \right)^2 = \left( \begin{array}{cc} \lambda^{m-k} & 0 \\ 0 & \lambda^{k-m} \end{array} \right) \left( \begin{array}{cc} \alpha & \beta \\ 1 & \gamma \end{array} \right) \left( \begin{array}{cc} \lambda^k & 0 \\ 0 & \lambda^{-k} \end{array} \right).
\]
This gives the system of equations
\[
\begin{aligned}
\alpha^2 - \alpha \lambda^m + \beta &= 0 \\
\beta (\alpha + \gamma) &= \beta \lambda^{m-2k} \\
\alpha + \gamma &= \lambda^{2k-m} \\
\gamma^2 - \gamma \lambda^{-m} + \beta &= 0 \\
\beta &= \alpha \gamma - 1.
\end{aligned}
\]
Since \(\beta \neq 0\), the second and third equations imply \(\lambda^{2(2k-m)} = 1\), which
holds because \(n\) divides \(2(2k - m)\). The system has the unique solution
given by
\[
\alpha = \frac{1}{\lambda^{m-2k} - \lambda^m}, \quad \beta = \frac{\lambda^{2(m-k)} - \lambda^{2m} - 1}{(\lambda^{m-2k} - \lambda^m)^2}, \quad \gamma = \frac{-\lambda^{2(m-k)}}{\lambda^{m-2k} - \lambda^m}.
\]
Assume \(\lambda^m(\lambda^{-2k} - 1) = 0\). Then \(|\lambda^m||\lambda^{-2k} - 1| = 0\), and so \(2k \equiv 0 \mod n\).
But \(G_n(m, k)\) is irreducible and so \(0 < m < k < n\). Since \(2(2k-m) \equiv 0 \mod n\), we deduce that \(2m \equiv 0 \mod n\), a contradiction. Hence \(\lambda^m(\lambda^{-2k} - 1) \neq 0\).

Let \(\tau(B)\) be the square of the trace of the matrix \(B\). Then
\[
\tau(B) = \frac{(1 - \lambda^{2(m-k)})^2}{(\lambda^{m-2k} - \lambda^m)^2} = \frac{1 + \lambda^{4(m-k)} - 2\lambda^{2(m-k)}}{1 + \lambda^{2m} - 2\lambda^{2(m-k)}} = 1
\]
since $\lambda^2(m - 2k) = 1$ and $\lambda^4(m - k) = \lambda^2m$ as $2(2k - m) \equiv 0 \mod n$. Hence $B$ is elliptic. By [1, p. 39], we determine the multiplier $M^2$ of $B$ by applying the quadratic formula

$$M^2 = \frac{1}{2}[\tau(B) - 2 \pm \sqrt{-4\tau(B) + \tau^2(B)}]$$

Since $M^2 = (-1 \pm i\sqrt{3})/2$, we conclude that $B$ has order 3. The statement follows.

(b) If $y^3 = 1$, then the relation $\theta^{k-m}y^2 = y\theta^k$ becomes $\theta^{k-m} = y\theta^ky$, hence $\theta^{2k-m} = (y\theta^k)^2$. Thus adding the relation $y^3 = 1$ gives a homomorphism from $E_n(m, k)$ onto $\langle \theta, y : \theta^m = y^3 = 1, \theta^{2k-m} = (y\theta^k)^2 \rangle$. If $(n, k) = 1$, then there exist integers $\xi$ and $\eta$ such that $\xi n + \eta k = 1$. Setting $u = y\theta^k$ and $v = \theta^{-k}$, we deduce that $E_n(m, k)$ covers the group defined in (b).

(c) If $n$ is odd, then by (b) $E_n(m, k)$ covers $\langle v, u : v^n = u^2 = (uv)^3 = 1 \rangle$. If $n$ is even, then the relation $v^{(n, 2k - m)} = 1$ implies a homomorphism of $E_n(m, k)$ onto the triangle group of type $((n, 2k - m), 2, 3)$. The infiniteness claim now follows from [8, §6.4].

Consider the case when $(n, k) = 1$ and $2(2k - m) \equiv 0 \mod n$. If $n$ is also odd, then $2k \equiv m \mod n$; since $G_n(m, k) \cong S(n)$, it is infinite for $n \geq 6$. If $n$ is even, then (c) has new consequences: for example, it implies that $G_{12}(4, 5)$ is infinite.

## 5 Investigating $G_n(m, k)$ for small values of $n$

We investigated the irreducible groups $G_n(m, k)$ for values of $n \leq 27$. We used implementations in MAGMA [3] of algorithms to perform coset enumerations, compute abelian quotient invariants and (normal) subgroups of low index, and construct presentations for subgroups and $p$-quotients of finitely-presented groups. We refer the interested reader to [10, Chapters 5 and 9] for details and references to these algorithms.

### 5.1 Isomorphism

We sought to solve the isomorphism problem among the irreducible $G_n(m, k)$ for small values of $n$. We applied the isomorphisms identified in Theorem 2, its corollaries, and Propositions 5-6 to obtain both an upper bound $U(n)$ to the value of $f(n)$, and a potentially redundant list of isomorphism types. We then used invariants of groups in the resulting list to obtain a lower bound $L(n)$ to the value of $f(n)$. These bounds frequently coincided, so allowing us to deduce the precise value of $f(n)$. 

In most cases, it sufficed to compute the abelian quotient invariants of a group and those of its derived group to distinguish it from any other on the list. We note the exceptional cases.

- We proved that $G_{14}(1,3)$ is not isomorphic to $G_{14}(1,5)$ by showing that, among their normal subgroups of index 16, the number of distinct abelian quotient invariants is 8 and 9 respectively.
- The $p$-class 2 241-quotient of the derived group of $G_{22}(1,5)$ has order $241^{22}$; the corresponding quotient of the derived group of $G_{22}(1,7)$ has order $241^{44}$.
- $PSL(2,5)$ is a homomorphic image of $G_{25}(1,3)$ but not of $G_{25}(1,6)$.
- $G_{26}(1,13)$ is finite, $G_{26}(13,14)$ is infinite.

We summarise our results in Table 1. For $n \in \{3, \ldots, 27\}$, we record the values of $L(n)$ and $U(n)$; for each of the $U(n)$ groups, we list one defining value of the parameters $(m, k)$. For $n \in \{17, 19, 21, 23\}$, the values of $L(n)$ and $U(n)$ differ by 1. The unresolved cases are listed in Table 2.

Table 1 demonstrates that Conjecture 8 is sharp. For $n \in \{28, \ldots, 200\}$, we computed $U(n)$ and counted the number of distinct abelian quotient invariants among $G_{n}(m,k)$. This provided additional evidence for the correctness of Conjecture 8; it also suggests that there is at most one coincidence among the values of the abelian quotient invariants of $G_{n}(m,k)$ when $n = p^\ell$.

### 5.2 Finiteness

We summarise the results of Gilbert & Howie [9] and Williams [19], with known isomorphisms applied.

**Theorem 13.**

(i) Suppose $(n,m) \notin \{(8,3),(9,3),(9,4),(9,7)\}$. Then $H(n,m)$ is finite if and only if $m = 0$ or 1, or $(n,m) = (2\ell, \ell + 1)$ where $\ell \geq 1$, or $(n,m) \in \{(3,2),(4,2),(5,2),(5,3),(6,3),(7,4)\}$.

(ii) Let $G = G_{n}(m,k)$ be strongly irreducible and assume $G \neq 1$. Then $G$ is finite if and only if $(m,k) = 1$ and $n = 2k$ or $n = 2(k-m)$, in which case $G \cong \mathbb{Z}_s$ where $s = 2^{n/2} - (-1)^{m+(n/2)}$.

The structure of (the then known) finite irreducible groups among $H(n,m)$ is recorded in [9, Table 1]. Some of the exceptions from Theorem 13(i) have since been resolved. We now know that $H(8,3) \cong G_{8}(3,1)$ is a soluble group of order $3^{10} \cdot 5$ and derived length 3. First established by R.M. Thomas, its order can now be determined by a routine coset enumeration in Magma.

We now prove that $H(9,3)$ is infinite. Recall first Newman’s extension [14] of the Golod-Safarević theorem, which we summarise for the prime 2.
<table>
<thead>
<tr>
<th>$n$</th>
<th>$L(n)$</th>
<th>$U(n)$</th>
<th>Parameters $(m, k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>(1, 2), (2, 3)</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>2</td>
<td>(1, k) $k \in {2, 3}$</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>5</td>
<td>(1, k) $k \in {2, 3}$, (2, 3), (3, 4), (4, 5)</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>3</td>
<td>(1, k) $k \in {2, 3, 4}$</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>6</td>
<td>(1, k) $k \in {2, 3, 4}$, (2, 3), (2, 5), (4, 5)</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>5</td>
<td>(1, k) $k \in {2, \ldots, 5}$, (3, 4)</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>8</td>
<td>(1, k) $k \in {2, \ldots, 5}$, (2, k) $k \in {3, 5}$, (4, 7), (5, 6)</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>5</td>
<td>(1, k) $k \in {2, \ldots, 6}$</td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>12</td>
<td>(1, k) $k \in {2, \ldots, 6}$, (2, k) $k \in {3, 7}$, (3, k) $k \in {4, 5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(4, k) $k \in {5, 7}$, (6, 7)</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>6</td>
<td>(1, k) $k \in {2, \ldots, 7}$</td>
</tr>
<tr>
<td>14</td>
<td>11</td>
<td>11</td>
<td>(1, k) $k \in {2, \ldots, 7}$, (2, k) $k \in {3, 5, 7}$, (4, 9), (7, 8)</td>
</tr>
<tr>
<td>15</td>
<td>12</td>
<td>12</td>
<td>(1, k) $k \in {2, \ldots, 8}$, (3, k) $k \in {4, 5, 7}$, (5, 6), (5, 7)</td>
</tr>
<tr>
<td>16</td>
<td>12</td>
<td>12</td>
<td>(1, k) $k \in {2, \ldots, 8}$, (2, k) $k \in {3, 5, 9}$, (4, 5), (8, 9)</td>
</tr>
<tr>
<td>17</td>
<td>7</td>
<td>8</td>
<td>(1, k) $k \in {2, \ldots, 9}$</td>
</tr>
<tr>
<td>18</td>
<td>17</td>
<td>17</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>(4, 11), (6, 7), (9, 10)</td>
</tr>
<tr>
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<td>8</td>
<td>9</td>
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<td>18</td>
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<td>(4, k) $k \in {5, 7, 11}$, (5, 6), (5, 8), (10, 11)</td>
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<tr>
<td>21</td>
<td>15</td>
<td>16</td>
<td>(1, k) $k \in {2, \ldots, 11}$, (3, k) $k \in {4, 5, 7, 8}$, (7, 8), (7, 9)</td>
</tr>
<tr>
<td>22</td>
<td>17</td>
<td>17</td>
<td>(1, k) $k \in {2, \ldots, 11}$, (2, k) $k \in {3, 5, 7, 9, 11}$, (4, 13), (11, 12)</td>
</tr>
<tr>
<td>23</td>
<td>10</td>
<td>11</td>
<td>(1, k) $k \in {2, \ldots, 12}$</td>
</tr>
<tr>
<td>24</td>
<td>26</td>
<td>26</td>
<td>(1, k) $k \in {2, \ldots, 12}$, (2, k) $k \in {3, 5, 7, 13}$, (3, k) $k \in {4, 5, 8, 10}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(4, k) $k \in {5, 7}$, (6, k) $k \in {7, 13}$, (8, k) $k \in {9, 13}$, (12, 13)</td>
</tr>
<tr>
<td>25</td>
<td>14</td>
<td>14</td>
<td>(1, k) $k \in {2, \ldots, 13}$, (5, 6), (5, 7)</td>
</tr>
<tr>
<td>26</td>
<td>20</td>
<td>20</td>
<td>(1, k) $k \in {2, \ldots, 13}$, (2, k) $k \in {3, 5, 7, 9, 11, 13}$, (4, 15), (13, 14)</td>
</tr>
<tr>
<td>27</td>
<td>17</td>
<td>17</td>
<td>(1, k) $k \in {2, \ldots, 14}$, (3, k) $k \in {4, 5, 10}$, (9, 10)</td>
</tr>
</tbody>
</table>

Table 1: Lower and upper bounds for $f(n)$ for $n \leq 27$
Table 2: Possible isomorphisms

<table>
<thead>
<tr>
<th>$n$</th>
<th>Parameters $(m, k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>$(1, 3), (1, 4)$</td>
</tr>
<tr>
<td>19</td>
<td>$(1, 3), (1, 6)$</td>
</tr>
<tr>
<td>21</td>
<td>$(1, 6), (1, 9)$</td>
</tr>
<tr>
<td>23</td>
<td>$(1, 3), (1, 7)$</td>
</tr>
</tbody>
</table>

Theorem 14. Let $G$ be a group with a finite presentation on $b$ generators and $r$ relations. Let $G_1 := [G, G]G^2$ and $G_2 := [G_1, G]G_1^2$, where the elementary abelian 2-groups $G/G_1$ and $G_1/G_2$ have rank $d$ and $e$ respectively. If $r - b \leq d^2/2 + d/2 - d - e + (e - d/2 - d^2/4)d/2$, then $G$ is infinite.

Lemma 15. The group $H := H(9, 3) \cong G_9(3, 4)$ is infinite.

Proof. The second derived group, $K$, of $H$ has index 448 in $H$. We obtain, using a Reidemeister-Schreier rewriting procedure [10, §2.5], a presentation for $K$ on 321 generators and 768 relations. Now $K$ has abelian quotient invariants $2^{36}4^7$. Let $Q$ denote its 2-quotient of $p$-class 2: $Q$ has order $2^{604}$, its Frattini quotient has rank $d = 43$, and so $e = 561$. Theorem 14 implies that $K$ is infinite. \qed

The other exceptions, $H(9, 4) \cong G_9(1, 3)$ and $H(9, 7) \cong G_9(1, 4)$, remain unresolved.

References


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