THE ALPERIN AND UNO CONJECTURES FOR THE FISCHER SIMPLE GROUP Fi_{22}

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Abstract

Using the local subgroup strategy of [3] and [4], we classify the radical subgroups and chains of the Fischer simple group Fi_{22} and verify the Alperin weight conjecture and the Uno reductive conjecture for this group; the latter is a refinement of the Dade reductive and Isaacs-Navarro conjectures.

1 Introduction

Let G be a finite group, p a prime and B a p-block of G. Alperin [1] conjectured that the number of B-weights equals the number of irreducible Brauer characters of B. Dade [11] generalized the Knörr-Robinson version of the Alperin weight conjecture and presented his ordinary conjecture exhibiting the number of ordinary irreducible characters of a fixed defect in B in terms of an alternating sum of related values for p-blocks of some p-local subgroups of G. Dade [12] presented several other forms of his conjecture and announced that his final conjecture needs only to be verified for finite non-abelian simple groups; in addition, if a finite group has a cyclic outer automorphism group, then the projective invariant conjecture is equivalent to the final conjecture. Recently, Isaacs and Navarro [16] proposed a new conjecture which is a refinement of the Alperin-McKay conjecture, and Uno [19] raised an alternating sum version of the Isaacs-Navarro conjecture which is a refinement of the Dade conjecture.

In [3] and [4], we presented a local subgroup strategy to decide the Alperin and Dade conjectures for the finite simple groups and demonstrated its computational effectiveness by using it to verify the Alperin and Dade conjectures for the Conway simple group Co_2 and the Fischer simple group Fi_{23}. In this paper, we apply the strategy to verify the Alperin and Uno conjectures for the Fischer simple group Fi_{22}. Although the outlines of our computations and proofs are similar to those of the Alperin and Dade conjectures for Co_2 and Fi_{23}, the details are significantly more complex, since we must verify Uno's invariant conjecture for Fi_{22}, the projective invariant conjecture

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for the 2-covering group $2.\mathrm{Fi}_{22}$, and the projective conjecture for the 3- and 6-covering groups, $3.\mathrm{Fi}_{22}$ and $6.\mathrm{Fi}_{22}$.

The challenge in deciding the conjectures for these groups is to determine the character tables of the normalizers of the radical 2- and 3-chains. Our approach to determining the character tables of the normalizers is similar to that employed in Fi_{23}, but the calculations are more complicated because of the large degrees, 185 328 and 370 656, of the faithful permutation representations of $3.\mathrm{Fi}_{22}$ and $6.\mathrm{Fi}_{22}$. If a relevant normalizer is a maximal subgroup of a covering group, then its character table is usually available in a character table library supplied with the computational algebra system GAP [18]. Otherwise, we constructed a "useful" representation of the normalizer and attempted to compute directly its character table; if this construction failed, we used lifting of characters of quotient groups, and induction and decomposition of characters of subgroups of the normalizer to obtain its character table. We present the details in Section 6.

The paper is organized as follows. In Section 2, we fix notation and state the Alperin weight conjecture, the Isaacs-Navarro conjecture, the Dade and Uno invariant, projective, and projective invariant conjectures, and state a useful lemma. In Section 3, we recall our (modified) local subgroup strategy and explain how we applied it to determine the radical subgroups of Fi_{22}. In Section 4, we classify the radical subgroups of Fi_{22} up to conjugacy and verify the Alperin weight conjecture. The classification of radical 2-subgroups of Fi_{22} was given in [4, p. 631]; we report these in detail in Section 4 because many subgroups there will be used in the next section to construct radical 2-chains. In Section 5, we do some cancellations in the alternating sum of Uno's conjecture when $p = 2$ or 3, and then determine radical chains (up to conjugacy) and their local structures. In Section 6, we verify Uno's invariant conjecture for Fi_{22}. In Section 7, we verify Uno's projective invariant conjecture for $2.\mathrm{Fi}_{22}$, and in Sections 8 and 9, we verify Uno's projective conjecture for $3.\mathrm{Fi}_{22}$ and $6.\mathrm{Fi}_{22}$, respectively.

2 The conjectures and a lemma

Let R be a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and let $\mathrm{Blk}(G)$ be the set of p-blocks, $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a B*-weight* if $\mathrm{d}(\varphi) = 0$ and $B(\varphi)^G = B$ (in the sense of Brauer), where $\mathrm{d}(\varphi) = \log_p(|G|_p) - \log_p(\varphi(1)_p)$ is the p-defect of φ and $B(\varphi)$ is the block of $N(R)$ containing φ. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights, and $\ell(B)$ the number of irreducible Brauer characters of B. Alperin conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$.

Given a p-subgroup chain

$$C : P_0 < P_1 < \cdots < P_n \tag{2.1}$$

of G, define $|C| = n$, the k-th subchain $C_k : P_0 < P_1 < \cdots < P_k$, and

$$N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \cdots \cap N(P_n). \tag{2.2}$$

The chain C is *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$ for $1 \le k \le n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G. Let $B \in \mathrm{Blk}(G)$ and let $D(B)$ be a defect group of B. The p-local rank (see [2]) of B is the number

$$plr(B) = \max\{|C| : C \in \mathcal{R},\ C : P_0 < P_1 < \cdots < P_n \le D(B)\}.$$

Let Z be a cyclic group and $\hat{G} = Z.G$ a central extension of Z by G, and $C \in \mathcal{R}(G)$. Denote by $N_{\hat{G}}(C)$ the preimage $\eta^{-1}(N(C))$ of $N(C)$ in $\hat{G}$, where η is the natural group homomorphism from $\hat{G}$ onto G with kernel Z. Let ρ be a faithful linear character of Z and $\hat{B}$ a block of $\hat{G}$ covering the block $B(\rho)$ of Z containing ρ. Denote by $\mathrm{Irr}(N_{\hat{G}}(C), \hat{B}, d, \rho)$ the set of irreducible characters ψ of $N_{\hat{G}}(C)$ such that ψ lies over ρ, $\mathrm{d}(\psi) = d$ and $B(\psi)^{\hat{G}} = \hat{B}$ and set $\mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \rho) = |\mathrm{Irr}(N_{\hat{G}}(C), \hat{B}, d, \rho)|$.

Dade's Projective Conjecture [12]. If $O_p(G) = 1$ and $\hat{B}$ is a p-block of $\hat{G}$ covering $B(\rho)$ with defect group $D(\hat{B}) \ne O_p(Z)$, then

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \rho) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

Let $\hat{H}$ be a subgroup of a finite group $\hat{G}$, $\varphi \in \mathrm{Irr}(\hat{H})$ and let $r(\varphi) = r_p(\varphi)$ be the integer $0 < r(\varphi) \le (p-1)$ such that the p'-part $(|\hat{H}|/\varphi(1))_{p'}$ of $|\hat{H}|/\varphi(1)$ satisfies

$$\left(\frac{|\hat{H}|}{\varphi(1)}\right)_{p'} \equiv r(\varphi) \pmod p.$$

Given $1 \le r \le (p-1)/2$, let $\mathrm{Irr}(\hat{H}, [r])$ be the subset of $\mathrm{Irr}(\hat{H})$ consisting of characters φ such that $r(\varphi) \equiv \pm r \pmod p$, and let $\mathrm{Irr}(\hat{H}, \hat{B}, d, \rho, [r]) = \mathrm{Irr}(\hat{H}, \hat{B}, d, \rho) \cap \mathrm{Irr}(\hat{H}, [r])$ and $\mathrm{k}(\hat{H}, \hat{B}, d, \rho, [r]) = |\mathrm{Irr}(\hat{H}, \hat{B}, d, \rho, [r])|$.

Suppose $Z = 1$ and let $\hat{B} = B \in \mathrm{Blk}(G)$ with a defect group $D = D(B)$ and the Brauer correspondent $b \in \mathrm{Blk}(N_G(D))$. Then $\mathrm{k}(N(D), B, d(B), [r])$ is the number of characters $\varphi \in \mathrm{Irr}(b)$ such that φ has height 0 and $r(\varphi) \equiv \pm r \pmod p$, where $\mathrm{d}(B)$ is the defect of B.

Isaacs-Navarro Conjecture [16, Conjecture B]. *In the notation above,*

$$\mathrm{k}(G, B, \mathrm{d}(B), [r]) = \mathrm{k}(N(D), B, \mathrm{d}(B), [r]).$$

The following refinement of Dade's conjecture is due to Uno.

Uno's Projective Conjecture [19, Conjecture 3.2]. *If $O_p(G) = 1$ and if $D(\hat{B}) \ne O_p(Z)$, then for all integers $d \ge 0$, faithful $\rho \in \mathrm{Irr}(Z)$ and $1 \le r \le (p-1)/2$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \rho, [r]) = 0. \tag{2.3}$$

Similarly, if $Z = 1$, then the projective conjecture is the ordinary conjecture.

Note that if $p = 2$ or 3, then the conjecture is equivalent to Dade's conjecture.

If, moreover, $\hat{E}$ is an extension of $\hat{G}$ centralizing Z and $N_{\hat{E}}(C, \psi)$ is the stabilizer of $(N_{\hat{G}}(C), \psi)$ in $\hat{E}$, then $N_{\hat{E}/\hat{G}}(C, \psi) = N_{\hat{E}}(C, \psi)/N_{\hat{G}}(C)$ is a subgroup of $\hat{E}/\hat{G}$. For a subgroup $\hat{U} \leq \hat{E}/\hat{G}$, denote by $\mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \hat{U}, \rho, [r])$ the number of characters ψ in $\mathrm{Irr}(N_{\hat{G}}(C), \hat{B}, d, \rho, [r])$ such that $N_{\hat{E}/\hat{G}}(C, \psi) = \hat{U}$. In the notation above, Uno's projective invariant conjecture is stated as follows.

Uno's Projective Invariant Conjecture. If $O_p(G) = 1$ and $\hat{B}$ is a p-block of $\hat{G}$ covering $B(\rho)$ with $D(\hat{B}) \neq O_p(Z)$, then

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \hat{U}, \rho, [r]) = 0. \tag{2.4}$$

In addition, if $\hat{E}/\hat{G}$ is cyclic and $u = |\hat{U}|$, then we set

$$\mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, u, \rho) = \mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \hat{U}, \rho).$$

In particular, if $Z = 1$ and ρ is the trivial character of Z, then $\hat{G} = G$ and $\hat{B}$ is a block B of G; we set $U = \hat{U}$ and

$$\mathrm{k}(N(C), B, d, U) = \mathrm{k}(N_{\hat{G}}(C), \hat{B}, \hat{U}, \rho).$$

Then the Projective Invariant Conjecture is equivalent to the Invariant Conjecture.

Uno's Invariant Conjecture. If $O_p(G) = 1$ and B is a p-block of G with defect $\mathrm{d}(B) > 0$, then

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N(C), B, d, U, [r]) = 0. \tag{2.5}$$

Remark 2.1 *Suppose $p = 5$ and $\hat{G}$ is a covering group of $G = \mathrm{Fi}_{22}$. Then an elementary abelian group p^2 is a Sylow subgroup of G. Let $B \in \mathrm{Blk}(\hat{G})$ with $D(B) \simeq p^2$, so that B has p-local rank two (see [2]). Let R be a radical subgroup of G such that $R < D(B)$ and $b \in \mathrm{Blk}(N_G(R))$ with $b^G = B$. Then b has p-local rank one or zero, and by [2, Theorem 1.4], Uno's projective conjecture holds for b. In particular,*

$$\mathrm{k}(N_{\hat{G}}(R), b, d, \rho, [r]) = \mathrm{k}(N_{\hat{G}}(D(b)), b, d, \rho, [r]) \tag{2.6}$$

for any $\rho \in \mathrm{Irr}(Z(\hat{G}))$. Thus Uno's projective conjecture for B is equivalent to the equation

$$\mathrm{k}(\hat{G}, B, d, \rho, [r]) = \mathrm{k}(N_{\hat{G}}(D(B)), B, d, \rho, [r]). \tag{2.7}$$

In Section 5, we will use the following lemma, whose proof is straightforward.

Lemma 2.2 *Let $\sigma : O_p(G) < P_1 < \ldots < P_{m-1} < Q = P_m < P_{m+1} < \ldots < P_\ell$ be a fixed radical p-chain of a finite group G, where $1 \leq m < \ell$. Suppose*

$$\sigma' : O_p(G) < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_\ell$$

is also a radical p-chain such that $N_G(\sigma) = N_G(\sigma')$. Let $\mathcal{R}^-(\sigma, Q)$ be the subfamily of $\mathcal{R}(G)$ consisting of chains C whose $(\ell-1)$-th subchain $C_{\ell-1}$ is conjugate to σ' in G, and $\mathcal{R}^0(\sigma, Q)$ the subfamily of $\mathcal{R}(G)$ consisting of chains C whose ℓ-th subchain C_ℓ is conjugate to σ in G. Then the map g sending any $O_p(G) < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_\ell < \ldots$ in $\mathcal{R}^-(\sigma, Q)$ to $O_p(G) < P_1 < \ldots < P_{m-1} < Q < P_{m+1} < \ldots < P_\ell < \ldots$ induces a bijection, denoted again by g, from $\mathcal{R}^-(\sigma, Q)$ onto $\mathcal{R}^0(\sigma, Q)$. Moreover, for any C in $\mathcal{R}^-(\sigma, Q)$, we have $|C| = |g(C)| - 1$ and $N_G(C) = N_G(g(C))$.

3 The modified local strategy

The maximal subgroups of Fi_{22} were classified by Flaass [14] and Wilson [21]. Using this classification, we know that each radical 2- or 3-subgroup R of Fi_{22} is radical in one of the nine maximal subgroups M of Fi_{22} and further that $N_{\mathrm{Fi}_{22}}(R) = N_M(R)$.

In [3] and [4], a modified local subgroup strategy was developed to classify the radical subgroups R. We review this method here. Suppose M is a subgroup of G such that $N_M(R) = N_G(R)$.

Step (1). We first consider the case where M is a p-local subgroup. Let $Q = O_p(M)$, so that $Q \leq R$. Choose a subgroup X of M. We explicitly compute the coset action of M on the cosets of X in M; we obtain a group W representing this action, a group homomorphism f from M to W, and the kernel K of f. For a suitable X, we have $K = Q$ and the degree of the action of W on the cosets is usually much smaller than that of M. We can now directly classify the radical p-subgroup classes of W, and the preimages in M of the radical subgroup classes of W are the radical subgroup classes of M.

Step (2). Now consider the case where M is not p-local. We may be able to find its radical p-subgroup classes directly. Alternatively, we find a (maximal) subgroup K of M such that $N_K(R) = N_M(R)$ for each radical subgroup R of M. If K is p-local, then we apply Step (1) to K. If K is not p-local, we can replace M by K and repeat Step (2).

Steps (1) and (2) constitute the *modified local strategy.* After applying the strategy, possible fusions among the resulting list of radical subgroups can be decided readily by testing whether the subgroups in the list are pairwise Fi_{22}-conjugate.

In investigating the conjectures for Fi_{22}, we used the minimal degree representation of Fi_{22} as a permutation group on 3510 points. Its maximal subgroups were constructed using the details supplied in [9] and the black-box algorithms of Wilson [20]. We also made extensive use of the algorithm described in [10] to construct random elements, and the procedures described in [3] and [4] for deciding the conjectures. We used the minimal degree representation of $\mathrm{Fi}_{22}.2$ as a permutation group on 3510 points, and that of $2.\mathrm{Fi}_{22}$ as a permutation subgroup of Fi_{23} on 31 671 points. In investigating the projective conjecture for $3.\mathrm{Fi}_{22}$ and $6.\mathrm{Fi}_{22}$, we constructed and used representations of $3.\mathrm{Fi}_{22}$ (and $3.\mathrm{Fi}_{22}.2$) and $6.\mathrm{Fi}_{22}$ as permutation groups on 185 328 and 370 656 points, respectively. The representation of $6.\mathrm{Fi}_{22}$ is available from the ATLAS of Finite Group Representations [6].

The computations reported in this paper were carried out using Magma V.2.6-2 on a Sun UltraSPARC Enterprise 4000 server.

4 Radical subgroups and weights of Fi_{22}

Let $\Phi(G,p)$ be a set of representatives for conjugacy classes of radical subgroups of G. For $H, K \leq G$, we write $H \leq_G K$ if $x^{-1}Hx \leq K$; and write $H \in_G \Phi(G,p)$ if $x^{-1}Hx \in \Phi(G,p)$ for some $x \in G$. We shall follow the notation of [9]. In particular, if p is odd, then $p_+^{1+2\gamma}$ is an extra-special group of order $p^{1+2\gamma}$ with exponent p; if δ is $+$ or $-$, then $2_\delta^{1+2\gamma}$ is an extra-special group of order $2^{1+2\gamma}$ with type δ. If X and Y

are groups, we use $X.Y$ and $X{:}Y$ to denote an extension and a split extension of X by Y, respectively. Given a positive integer n, we use E_{p^n} or simply p^n to denote the elementary abelian group of order p^n, $\mathbb{Z}_n$ or simply n to denote the cyclic group of order n, and D_{2n} to denote the dihedral group of order $2n$.

Let G be the simple Fischer group Fi_{22} and $E = Aut(G) = G.2$. Then

$$|G| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13,$$

and we may suppose $p \in \{2, 3, 5\}$, since both conjectures hold for a block with a cyclic defect group by [11, Theorem 7.1] and [2, Theorem 5.1].

We denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H and by $\mathrm{d}(H)$ the number $\log_p(|H|)$. Given $R \in \Phi(G, p)$, let $C(R) = C_G(R)$ and $N = N_G(R)$. If $B_0 = B_0(G)$ is the principal p-block of G, then (c.f. (4.1) of [3])

$$\mathcal{W}(B_0) = \sum_R |\mathrm{Irr}^0(N/C(R)R)|, \tag{4.1}$$

where the summation runs over the subgroups R in $\Phi(G, p)$ for which $\mathrm{d}(C(R)R/R) = 0$. The character table of $N/C(R)R$ can be calculated by Magma, and so we find $|\mathrm{Irr}^0(N/C(R)R)|$.

Proposition 4.1 *Let $G = \mathrm{Fi}_{22}$ and $E = \mathrm{Aut}(G) = G.2$. Then the non-trivial radical p-subgroups R of G (up to conjugacy) and their local structures are given in Tables 1 and 2 according as $p \geq 3$ or $p = 2$, where $S \in \mathrm{Syl}_3(G)$ is a Sylow 3-subgroup, H^* denotes a subgroup of G such that $H^* \simeq H$ and $H^* \neq_G H$. If $p = 3$, then τ permutes the pairs $(3^{3+3}, (3^{3+3})^*)$ and $(3_+^{1+6}.3, (3_+^{1+6}.3)^*)$ for some $\tau \in E \backslash G$. If $p = 2$, then $N_E(R) = N.2$ for all radical subgroups R.*

R	$C(R)$	$N_G(R)$	$N_E(R)$	$\|\mathrm{Irr}^0(N/C(R)R)\|$
5	$5 \times S_5$	$5{:}4 \times S_5$	$2 \times 5{:}4 \times S_5$	
5^2	5^2	$5^2{:}4S_4$	$2 \times 5^2{:}4S_4$	16
3	$3 \times U_4(3){:}2$	$S_3 \times U_4(3){:}2$	$N.2$	
3^5	3^5	$3^5{:}U_4(2){:}2$	$N.2$	2
3^{3+3}	3^3	$3^{3+3}{:}L_3(3)$	N	1
$(3^{3+3})^*$	3^3	$(3^{3+3})^*{:}L_3(3)$	N	1
3_+^{1+6}	3	$3_+^{1+6}.2^{3+4}{:}3^2{:}2$	$N.2$	4
$3_+^{1+6}.3$	3	$3_+^{1+6}.3{:}2S_4$	N	2
$3^5.3^3$	3	$3^5.3^3.(2 \times S_4)$	$N.2$	4
$(3_+^{1+6}.3)^*$	3	$(3_+^{1+6}.3)^*.2S_4$	N	2
$3^5{:}3_+^{1+2}$	3^2	$3^5{:}3_+^{1+2}.2S_4$	$N.2$	2
S	3	$S.2^2$	$N.2$	4

Table 1: Non-trivial radical p-subgroups of Fi_{22} with $p \geq 3$

Proof: Case (1). Suppose $p = 5$, so that by [9, p. 156], $G = \mathrm{Fi}_{22}$ has a unique class of elements x of order 5 and $C(x) = 5 \times S_5$ and $N(\langle x \rangle) = 5{:}4 \times S_5$ (see [21, p. 207]). Thus $S \in \mathrm{Syl}_5(G)$ is elementary abelian of order 25, $N(S) = 5^2{:}\,4S_4$ and $N_{N(\langle x \rangle)}(S) = 5{:}4 \times 5{:}4$.

Case (2). Suppose $p = 3$, $i \in \{1, \dots, 4\}$, and M_i is a maximal subgroup of G where $M_1 = N(3A) \simeq S_3 \times U_4(3){:}\,2$, $M_2 = N(3B) \simeq 3_+^{1+6}{:}2^{3+4}{:}3^2{:}2$, $M_3 \simeq O_7(3) \simeq M_4$. By [21, p. 203], we may suppose a 3-local subgroup R of G is a subgroup of some M_i with $N_G(R) = N_{M_i}(R)$.

The subgroups M_1 and M_2 are normalizers of some $3A$ and $3B$ elements, so we can easily construct them in G. The generators of a maximal subgroup, say $M_3 \simeq O_7(3)$ are identified by Wilson [6], using standard generators of Fi_{22}. Construct a maximal subgroup $K \simeq 3^5{:}\,U_4(2){:}\,2$ of M_3 and then use the modified local strategy to obtain all radical subgroups of K (see Case (2.3) below). One of the radical subgroups R of K is isomorphic to $3^5{:}\,3_+^{1+2}$ with center $Z(R) = 3^2$. The group R has exactly two conjugacy classes of subgroups, say Q_1 and Q_2, satisfying the following conditions: $|Q_1| = |Q_2| = 3^6$, $Q_1 \simeq Q_2 \simeq 3^{3+3}$, $N_{M_3}(Q_1) = N_G(Q_1) \simeq 3^{3+3}{:}\,L_3(3)$ and $N_{M_3}(Q_2) \neq N_G(Q_2) \simeq 3^{3+3}{:}\,L_3(3)$. The group $N_G(Q_2)$ is a maximal subgroup of $M_4 \simeq O_7(3)$, and M_4 can be constructed using $N_G(Q_2)$ and some random element of G. We then apply the modified local strategy to each M_i.

Case (2.1) We may take

$$\Phi(M_2, 3) = \{3_+^{1+6}, 3_+^{1+6}.3, 3^5.3^3, (3_+^{1+6}.3)^*, S\}, \tag{4.2}$$

and moreover, $N(R) = N_{M_2}(R)$ for each $R \in \Phi(M_2, 3)$. We may suppose $\Phi(M_2, 3) \subseteq \Phi(G, 3)$. Since $N_E(3_+^{1+6}.3) = N_G(3_+^{1+6}.3)$ and $N_E(3_+^{1+6}) = N_G(3_+^{1+6}).2$, it follows that τ permutes the pair $(3_+^{1+6}.3, (3_+^{1+6}.3)^*)$.

Case (2.2) We may take

$$\Phi(M_1, 3) = \{3, 3^5, 3 \times 3_+^{1+4}, S'\}, \tag{4.3}$$

and moreover, $N(R) \neq N_{M_1}(R)$ for $R \in \Phi(M_1, 3) \backslash \{3\}$ and $N_E(R) = N_G(R).2$, where $S' \in \mathrm{Syl}_3(M_1)$. In addition, $C(3^5) = 3^5$, $N(3^5) \simeq S_3 \times 3^4{:}\,S_6$, $C(3 \times 3_+^{1+4}) = 3^2$, $N(3 \times 3_+^{1+4}) \simeq S_3 \times 3_+^{1+4}.2S_4{:}\,2$, $C(S') = 3^2$, $N(S') \simeq S_3 \times 3_+^{1+4}.3.D_8$.

Case (2.3) Let $L_1 \simeq 3_+^{1+6}{:}(2A_4 \times A_4).2$, $L_2 \simeq 3^{3+3}{:}L_3(3)$ and $L_3 = 3^5{:}U_4(2){:}2$ be the maximal subgroups of $M_3 = O_7(3)$ (cf. [9, p. 109]), so that by the Borel-Tits theorem [8], we may suppose each radical subgroup of $O_7(3)$ is a subgroup of $\Phi(L_i, 3)$ for some i with $N(R) = N_{L_i}(R)$.

We may take

$$\Phi(L_1, 3) = \{3_+^{1+6}, 3^5{:}\,3^3, (3_+^{1+6}.3)^*, S\}, \tag{4.4}$$

and moreover, $N(R) = N_{M_3}(R) = N_{L_1}(R)$ for $R \in \Phi(L_1, 3) \backslash \{3_+^{1+6}\}$.

We may take

$$\Phi(L_2, 3) = \{3^{3+3}, 3_+^{1+6}.3, 3^5{:}3_+^{1+2}, S\} \subseteq_G \Phi(G, 3), \tag{4.5}$$

and then $N(R) = N_{M_3}(R)$ for all $R \in \Phi(L_2, 3)$.

We may take

$$\Phi(L_3,3)=\{3^5, 3^5{:}\,3^3, 3^5{:}3_+^{1+2}, S\}\subseteq_G \Phi(G,3), \tag{4.6}$$

and so $N(R)=N_{M_3}(R)=N_{L_3}(R)$ for all $R\in\Phi(L_3,3)$.

It follows that

$$\Phi(M_3,3)=\{3^5, 3^{3+3}, 3_+^{1+6}, (3_+^{1+6}.3)^*, 3^5{:}\,3^3, 3^5{:}\,3_+^{1+2}, S\},$$

and $N(R)=N_{M_3}(R)$ for $R\in\Phi(M_3,3)\backslash\{3_+^{1+6}\}$.

Case (2.4) Since M_3 and M_4 are conjugate in E and $O_3(M_2)=_G O_3(L_1)$, it follows that

$$\Phi(M_4,3)=\{3^5, (3^{3+3})^*, 3_+^{1+6}, 3_+^{1+6}.3, 3^5{:}\,3^3, 3^5{:}\,3_+^{1+2}, S\}.$$

R	$C(R)$	N	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
2	$2.U_6(2)$	$2.U_6(2)$	
2^6	2^6	$2^6{:}\,S_6(2)$	1
2^{10}	2^{10}	$2^{10}{:}\,M_{22}$	0
$2\times 2_+^{1+8}$	2^2	$(2\times 2_+^{1+8}{:}U_4(2)){:}2$	0
$2^6.2^5$	2	$2^6.2^5.S_6$	1
$2^{10}.2^3$	2^3	$2^{10}.2^3.L_3(2)$	1
2^{5+8}	2^5	$2^{5+8}.(S_3\times A_6)$	2
$2^{5+8}.2$	2^5	$2^{5+8}.2.A_6$	2
$2^{10}.2^4$	2^2	$2^{10}.2^4.S_5$	0
$2^{5+8}.2^2$	2^2	$2^{5+8}.2^2.(S_3\times S_3)$	1
$(2\times 2_+^{1+8}).2.2^4$	2^2	$(2\times 2_+^{1+8}).2.2^4.(S_3\times S_3)$	1
$2^{10}.2^2.2^3$	2	$2^{10}.2^2.2^3.S_3$	1
$2^{10}.2^2.2^4$	2^2	$2^{10}.2^2.2^4.S_3$	1
$2^{5+8}.D_8$	2	$2^{5+8}.D_8.S_3$	1
$2^{5+8}.2^3$	2^2	$2^{5+8}.2^3.S_3$	1
$2^{5+8}.D_8.2$	2	$2^{5+8}.D_8.2$	1

Table 2: Non-trivial radical 2-subgroups of Fi_{22}

Case (3) For $1\le i\le 5$, let M_i be the maximal subgroups of G such that $M_1\simeq 2.U_6(2)$, $M_2\simeq 2^{10}{:}M_{22}$, $M_3\simeq 2^6.S_6(2)$, $M_4\simeq(2\times 2_+^{1+8}{:}\,U_4(2)){:}2$ and $M_5\simeq 2^{5+8}{:}(S_3\times A_6)$. If R is a non-trivial radical 2-subgroup of G, then by Wilson [21, Proposition 4.4] or Flaass [14], we may suppose $R\in\Phi(M_i,2)$ such that $N(R)=N_{M_i}(R)$ for some i.

Case (3.1) We may take

$$\Phi(M_4,2)=\{2{\times}2_+^{1+8}, 2^6.2^5, 2^{10}.2^4, (2{\times}2_+^{1+8}).2.2^4, 2^{10}.2^2.2^3, 2^{10}.2^2.2^4, 2^{5+8}.D_8, 2^{5+8}.D_8.2\}$$

and moreover, $N(R)=N_{M_4}(R)$ for each $R\in\Phi(M_4,2)$.

Case (3.2) Let $K_1 = (2 \times 2_+^{1+8}){:}U_4(2)$, $K_2 = 2^{10}{:}L_3(4)$, $K_3 = 2^{5+8}.(3 \times A_5).2$ be maximal subgroups of $M_1 \simeq 2.U_6(2)$. Then we may suppose each $R \in \Phi(M_1, 2)$ is a subgroup of $\Phi(K_i, 2)$ for some i such that $N_{M_1}(R) = N_{K_i}(R)$.

We may take

$$\Phi(K_1, 2) = \{2 \times 2_+^{1+8}, 2^{10}.2^4, (2 \times 2_+^{1+8}).2.2^4, 2^{10}.2^2.2^4\},$$

and so $N_{K_1}(R) = N_{M_1}(R) \neq N(R)$ for each $R \in \Phi(K_1, 2)$. Moreover, $C(R) \simeq 2^2$ and

$$N_{M_1}(R) = \begin{cases} (2 \times 2_+^{1+8}){:}\,U_4(2) & \text{if } R = 2 \times 2_+^{1+8}, \\ 2^{10}.2^4.A_5 & \text{if } R = 2^{10}.2^4, \\ (2 \times 2_+^{1+8}).2.(A_4 \times A_4).2 & \text{if } R = (2 \times 2_+^{1+8}).2.2^4, \\ 2^{10}.2^2.2^4.3 & \text{if } R = 2^{10}.2^2.2^4. \end{cases}$$

We may take

$$\Phi(K_2, 2) = \{2^{10}, 2^{10}.2^4, 2^{5+8}.2, 2^{10}.2^2.2^4\},$$

and moreover, $N_{K_2}(R) = N_{M_1}(R) \neq N(R)$ for each $R \in \Phi(K_2, 2)$, $C(2^{10}) = 2^{10}$, $C(2^{5+8}.2) = 2^5$, $C(2^{10}.2^4) = 2^2 = C(2^{10}.2^2.2^4)$, and $N_{M_1}(2^{5+8}.2) = 2^{5+8}.2A_5$.

We may take

$$\Phi(K_3, 2) = \{2^{5+8}, 2^{5+8}.2, (2 \times 2_+^{1+8})2.2^4, 2^{10}.2^2.2^4\},$$

and moreover, $N_{K_3}(R) = N_{M_2}(R) \neq N(R)$ for each $R \in \Phi(K_3, 2)$.

It follows that

$$\Phi(M_1, 2) = \{2, 2 \times 2_+^{1+8}, 2^{10}, 2^{5+8}, 2^{10}.2^4, 2^{5+8}.2, (2 \times 2_+^{1+8}).2.2^4, 2^{10}.2^2.2^4\},$$

and $N_{M_1}(R) \neq N(R)$ for each $R \in \Phi(M_1, 2) \backslash \{2\}$.

Case (3.3) We may take

$$\Phi(M_2, 2) = \{2^{10}, 2^{10}.2^3, 2^{10}.2^4, 2^{5+8}.2, 2^{10}.2^2.2^3, 2^{5+8}.2^3, 2^{10}.2^2.2^4, 2^{5+8}.D_8.2\},$$

and moreover, $N(R) = N_{M_2}(R)$ for each $R \in \Phi(M_2, 2)$.

Case (3.4) We may take

$$\Phi(M_5, 2) = \{2^{5+8}, 2^{5+8}.2, 2^{5+8}.2^2, (2 \times 2_+^{1+8}).2.2^4, 2^{10}.2^2.2^4, 2^{5+8}.2^3, 2^{5+8}.D_8, 2^{5+8}.D_8.2\}$$

and moreover, for $R \in \Phi(M_5, 2)$, $N(R) = N_{M_5}(R)$.

Case (3.5) We may take

$$\Phi(M_3, 2) = \{2^6, 2^6.2^5, 2^6.2^6, 2^6.2^3.2^4, 2^6.2^3.2^4.2, 2^6.2^5.2^3, (2^6.2^5.2^3)^*, S'\},$$

where $S' \in \mathrm{Syl}_2(M_3)$. In addition, $N(R) = N_{M_3}(R)$ for $R \in \{2^6, 2^6.2^5\}$ and $N(R) \neq N_{M_3}(R)$ for $R \in \Phi(M_3, 2) \backslash \{2^6, 2^6.2^5\}$. Moreover,

$$N_{M_3}(R) = \begin{cases} 2^6.2^6.L_3(2) & \text{if } R = 2^6.2^6, \\ 2^6.2^3.2^4.(S_3 \times S_3) & \text{if } R = 2^6.2^3.2^4, \\ 2^6.2^3.2^4.2.S_3 & \text{if } R = 2^6.2^3.2^4.2, \\ 2^6.2^5.2^3.S_3 & \text{if } R = 2^6.2^5.2^3, \\ (2^6.2^5.2^3)^*.S_3 & \text{if } R = (2^6.2^5.2^3)^*, \\ S' & \text{if } R = S'. \end{cases}$$

This completes the classification of radical 2-subgroups of G. The centralizers and normalizers of $R \in \Phi(G,2)$ are given by MAGMA. $\square$

Lemma 4.2 *Suppose $p = 2, 3$ or 5, and suppose G is a covering group of Fi_{22}, and ρ a faithful linear character of $Z(G)$. Let $B_0 = B_0(G)$ be the principal block of G, $\mathrm{Irr}^+(G \mid \rho)$ the set of characters of $\mathrm{Irr}(G)$ with positive p-defect and covering ρ, and $\mathrm{Blk}^*(G, p, \rho)$ the set of p-blocks of G with a non-cyclic defect group and covering the block $B(\rho)$. If a defect group $D(B)$ of $B \in \mathrm{Blk}(G)$ is cyclic, then $\mathrm{Irr}(B)$ and $\ell(B)$ are given by [15, p. 218].*

(a) *If $G = \mathrm{Fi}_{22}$, then ρ is the trivial character, $\mathrm{Blk}^*(G, p, \rho) = \{B_0\}$, $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G \mid \rho) \backslash (\cup_{B'} \mathrm{Irr}(B'))$ and*

$$\ell(B_0) = \begin{cases} 16 & \textit{if } p = 5, \\ 22 & \textit{if } p = 3, \\ 14 & \textit{if } p = 2, \end{cases}$$

where B' runs over the blocks of G with cyclic defect groups.

(b) *Suppose $G = 2.\mathrm{Fi}_{22}$, so that $Z(G)$ has a unique faithful linear character ξ. If $p = 3, 5$, then $\mathrm{Blk}^*(G, p, \xi) = \{B_1\}$ such that $\mathrm{Irr}(B_1) = \mathrm{Irr}^+(G \mid \xi) \backslash (\cup_{B'} \mathrm{Irr}(B'))$; if $p = 2$, then $\mathrm{Blk}^*(G, p, \xi) = \{B_0, B_1\}$ and in the notation of* [9, p. 156], $\mathrm{Irr}(B_1) = \{\chi_{63}, \chi_{64}, \chi_{113}, \chi_{114}\}$, *$D(B_1) \simeq 2^2$ and $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash (\mathrm{Irr}(B_1) \cup (\cup_{B'} \mathrm{Irr}(B')))$, where $\mathrm{Irr}^+(G)$ is the character in $\mathrm{Irr}(G)$ with positive defect and B' runs over the blocks of G with cyclic defect groups.*

(c) *Suppose $G = 3.\mathrm{Fi}_{22}$, so that $Z(G)$ has two faithful linear characters ζ_1 and ζ_2. If $p = 2, 5$, then $\mathrm{Blk}^*(G, p, \zeta_i) = \{B_i\}$, and $\mathrm{Irr}(B_i) = \mathrm{Irr}^+(G \mid \zeta_i) \backslash (\cup_{B'} \mathrm{Irr}(B'))$; if $p = 3$, then $\mathrm{Blk}^*(G, p, \zeta_i) = \{B_0, B_i\}$ and in the notation of* [9, p. 156], *we may suppose*

$$\mathrm{Irr}(B_i) = \begin{cases} \{\chi_j : j \in \{29, 38, 49, 98, 99, 114, 115, 126, 127\}\} & \textit{if } i = 1, \\ \{\chi_j : j \in \{48, 58, 65, 124, 125, 156, 157, 162, 163\}\} & \textit{if } i = 2, \end{cases}$$

and $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash (\mathrm{Irr}(B_1) \cup \mathrm{Irr}(B_2) \cup (\cup_{B'} \mathrm{Irr}(B')))$, where B' runs over the blocks of G with cyclic defect groups.

(d) *Suppose $G = 6.\mathrm{Fi}_{22}$, so that $Z(G)$ has two faithful linear characters η_1 and η_2. If $p = 2, 5$, then $\mathrm{Blk}^*(G, p, \eta_i) = \{B_i\}$ and $\mathrm{Irr}(B_i) = \mathrm{Irr}^+(G \mid \eta_i) \backslash (\cup_{B'} \mathrm{Irr}(B'))$; if $p = 3$, then $\mathrm{Blk}^*(G, p, \eta_i) = \{B_1\}$ such that*

$$(\mathrm{Irr}^+(G \mid \eta_i) \backslash (\cup_{B'} \mathrm{Irr}(B'))) \subseteq \mathrm{Irr}(B_1)$$

for $i = 1, 2$, where B' runs over the blocks of G with cyclic defect groups.

PROOF: If $B \in \mathrm{Blk}^+(G, p)$ is non-principal with $D = D(B)$, then $\mathrm{Irr}^0(C(D)D/D)$ has a non-trivial character θ and $N(\theta)/C(D)D$ is a p'-group, where $N(\theta)$ is the stabilizer of θ in $N(D)$. By Proposition 4.1, $D \in_G \{5, 3, 2\}$ and D is cyclic. In particular, $\mathrm{Irr}(B)$ and $\ell(B)$ are given by [15, p.218].

If $\ell_p(G)$ is the number of p-regular G-conjugacy classes, then $\ell(B_0)$ can be calculated by the following equation due to Brauer:

$$\ell_p(G) = \bigcup_{B \in \mathrm{Blk}^+(G,p)} \ell(B) + |\mathrm{Irr}^0(G)|.$$

Suppose $B \in \mathrm{Blk}^*(G, p, \rho)$. Using central characters one can show $\mathrm{Irr}(B)$ has the indicated description.

□

Theorem 4.3 *Let B be a p-block of $G = \mathrm{Fi}_{22}$ with a non-cyclic defect group. Then $\mathcal{W}(B) = \ell(B)$.*

PROOF: By Lemma 4.2, $B = B_0$ and the theorem follows by Proposition 4.1, Lemma 4.2 and (4.1). □

5 Radical chains of Fi_{22}

Let $G = \mathrm{Fi}_{22}$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$. In this section, we do some cancellations in the alternating sum of Uno's conjecture. First we list some radical p-chains $C(i)$ with their normalizers, then reduce the proof of the conjecture to the subfamily $\mathcal{R}^0(G)$ of $\mathcal{R}(G)$ consisting of the union of G-orbits of all $C(i)$. The subgroups of the p-chains in Tables 3 and 4 are given either by Tables 1 and 2 or in the proofs of Proposition 4.1 and Lemma 5.1.

Lemma 5.1 *Let $G = \mathrm{Fi}_{22}$, $E = G.2$ and let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that*

$$\mathcal{R}^0(G)/G = \begin{cases} \{C(i) : 1 \leq i \leq 16\} & \text{with } C(i) \text{ given in Table 3 if } p = 3, \\ \{C(i) : 1 \leq i \leq 24\} & \text{with } C(i) \text{ given in Table 4 if } p = 2, \end{cases}$$

Then

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d, [r]) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d, [r]) \qquad (5.1)$$

for all integers $d, r \geq 0$. If $p = 3$, then an element $\tau \in E \backslash G$ permutes the pairs $(C(i), C(j))$ for $(i, j) \in \{(7, 9), (8, 10), (11, 15), (14, 16)\}$. If $p = 2$, then $N_E(C(i)) = N(C(i)).2$ for each i.

PROOF: Suppose C' is a radical chain such that

$$C' : 1 < P_1' < \ldots < P_m'. \qquad (5.2)$$

C		$N(C)$	$N_E(C)$
$C(1)$	1	Fi_{22}	$\mathrm{Fi}_{22}.2$
$C(2)$	$1 < 5$	$5{:}4 \times S_5$	$N(C) \times 2$
$C(3)$	$1 < 5 < 5^2$	$5{:}4 \times 5{:}4$	$N(C) \times 2$
$C(4)$	$1 < 5^2$	$5^2{:}4S_4$	$N(C) \times 2$
$C(1)$	1	Fi_{22}	$\mathrm{Fi}_{22}.2$
$C(2)$	$1 < 3$	$S_3 \times U_4(3){:}\,2$	$N(C).2$
$C(3)$	$1 < 3 < 3^5$	$S_3 \times 3^4{:}\,S_6$	$N(C).2$
$C(4)$	$1 < 3 < 3^5 < S'$	$S_3 \times 3_+^{1+4}.3.D_8$	$N(C).2$
$C(5)$	$1 < 3 < 3 \times 3_+^{1+4}$	$S_3 \times 3_+^{1+4}.2S_4{:}\,2$	$N(C).2$
$C(6)$	$1 < 3^5$	$3^5{:}\,U_4(2){:}\,2$	$N(C).2$
$C(7)$	$1 < 3^{3+3} < 3^5{:}\,3_+^{1+2}$	$3^5{:}\,3_+^{1+2}{:}\,2S_4$	$N(C)$
$C(8)$	$1 < 3^{3+3}$	$3^{3+3}{:}\,L_3(3)$	$N(C)$
$C(9)$	$1 < (3^{3+3})^* < 3^5{:}\,3_+^{1+2}$	$3^5{:}\,3_+^{1+2}{:}\,2S_4$	$N(C)$
$C(10)$	$1 < (3^{3+3})^*$	$(3^{3+3})^*{:}\,L_3(3)$	$N(C)$
$C(11)$	$1 < 3_+^{1+6} < 3_+^{1+6}.3$	$3_+^{1+6}.3.2S_4$	$N(C)$
$C(12)$	$1 < 3_+^{1+6}$	$3_+^{1+6}.2^{3+4}{:}\,3^2{:}\,2$	$N(C).2$
$C(13)$	$1 < 3_+^{1+6} < 3^5{:}\,3^3$	$3^5.3^3{:}\,(2 \times S_4)$	$N(C).2$
$C(14)$	$1 < 3_+^{1+6} < 3_+^{1+6}.3 < S$	$S.2^2$	$N(C)$
$C(15)$	$1 < 3_+^{1+6} < (3_+^{1+6}{:}\,3)^*$	$(3_+^{1+6}{:}\,3)^*{:}\,2S_4$	$N(C)$
$C(16)$	$1 < 3_+^{1+6} < (3_+^{1+6}.3)^* < S$	$S.2^2$	$N(C)$

Table 3: Some radical p-chains of Fi_{22} with p odd

Let $C \in \mathcal{R}(G)$ be given by (2.1) with $P_1 \in \Phi(G, 3)$.

Case (1) Suppose $p = 3$, and $R \in \Phi(L_3, 3)\backslash\{3^5\}$ given by (4.6). Let $\sigma(R) : 1 < Q = 3^5 < R$, so that $\sigma(R)' : 1 < R$. Then $\sigma(R)$ and $\sigma(R)'$ satisfy the conditions of Lemma 2.2, so there is a bijection g from $\mathcal{R}^-(\sigma(R), 3^5)$ onto $\mathcal{R}^0(\sigma(R)', 3^5)$ such that $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C'))$ and $|C'| = |g(C')| - 1$ for each $C' \in \mathcal{R}^-(\sigma(R), 3^5)$. Thus

$$\mathrm{k}(N(C'), B, d, u, [r]) = \mathrm{k}(N(g(C')), B, d, u, [r]), \tag{5.3}$$

and we may suppose

$$C \notin \bigcup_{R \in \Phi(L_3,3)\backslash\{3^5\}} (\mathcal{R}^-(\sigma(R), 3^5) \cup \mathcal{R}^0(\sigma(R), 3^5)).$$

Thus $P_1 \notin \{3^5{:}\,3^3, 3^5{:}\,3_+^{1+2}, S\}$, and if $P_1 = 3^5$ then $C =_G C(6)$. We may suppose

$$P_1 \in \{3, 3^{3+3}, (3^{3+3})^*, 3_+^{1+6}, 3_+^{1+6}.3, (3_+^{1+6}.3)^*\} \subseteq \Phi(G, 3).$$

Let $C' : 1 < 3 < S'$ and $g(C') : 1 < 3 < 3 \times 3_+^{1+4} < S'$, where $S', 3 \times 3_+^{1+4} \in \Phi(M_1, 3)$. Then $N(C') = N(g(C'))$ and $N_E(C') = N_E(g(C')) = S'.2$ and we may suppose $C \neq_G$

$C', g(C')$. Since $\Phi(S_3 \times 3^4{:}\, S_6, 3) = \{3^5, S'\} \subseteq_G \Phi(M_1, 3)$, it follows that if $P_1 = 3$, then $C \in_G \{C(2), C(3), C(4), C(5)\}$.

By (4.5), $\Phi((3^{3+3})^*{:}\, L_3(3), 3) = \{(3^{3+3})^*, 3_+^{1+6}.3, 3^5{:}\, 3_+^{1+2}, S\}$. Let $\sigma : 1 < Q = (3^{3+3})^* < 3_+^{1+6}.3$, so that $\sigma' : 1 < 3_+^{1+6}.3$. A similar proof to above shows that there is a bijection g from $\mathcal{R}^-(\sigma, (3^{3+3})^*)$ onto $\mathcal{R}^0(\sigma, (3^{3+3})^*)$ such that $N_E(C') = N(C') = N(g(C')) = N_E(g(C'))$ and $|C'| = |g(C')| - 1$ for each $C' \in \mathcal{R}^-(\sigma, (3^{3+3})^*)$, so that (5.3) holds. We may suppose $P_1 \neq_G 3_+^{1+6}.3$ and if $P_1 =_G (3^{3+3})^*$, then $P_2 \in_G \{3^5{:}\, 3_+^{1+2}, S\}$.

Let $C' : 1 < (3^{3+3})^* < S$ and $g(C') : 1 < (3^{3+3})^* < 3^5{:}3_+^{1+2} < S$. Then $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C'))$ and we may suppose $C \neq_G C', g(C')$. Thus if $P_1 = (3^{3+3})^*$, then $C \in_G \{C(9), C(10)\}$.

Let $\sigma : 1 < Q = 3^{3+3} < (3_+^{1+6}.3)^*$, so that $\sigma' : 1 < (3_+^{1+6}.3)^*$. A similar proof to above shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 3^{3+3}) \cup \mathcal{R}^0(\sigma, 3^{3+3})).$$

In particular, we may suppose $P_1 \neq_G (3_+^{1+6}.3)^*$ and moreover, if $P_1 = 3^{3+3}$, then $P_2 \neq_G (3_+^{1+6}.3)^*$.

Let $C' : 1 < 3^{3+3} < S$ and $g(C') : 1 < 3^{3+3} < 3^5{:}3_+^{1+2} < S$. Then we may suppose $C \neq_G C', g(C')$. So if $P_1 = 3^{3+3}$, then $C \in_G \{C(7), C(8)\}$.

Let $C' : 1 < 3_+^{1+6} < S$ and $g(C') : 1 < 3_+^{1+6} < 3^5{:}\, 3^3 < S$. Then $N(C') = N(g(C'))$ and $N_E(C) = N_E(g(C')) = N(C').2$, so that (5.3) holds. Thus if $P_1 = 3_+^{1+6}$, then $C \in_G \{C(i) : 11 \leq i \leq 16\}$.

Case (2) Let $R \in \Phi(M_2, 2) \backslash \{2^{10}\}$, and $\sigma(R) : 1 < Q = 2^{10} < R$, so that $\sigma(R) : 1 < R$. A similar proof to that of Case (1) shows that we may suppose

$$C \notin \bigcup_{R \in \Phi(M_2,2) \backslash \{2^{10}\}} (\mathcal{R}^-(\sigma(R), 2^{10}) \cup \mathcal{R}^0(\sigma(R), 2^{10})). \tag{5.4}$$

Thus if $P_1 = 2^{10}$, then $C =_G C(14)$. We may assume

$$P_1 \in \{2, 2^6, 2 \times 2^{1+8}, 2^6.2^5, 2^{5+8}, 2^{5+8}.2^2, (2 \times 2_+^{1+8}).2.2^4, 2^{5+8}.D_8\}.$$

Case (2.1) Let

$$\Omega = \Phi(K_1, 2) \backslash \{2 \times 2_+^{1+8}\} = \{2^{10}.2^4, (2 \times 2_+^{1+8}).2.2^4, 2^{10}.2^2.2^4\} \subseteq \Phi(M_1, 2),$$

$R \in \Omega$, and let $\sigma(R) : 1 < 2 < Q = 2 \times 2_+^{1+8} < R$, so that $\sigma(R)' : 1 < 2 < R$. A similar proof to Case (1) shows that we may suppose (5.4) holds with 2^{10} replaced by $2 \times 2_+^{1+8}$ and $\Phi(M_2, 2) \backslash \{2^{10}\}$ by Ω.

Let $2^{5+8}.2 \in \Phi(K_2, 2)$ and let $\sigma : 1 < 2 < Q = 2^{10} < 2^{5+8}.2$, so that $\sigma' : 1 < 2 < 2^{5+8}.2$. Since $N_{N_E(2)}(2^{5+8}.2) \leq N_{N_E(2)}(2^{10})$, it follows that we may suppose $C \notin \mathcal{R}^-(\sigma, 2^{10}) \cup \mathcal{R}^0(\sigma, 2^{10})$.

Let $C' : 1 < 2 < 2^{10} < 2^{10}.2^2.2^4$, $g(C') : 1 < 2 < 2^{10} < 2^{10}.2^4 < 2^{10}.2^2.2^4$, $C'' : 1 < 2 < 2^{5+8} < 2^{10}.2^2.2^4$ and $g(C'') : 1 < 2 < 2^{5+8} < (2 \times 2_+^{1+8}).2.2^4 < 2^{10}.2^2.2^4$. Then $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C')) = N(C').2$, $N(C'') = N(g(C''))$ and $N_E(C'') = N_E(g(C'')) = N(C'').2$, so that we may suppose $C \notin \{C', g(C'), C'', g(C'')\}$. It follows that if $P_1 = 2$, then $C \in_G \{C(j) : 2 \leq j \leq 9\}$.

C		$N(C)$
$C(1)$	1	Fi_{22}
$C(2)$	$1 < 2$	$2.U_6(2)$
$C(3)$	$1 < 2 < 2 \times 2_+^{1+8}$	$(2 \times 2_+^{1+8}){:}\,U_4(2)$
$C(4)$	$1 < 2 < 2^{10} < 2^{10}.2^4$	$2^{10}.2^4.A_5$
$C(5)$	$1 < 2 < 2^{10}$	$2^{10}{:}\,L_3(4)$
$C(6)$	$1 < 2 < 2^{5+8} < 2^{5+8}.2$	$2^{5+8}.2.A_5$
$C(7)$	$1 < 2 < 2^{5+8}$	$2^{5+8}.(3 \times A_5).2$
$C(8)$	$1 < 2 < 2^{5+8} < (2 \times 2_+^{1+8}).2.2^4$	$(2 \times 2_+^{1+8}).2.(A_4 \times A_4).2$
$C(9)$	$1 < 2 < 2^{5+8} < 2^{5+8}.2 < 2^{10}.2^2.2^4$	$2^{10}.2^2.2^4.3$
$C(10)$	$1 < 2^6$	$2^6.S_6(2)$
$C(11)$	$1 < 2^6 < 2^6.2^6$	$2^6.2^6.L_3(2)$
$C(12)$	$1 < 2^6 < 2^6.2^3.2^4 < 2^6.2^3.2^4.2$	$2^6.2^3.2^4.2.S_3$
$C(13)$	$1 < 2^6 < 2^6.2^3.2^4$	$2^6.2^3.2^4(S_3 \times S_3)$
$C(14)$	$1 < 2^{10}$	$2^{10}{:}\,M_{22}$
$C(15)$	$1 < 2 \times 2_+^{1+8} < 2^6.2^5$	$2^6.2^5.S_6$
$C(16)$	$1 < 2 \times 2_+^{1+8}$	$(2 \times 2_+^{1+8}{:}\,U_4(2)){:}\,2$
$C(17)$	$1 < 2 \times 2_+^{1+8} < 2^{10}.2^4$	$2^{10}.2^4.S_5$
$C(18)$	$1 < 2 \times 2_+^{1+8} < 2^6.2^5 < 2^6.2^5.2^3$	$2^6.2^5.2^3.S_3$
$C(19)$	$1 < 2 \times 2_+^{1+8} < 2^6.2^5 < 2^6.2^5.2^3 < 2^6.2^5.2^3.2$	$2^6.2^5.2^3.2$
$C(20)$	$1 < 2 \times 2_+^{1+8} < 2^6.2^5 < (2^6.2^5.2^3)^*$	$(2^6.2^5.2^3)^*.S_3$
$C(21)$	$1 < 2^{5+8} < 2^{5+8}.2$	$2^{5+8}.2A_6$
$C(22)$	$1 < 2^{5+8}$	$2^{5+8}.(S_3 \times A_6)$
$C(23)$	$1 < 2^{5+8} < (2 \times 2_+^{1+8}).2.2^4$	$(2 \times 2_+^{1+8}).2.2^4.(S_3 \times S_3)$
$C(24)$	$1 < 2^{5+8} < (2 \times 2_+^{1+8}).2.2^4 < 2^{10}.2^2.2^4$	$2^{10}.2^2.2^4.S_3$

Table 4: Some radical 2-chains of Fi_{22}

Case (2.2) If $2^6.2^6 \in \Phi(M_3, 2)$, then $N_{M_3}(2^6.2^6) = 2^6.2^6.L_3(2)$ and $N_{M_3.2}(2^6.2^6) = N_E(2^6.2^6) = N(2^6.2^6).2$. We may take

$$\Phi(2^6.2^6.L_3(2), 2) = \{2^6.2^6, 2^6.2^3.3^4.2, 2^6.2^5.2^3, S'\} \subseteq \Phi(M_3, 2),$$

and $N_{2^6.2^6.L_3(2)}(R) = N_{M_3}(R)$, $N_{M_3.2}(R) = N_{M_3}(R).2 \leq N_{M_3.2}(2^6.2^6)$ for each $R \in \Phi(2^6.2^6.L_3(2), 2)$.

Let $2^6.2^5 \in \Phi(M_3, 2)$, and $\sigma : 1 < Q = 2^6 < 2^6.2^5$, so that $\sigma' : 1 < 2^6.2^5$. A similar proof to Case (1) shows that we may suppose $C \notin \mathcal{R}^-(\sigma, 2^6) \cup \mathcal{R}^0(\sigma, 2^6)$, since $N(2^6.2^5) = N_{M_3}(2^6.2^5) = 2^6.2^5.S_6$ and $N_E(2^6.2^5) = N_{M_3.2}(2^6.2^5) = N(2^6.2^5).2$. In particular, we may suppose $P_1 \neq_G 2^6.2^5$.

Let $R \in \Phi(2^6.2^6.L_3(2), 2) \backslash \{2^6.2^6\}$, and $\sigma(R) : 1 < 2^6 < Q = 2^6.2^6 < R$, so that $\sigma(R)' : 1 < 2^6 < R$. Then we may suppose $C \notin \mathcal{R}^-(\sigma(R), 2^6.2^6) \cup \mathcal{R}^0(\sigma(R), 2^6.2^6)$.

Similarly, for $2^6.2^3.2^4 \in \Phi(M_3,2)$, $N_{M_3}(2^6.2^3.2^4) = 2^6.2^3.2^4.(S_3 \times S_3)$ and we may take

$$\Phi(2^6.2^3.2^4.(S_3 \times S_3),2) = \{2^6.2^3.2^4, 2^6.2^3.2^4.2, (2^6.2^5.2^3)^*, S'\} \subseteq \Phi(M_3,2);$$

in addition, $N_{2^6.2^3.2^4.(S_3\times S_3)}(R) = N_{M_3}(R)$ and

$$N_{M_3.2}(R) = N_{M_3}(R).2 \leq N_{M_3.2}(2^6.2^3.2^4)$$

for each $R \in \Phi(2^6.2^3.2^4.(S_3 \times S_3),2)$.

Let $\sigma : 1 < 2^6 < Q = 2^6.2^3.2^4 < (2^6.2^5.2^3)^*$, so that $\sigma' : 1 < 2^6 < (2^6.2^5.2^3)^*$. Then we may suppose $C \notin \mathcal{R}^-(\sigma, 2^6.2^3.2^4) \cup \mathcal{R}^0(\sigma, 2^6.2^3.2^4)$.

Let $C' : 1 < 2^6 < 2^6.2^3.2^4 < S'$ and $g(C') : 1 < 2^6 < 2^6.2^3.2^4 < 2^6.2^3.2^4.2 < S'$. Then $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C')) = S'.2$ and we may suppose $C \notin_G \{C', g(C')\}$. It follows that if $P_1 = 2^6$, then $C \in_G \{C(10), C(11), C(12), C(13)\}$.

Case (2.3) Let $R \in \{(2 \times 2_+^{1+8}).2.2^4, 2^{5+8}.D_8\} \subseteq \Phi(M_4,2)$, and $\sigma(R) : 1 < Q = 2 \times 2_+^{1+8} < R$, so that $\sigma(R)' : 1 < R$. Since $N(R) = N_{M_4}(R)$ and $N_E(R) = N_{M_4.2}(R) = N(R).2$, it follows that we may suppose $C \notin \mathcal{R}^-(\sigma(R), 2\times 2_+^{1+8}) \cup \mathcal{R}^0(\sigma(R), 2 \times 2_+^{1+8})$. In particular, $P_1 \neq_G \{(2 \times 2_+^{1+8}).2.2^4, 2^{5+8}.D_8\}$ and if $P_1 = 2 \times 2_+^{1+8}$ with $|C| \geq 2$, then $P_2 \in \Phi((2 \times 2_+^{1+8}{:}\, U_4(2)){:}\, 2, 2) \backslash \{2 \times 2_+^{1+8}, (2 \times 2_+^{1+8}).2.2^4, 2^{5+8}.D_8\}$.

Let $2^{10}.2^4 \in \Phi(M_4,2)$, so that $N_{M_4}(2^{10}.2^4) = N(2^{10}.2^4) = 2^{10}.2^4.S_5$ and $N_E(2^{10}.2^4) = N_{M_4.2}(2^{10}.2^4) = N(2^{10}.2^4).2$. We may take

$$\Phi(2^{10}.2^4.S_5,2) = \{2^{10}.2^4, 2^{10}.2^2.2^3, 2^{10}.2^2.2^4, S\}$$

and in addition, $N_{2^{10}.2^4.S_5}(R) = N_{M_4}(R)$ and

$$N_{M_4.2}(R) = N_{M_4}(R).2 \leq N_{M_4.2}(2^{10}.2^4)$$

for each $R \in \Phi(2^{10}.2^4.S_5,2)$.

Let $R \in \Phi(2^{10}.2^4.S_5,2)\backslash\{2^{10}.2^4\}$, and $\sigma(R) : 1 < 2 \times 2_+^{1+8} < Q = 2^{10}.2^4 < R$, so that $\sigma(R)' : 1 < 2 \times 2_+^{1+8} < R$. A similar proof to above shows that we may suppose $C \notin \mathcal{R}^-(\sigma(R), 2^{10}.2^4) \cup \mathcal{R}^0(\sigma(R), 2^{10}.2^4)$. Thus we may suppose $P_2 \notin \Phi(2^{10}.2^4.S_5,2)\backslash\{2^{10}.2^4\}$ and if $P_2 = 2^{10}.2^4$, then $|C| = 2$ and $C =_G C(17)$.

Let $2^6.2^5 \in \Phi(M_4,2)$, so that $N_{M_4}(2^6.2^5) = N(2^6.2^5) = 2^6.2^5.S_6$ and $N_E(2^6.2^5) = N_{M_4.2}(2^6.2^5) = 2^6.2^5.(S_6 \times 2)$. We may take

$$\Phi(2^6.2^5.S_6,2) = \{2^6.2^5, 2^6.2^5.2^3, (2^6.2^5.2^3)^*, 2^6.2^5.2^3.2\}$$

and in addition, $N_{2^6.2^5.S_6}(R) = R.S_3$ or R according as $R \in \{2^6.2^5.2^3, (2^6.2^5.2^3)^*\}$ or $R = 2^6.2^5.2^3.2$ and $N_{N_E(2^6.2^5)}(R) = N_{2^6.2^5.S_6}(R).2$ for each $R \in \Phi(2^6.2^5.S_6,2)$.

Let $C' : 1 < 2 \times 2_+^{1+8} < 2^6.2^5 < 2^6.2^5.2^3.2$ and $g(C') : 1 < 2 \times 2_+^{1+8} < 2^6.2^5 < (2^6.2^5.2^3)^* < 2^6.2^5.2^3.2$. Then $N(C') = N(g(C'))$, $N_E(C') = N_E(g(C')) = N(C').2$ and we may suppose $C \notin_G \{C', g(C')\}$. It follows that if $P_1 = 2 \times 2_+^{1+8}$, then

$$C \in_G \{C(j) : 15 \leq j \leq 20\}.$$

Case (2.4) Let $2^{5+8}.2^2 \in \Phi(M_5,2)$, and $\sigma : 1 < Q = 2^{5+8} < 2^{5+8}.2^2$. Since $N_E(2^{5+8}.2^2) = N_{M_5.2}(2^{5+8}.2^2)$, it follows that we may suppose $C \notin \mathcal{R}^-(\sigma, 2^{5+8}) \cup$

$\mathcal{R}^0(\sigma, 2^{5+8})$, so we may suppose $P_1 \neq_G 2^{5+8}.2$ and if $P_1 = 2^{5+8}$ with $|C| \geq 2$, then $P_2 \neq_G 2^{5+8}.2^2$.

If $2^{5+8}.2 \in \Phi(M_5, 2)$, then $N(2^{5+8}.2) = N_{M_5}(2^{5+8}.2) = 2^{5+8}.2.A_6$, $N_E(2^{5+8}.2) = N_{M_5.2}(2^{5+8}.2) = N(2^{5+8}.2).2$ and we may take

$$\Phi(2^{5+8}.2.A_6, 2) = \{2^{5+8}.2, 2^{10}.2^2.2^4, 2^{5+8}.2^3, S\}.$$

In addition, $N_{2^{5+8}.2.A_6}(R) = N_{M_5}(R)$ and $N_{M_5.2}(R) = N_{M_5}(R).2 \leq N_{M_5.2}(2^{5+8}.2)$ for each $R \in \Phi(2^{5+8}.2.A_6, 2)$.

Let $R \in \{2^{10}.2^2.2^4, 2^{5+8}.2^3\}$, and let $\sigma(R) : 1 < 2^{5+8} < Q = 2^{5+8}.2 < R$. Then we may suppose $C \notin \mathcal{R}^-(\sigma(R), 2^{5+8}.2) \cup \mathcal{R}^0(\sigma(R), 2^{5+8}.2)$.

If $(2 \times 2_+^{1+8}).2.2^4 \in \Phi(M_5, 2)$, then

$$N_E((2 \times 2_+^{1+8}).2.2^4) = N_{M_5.2}((2 \times 2_+^{1+8}).2.2^4) = N((2 \times 2_+^{1+8}).2.2^4).2,$$

$N((2 \times 2_+^{1+8}).2.2^4) = N_{M_5}((2 \times 2_+^{1+8}).2.2^4) = (2 \times 2_+^{1+8}).2.2^4.(S_3 \times S_3)$ and we may take

$$\Phi((2 \times 2_+^{1+8}).2.2^4.(S_3 \times S_3), 2) = \{(2 \times 2_+^{1+8}).2.2^4, 2^{10}.2^2.2^4, 2^{5+8}.D_8, S\}.$$

In addition, $N_{(2\times 2_+^{1+8}).2.2^4.(S_3\times S_3)}(R) = N_{M_5}(R)$ and

$$N_{M_5.2}(R) = N_{M_5}(R).2 \leq N_{M_5.2}((2 \times 2_+^{1+8}).2.2^4)$$

for each $R \in \Phi((2 \times 2_+^{1+8}).2.2^4.(S_3 \times S_3), 2)$.

Let $R \in \{2^{5+8}.D_8, S\}$, and $\sigma(R) : 1 < 2^{5+8} < Q = (2 \times 2_+^{1+8}).2.2^4 < R$. Then we may suppose $C \notin \mathcal{R}^-(\sigma(R), (2 \times 2_+^{1+8}).2.2^4) \cup \mathcal{R}^0(\sigma(R), (2 \times 2_+^{1+8}).2.2^4)$.

Let $C' : 1 < 2^{5+8} < 2^{5+8}.2 < S$ and $g(C') : 1 < 2^{5+8} < (2 \times 2_+^{1+8}).2.2^4 < 2^{10}.2^2.2^4 < S$. Then (5.3) holds, and we may suppose $C \notin_G \{C', g(C')\}$. It follows that if $P_1 = 2^{5+8}$, then $C \in_G \{C(i) : 21 \leq i \leq 24\}$. $\square$

Remark 5.2 *Let $\hat{G}$ be a covering group of $G = \mathrm{Fi}_{22}$, ρ a faithful linear character of $Z(\hat{G})$ and $\hat{B}$ a block of $\hat{G}$ covering the block $B(\rho)$ containing ρ. If $D(\hat{B}) \neq O_p(Z(\hat{G}))$ and $p = 2, 3$, then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|}\mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \rho, [r]) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|}\mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \rho, [r])$$

for all integers $d, u \geq 0$.

The proof of the Remark is the same as that of Lemma 5.1, since $N(C') = N(g(C'))$ implies $N_{\hat{G}}(C') = N_{\hat{G}}(g(C'))$.

6 Uno's invariant conjecture for Fi_{22}

Let $N(C)$ be the normalizer of a radical p-chain C. If $N(C)$ is a maximal subgroup of Fi_{22}, then the character table of $N(C)$ can be found in the library of character tables distributed with GAP. If this is not the case, we construct a "useful" description of $N(C)$ and attempt to compute directly its character table using MAGMA.

If $N(C)$ is soluble, we construct a power-conjugate presentation for $N(C)$ and use this presentation to obtain the character table.

If $N(C)$ is insoluble, we construct faithful permutation representations for $N(C)$ and use these as input to the character table construction function. We employ two strategies to obtain faithful representations of $N(C)$.

1. Construct the actions of $N(C)$ on the cosets of soluble subgroups of $N(C)$.

2. Construct the orbits of $N(C)$ on the underlying set of Fi_{22}; for the stabilizer of an orbit representative, construct the action of $N(C)$ on its cosets.

In several cases, however, none of the representations constructed was of sufficiently small degree to allow us to construct the required character table.

In these cases, we directly calculate the character table of $N(C)$ as follows: first calculate the character tables of some subgroups and quotient groups of $N(C)$; next induce or lift these characters to $N(C)$, so the liftings and the irreducible characters from the induction form a partial character table T of $N(C)$; finally decompose the remaining inductions or the tensor products of the inductions using the table T.

The tables listing degrees of irreducible characters referenced in the proof of Theorem 6.1 are available electronically [5].

Theorem 6.1 *Let B be a p-block of $G = \mathrm{Fi}_{22}$ with positive defect. Then B satisfies the Uno's invariant conjecture.*

PROOF: Let $E = \mathrm{Aut}(G) = \mathrm{Fi}_{22}.2$ and we may suppose $D(B)$ is non-cyclic, so that $B = B_0$ by Lemma 4.2.

Case (1) Suppose $p = 5$ and let $C = C(2)$, $C' = C(3)$, so that $N(C) \simeq 5{:}4 \times S_5$, $N_E(C) = N(C) \times 2$, $N(C') \simeq 5{:}4 \times 5{:}4$ and $N_E(C') = N(C') \times 2$. It follows by (2.6) that for all integers u, d

$$\mathrm{k}(N(C), B_0, d, u, [r]) = \mathrm{k}(N(C'), B_0, d, u, [r]).$$

Now $N(D(B)) = N(C(4)) \simeq 5^2{:}4S_4$, $N_E(D(B)) = 5^2{:}4S_4 \times 2$, and so

$$\mathrm{k}(G, B_0, d, u, [r]) = \mathrm{k}(N(B(D)), B_0, d, u, [r]) = \begin{cases} 10 & \text{if } d = 2,\ u = 2 \text{ and } r = 1, \\ 10 & \text{if } d = 2,\ u = 2 \text{ and } r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

This proves the theorem when $p = 5$.

Case (2) Suppose $p = 3$, so that Dade's invariant conjecture is equivalent to Uno's invariant conjecture. We set $\mathrm{k}(i, d, u) = \mathrm{k}(N(C(i)), B_0, d, u)$ for integers i, d, u.

First, we consider the radical 3-chains $C(j)$ with $\mathrm{d}(N(C(j))) = 7$, so that $2 \leq j \leq 5$. The values $\mathrm{k}(i, d, u)$ are given in Table 5.

Defect d	7	7	6	6	5	5	4	otherwise
Value u	2	1	2	1	2	1	2	otherwise
k(2, d, u)	36	18	18	0	18	0	6	0
k(3, d, u)	36	18	15	12	18	0	0	0
k(4, d, u)	36	18	15	12	18	18	0	0
k(5, d, u)	36	18	18	0	18	18	6	0

Table 5: Values of $\mathrm{k}(i, d, u)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 7$

It follows that

$$\sum_{i=2}^{5}(-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = 0. \tag{6.1}$$

Next we consider the chain $C(j)$ with $\mathrm{d}(N(C(j))) = 9$, so that $j = 1$ or $6 \leq j \leq 16$. The values $\mathrm{k}(i, d, u)$ are given in Table 6.

Defect d	9	9	8	8	7	7	6	6	5	5	4	otherwise
Value u	2	1	2	1	2	1	2	1	2	1	2	otherwise
k(1, d, u)	12	6	19	2	6	2	1	2	6	0	2	0
k(6, d, u)	12	6	19	2	16	12	1	2	6	0	0	0
k(7, d, u) = k(9, d, u)	0	18	0	12	0	46	0	3	0	0	0	0
k(8, d, u) = k(10, d, u)	0	18	0	12	0	13	0	3	0	0	0	0
k(11, d, u) = k(15, d, u)	0	18	0	21	0	13	0	18	0	3	0	0
k(12, d, u)	12	6	22	8	6	2	6	12	6	6	2	0
k(13, d, u)	12	6	22	8	16	12	6	12	6	0	0	0
k(16, d, u) = k(14, d, u)	0	18	0	21	0	46	0	18	0	0	0	0

Table 6: Values of $\mathrm{k}(i, d, u)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 9$

It follows that

$$\sum_{i=1,6\leq i\leq 16}(-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_0, d, u) = 0 \tag{6.2}$$

and the theorem follows.

Case (3) First, we consider the 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 15$, so that $j \in \{10, 11, 12, 13, 15, 18, 19, 20\}$. In this case, each element of $N_E(C)\backslash N(C)$ stabilizes each character of $\mathrm{Irr}(N(C))$. The values $\mathrm{k}(i, d, u)$ are given in Table 7.

It follows that

$$\sum_{i\in\{11,13,15,19\}}\mathrm{k}(N(C(i)), B_0, d, u) = \sum_{i\in\{10,12,18,20\}}\mathrm{k}(N(C(i)), B_0, d, u). \tag{6.3}$$

Next we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 17$, so that $j \in \{1, 14, 16, 17, 21, 22, 23, 24\}$. The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 8.

Defect d	15	14	13	12	11	10	9	6	otherwise
Value u	2	2	2	2	2	2	2	2	otherwise
k(10, d, u)	32	16	4	12	2	0	0	1	0
k(11, d, u)	32	16	12	20	4	0	0	0	0
k(12, d, u)	32	32	60	28	20	4	0	0	0
k(13, d, u)	32	32	52	20	18	4	0	0	0
k(15, d, u)	32	56	12	20	6	8	2	1	0
k(18, d, u)	32	56	44	28	12	8	2	0	0
k(19, d, u)	32	72	92	52	28	12	2	0	0
k(20, d, u)	32	72	60	44	22	12	2	0	0

Table 7: Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 15$

Defect d	17	16	15	14	14	13	13	12	12	11	11	10	10	9	9	6
Value u	2	2	2	2	1	2	1	2	1	2	1	2	1	2	1	2
k(1, d, u)	16	12	2	4	4	3	6	9	0	2	0	1	0	0	2	1
k(14, d, u)	16	12	2	4	4	2	2	0	0	0	0	1	0	0	0	0
k(16, d, u)	16	12	10	20	4	10	6	14	4	8	4	3	0	1	2	1
k(17 d, u)	16	12	10	24	4	11	2	9	4	2	0	3	0	0	0	0
k(21, d, u)	16	12	18	10	4	18	2	4	0	0	0	0	4	0	0	0
k(22, d, u)	16	12	18	10	4	19	6	13	0	2	0	0	4	0	2	0
k(23, d, u)	16	12	26	26	4	26	6	26	4	8	4	2	4	1	2	0
k(24, d, u)	16	12	26	30	4	27	2	21	4	2	0	2	4	0	0	0

Table 8: Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 17$

It follows that

$$\sum_{i \in \{1,17,21,23\}} \mathrm{k}(N(C(i)), B_0, d, u) = \sum_{i \in \{14,16,22,24\}} \mathrm{k}(N(C(i)), B_0, d, u). \tag{6.4}$$

Now we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 16$, so that $2 \leq j \leq 9$. The non-zero values $\mathrm{k}(i, d, u)$ are given in Table 9.

It follows that

$$\sum_{i \in \{3,5,7,9\}} \mathrm{k}(N(C(i)), B_0, d, u) = \sum_{i \in \{2,4,6,8\}} \mathrm{k}(N(C(i)), B_0, d, u). \tag{6.5}$$

Thus the theorem follows by (6.3), (6.4) and (6.5). This completes the proof. □

Defect d	16	16	15	15	14	14	13	13	12	12	11	11	10	10	9	9	6
Value u	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
k(2, d, u)	8	8	4	4	4	0	12	4	5	4	6	4	6	2	2	0	2
k(3, d, u)	8	8	4	4	4	0	12	12	13	12	8	12	6	2	2	4	2
k(4, d, u)	8	8	4	4	4	0	14	6	13	10	6	0	4	2	0	0	0
k(5, d, u)	8	8	4	4	4	0	10	2	5	2	0	0	4	2	0	0	0
k(6, d, u)	8	8	4	4	12	12	10	2	13	10	4	0	4	0	0	0	0
k(7, d, u)	8	8	4	4	12	12	12	4	13	12	10	4	6	0	2	0	0
k(8, d, u)	8	8	4	4	12	12	12	12	21	20	12	12	6	8	2	4	0
k(9, d, u)	8	8	4	4	12	12	14	6	21	18	10	0	4	8	0	0	0

Table 9: Values of $\mathrm{k}(i, d, u)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 16$

7 Uno's projective invariant conjecture for $2.\mathrm{Fi}_{22}$

Let C be a radical p-chain of Fi_{22}. The character tables of $N_{2.\mathrm{Fi}_{22}}(C)$ and $N_{2.\mathrm{Fi}_{22}.2}(C)$ can either be found in the library of character tables distributed with GAP or computed directly using MAGMA as in Section 6.

Let $H = N_{2.\mathrm{Fi}_{22}}(C)$ or $N_{2.\mathrm{Fi}_{22}.2}(C)$ and let ξ be the faithful linear character of $Z(2.\mathrm{Fi}_{22})$. Denote by $\mathrm{Irr}(H \mid \xi)$ the subset of $\mathrm{Irr}(H)$ consisting of characters covering ξ. The tables listing degrees of irreducible characters referenced in the proof of Theorem 7.1 are available electronically [5].

Theorem 7.1 *Let B be a p-block of $G = 2.\mathrm{Fi}_{22}$ with $D(B) \neq O_p(G)$. Then B satisfies Uno's projective invariant conjecture.*

PROOF: We may suppose B has a non-cyclic defect group. Let $N(C) = N_G(C)$ for each $C \in \mathcal{R}(\mathrm{Fi}_{22})$ and let $E = G.2 = 2.\mathrm{Fi}_{22}.2$.

Case (1) Suppose $p = 5$, so that $B = B_1$ given by Lemma 4.2. Thus

$$\mathrm{k}(G, B_1, d, u, \xi) = \mathrm{k}(N(C(4)), B_1, d, u, \xi) = \begin{cases} 2 & \text{if } d = 2,\ u = 2 \text{ and } r = 2, \\ 2 & \text{if } d = 2,\ u = 2 \text{ and } r = 1, \\ 8 & \text{if } d = 2,\ u = 1 \text{ and } r = 2, \\ 8 & \text{if } d = 2,\ u = 1 \text{ and } r = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and when $j = 1$ or 2,

$$\mathrm{k}(N(C(j)), B_1, d, u, \xi, [r]) = \begin{cases} 1 & \text{if } d = 2,\ u = 2 \text{ and } r = 1, \\ 24 & \text{if } d = 2,\ u = 1 \text{ and } r = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This implies the theorem when $p = 5$.

Case (2) Suppose $p = 3$, so that the projective invariant conjecture of Uno is equivalent to that of Dade. By Lemma 4.2, $B = B_1$ and we set $\mathrm{k}(j, d, u) = \mathrm{k}(N(C(j)), B, d, u, \xi)$.

We first consider the radical 3-chains $C(j)$ with $\mathrm{d}(N(C(j))) = 7$, so that $2 \le j \le 5$. The values $\mathrm{k}(j, d, u)$ are given in Table 10.

Defect d	7	7	6	6	5	5	4	4	otherwise
Value u	2	1	2	1	2	1	2	1	otherwise
k(2, d, u)	4	50	3	6	0	18	2	4	0
k(3, d, u)	4	50	1	26	0	18	0	0	0
k(4, d, u)	4	50	1	26	6	30	0	0	0
k(5, d, u)	4	50	3	6	6	30	2	4	0

Table 10: Values of $\mathrm{k}(N(C(i)), B_1, d, u, \xi)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 7$

It follows that

$$\sum_{i=2}^{5}(-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_1, d, u, \xi) = 0.$$

Next we consider the chain $C(j)$ with $\mathrm{d}(N(C(j))) = 9$, so that $j = 1$ or $6 \le j \le 16$. The values $\mathrm{k}(i, d, u)$ are given in Table 11.

Defect d	9	9	8	8	7	7	6	6	5	otherwise
Value u	2	1	2	1	2	1	2	1	1	otherwise
k(1, d, u)	4	14	3	18	1	0	1	2	6	0
k(6, d, u)	4	14	3	18	4	24	1	2	6	0
k(7, d, u) = k(9, d, u)	0	18	0	12	0	46	0	3	0	0
k(8, d, u) = k(10, d, u)	0	18	0	12	0	13	0	3	0	0
k(11, d, u) = k(15, d, u)	0	18	0	21	0	13	0	18	3	0
k(12, d, u)	4	14	4	26	1	0	0	18	12	0
k(13, d, u)	4	14	4	26	4	24	0	18	6	0
k(16, d, u) = k(14, d, u)	0	18	0	21	0	46	0	18	0	0

Table 11: Values of $\mathrm{k}(N(C(i)), B_1, d, u, \xi)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 9$

It follows that

$$\sum_{i=1,6\le i\le 16}(-1)^{|C(i)|}\mathrm{k}(N(C(i)), B_1, d, u, \xi) = 0$$

and the theorem follows.

Case (3) Suppose $p = 2$, so that by Lemma 4.2, $B = B_0$ or B_1. If $B = B_1$, then

$$\mathrm{k}(G, B_1, d, u, \xi) = \mathrm{k}(N(C(2)), B_1, d, u, \xi) = \begin{cases} 2 & \text{if } d = 2 \text{ and } u = 1, \\ 0 & \text{otherwise} \end{cases}$$

and the theorem follows when $B = B_1$.

Set $\mathrm{k}(j, d, u) = \mathrm{k}(N(C(j)), B_0, d, u, \xi)$. We first consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 16$, so that $j \in \{10, 11, 12, 13, 15, 18, 19, 20\}$. The values $\mathrm{k}(i, d, u)$ are given in Table 12.

Defect d	14	13	13	12	12	11	11	10	8	otherwise
Value u	1	2	1	2	1	2	1	2	1	otherwise
k(10, d, u)	16	4	8	2	8	0	4	1	4	0
k(11, d, u)	16	12	8	2	8	0	4	1	0	0
k(12, d, u)	16	12	40	2	24	4	4	1	0	0
k(13, d, u)	16	4	40	2	24	4	4	1	0	0
k(15, d, u)	16	4	8	2	40	0	8	0	4	0
k(18, d, u)	16	12	8	10	40	0	8	0	0	0
k(19, d, u)	16	12	40	10	56	4	8	0	0	0
k(20, d, u)	16	4	40	2	56	4	8	0	0	0

Table 12: Values of $\mathrm{k}(N(C(j)), B_0, d, u, \xi)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 16$

It follows that

$$\sum_{i \in \{11,13,15,19\}} \mathrm{k}(N(C(i)), B_0, d, u, \xi) = \sum_{i \in \{10,12,18,20\}} \mathrm{k}(N(C(i)), B_0, d, u, \xi). \tag{7.1}$$

Next we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 18$, so that $j \in \{1, 14, 16, 17, 21, 22, 23, 24\}$. The values $\mathrm{k}(i, d, u)$ are given in Table 13.

Defect d	14	13	13	12	12	11	11	10	8	otherwise
Value u	2	2	1	2	1	2	1	1	1	otherwise
k(1, d, u)	8	2	12	4	12	2	0	2	4	0
k(14, d, u)	8	2	12	2	0	2	0	0	0	0
k(16, d, u)	24	2	16	0	20	2	4	2	4	0
k(17 d, u)	24	2	16	2	8	2	0	0	0	0
k(21, d, u)	8	2	28	2	0	0	4	0	0	0
k(22, d, u)	8	2	28	4	12	0	4	2	0	0
k(23, d, u)	24	2	32	0	20	0	8	2	0	0
k(24, d, u)	24	2	32	2	8	0	4	0	0	0

Table 13: Values of $\mathrm{k}(N(C(i)), B_0, d, u, \xi)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 18$

It follows that

$$\sum_{i \in \{1,17,21,23\}} \mathrm{k}(N(C(i)), B_0, d, u, \xi) = \sum_{i \in \{14,16,22,24\}} \mathrm{k}(N(C(i)), B_0, d, u, \xi).$$

Now we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 17$, so that $2 \le j \le 9$. The values $\mathrm{k}(i, d, u)$ are given in Table 14.

Defect d	14	13	12	11	10	7	otherwise
Value u	1	1	1	1	1	1	otherwise
k(2, d, u)	16	12	20	8	2	2	0
k(3, d, u)	16	28	28	8	6	2	0
k(4, d, u)	16	28	8	8	0	0	0
k(5, d, u)	16	12	0	8	0	0	0
k(6, d, u)	16	44	0	4	0	0	0
k(7, d, u)	16	44	20	4	2	0	0
k(8, d, u)	16	60	28	12	6	0	0
k(9, d, u)	16	60	8	12	0	0	0

Table 14: Values of $\mathrm{k}(N(C(i)), B_0, d, u, \xi)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 17$

It follows that

$$\sum_{i \in \{3,5,7,9\}} \mathrm{k}(N(C(i)), B_0, d, u, \xi) = \sum_{i \in \{2,4,6,8\}} \mathrm{k}(N(C(i)), B_0, d, u, \xi).$$

The theorem follows. $\square$

8 Uno's projective conjecture for $3.\mathrm{Fi}_{22}$

Let C be a radical p-chain of Fi_{22} and $N_{3.\mathrm{Fi}_{22}}(C) = 3.N_{\mathrm{Fi}_{22}}(C)$. The character table of $N_{3.\mathrm{Fi}_{22}}(C)$ can either be found in the library of character tables distributed with GAP or computed directly using MAGMA as in Section 6.

Let $H = N_{3.\mathrm{Fi}_{22}}(C)$ and let ζ_1 and let ζ_2 be the faithful linear characters of $Z(3.\mathrm{Fi}_{22})$. Denote by $\mathrm{Irr}(H \mid \zeta_i)$ the subset of $\mathrm{Irr}(H)$ consisting of characters covering ζ_i. The tables listing degrees of irreducible characters referenced in the proof of Theorem 8.1 are available electronically [5].

Theorem 8.1 *Let B be a p-block of $G = 3.\mathrm{Fi}_{22}$ with $D(B) \neq O_p(G)$. Then B satisfies Uno's projective conjecture.*

PROOF: We may suppose B has a non-cyclic defect group and let $N(C) = N_G(C)$ for each $C \in \mathcal{R}(\mathrm{Fi}_{22})$.

Case (1) Suppose $p = 5$, so that $B = B_1$ or B_2. By [9, pp. 156-159], $\mathrm{k}(G, B_\ell, d, \zeta_\ell) = \sum_{u \ge 0} \mathrm{k}(G, B_0(\mathrm{Fi}_{22}), d, u)$ for each ℓ.

If R is a non-trivial radical p-subgroup of G, then $R = 5$ or $R = 5^2 \in \mathrm{Syl}_5(G)$, and $N_G(R) = 3 \times N_{\mathrm{Fi}_{22}}(R)$, so that for $i \geq 2$, $N(C(i)) = 3 \times N_{\mathrm{Fi}_{22}}(C(i))$, where $C(i)$ is a radical chain of Fi_{22} given by Table 3. Thus

$$\mathrm{k}(N(C(i)), B_\ell, d, \zeta_\ell) = \sum_{u \geq 0} \mathrm{k}(N_{\mathrm{Fi}_{22}}(C), B_0(\mathrm{Fi}_{22}), d, u)$$

and the proof (1) of Theorem 6.1 can be applied here for blocks $B = B_1$ and B_2. This proves the theorem when $p = 5$.

Case (2) Suppose $p = 3$, so that $B = B_0, B_1$ or B_2 given by Lemma 4.2 (c).

Suppose $B = B_1$ or B_2. Then $D(B) \simeq 3^2$, $N(D(B)) = N(C(2))$ and the theorem follows by Remark 2.1 and

$$\mathrm{k}(G, B, d, \zeta_\ell) = \mathrm{k}(N_G(C(2)), B, d, \zeta_\ell) = \begin{cases} 6 & \text{if } d = 2, \\ 0 & \text{otherwise} \end{cases}$$

for $\ell = 1, 2$.

Suppose $B = B_0$ and set $\mathrm{k}(j, d) = \mathrm{k}(N(C(j)), B_0, d, \zeta_\ell)$. We first consider the radical 3-chains $C(j)$ with $\mathrm{d}(N(C(j))) = 8$, so that $2 \leq j \leq 5$. The values $\mathrm{k}(j, d)$ are given in Table 15.

Defect d	7	6	5	otherwise
k(2, d)	54	15	12	0
k(3, d)	54	15	0	0
k(4, d)	54	33	0	0
k(5, d)	54	33	12	0

Table 15: Values of $\mathrm{k}(N(C(i)), B_0, d, \zeta_\ell)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 8$

It follows that

$$\sum_{i=2}^{5} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d, \zeta_\ell) = 0.$$

Next we consider the chain $C(j)$ with $\mathrm{d}(N(C(j))) = 10$, so that $j = 1$ or $6 \leq j \leq 16$. The values $\mathrm{k}(i, d)$ are given in Table 16.

It follows that

$$\sum_{i=1, 6 \leq i \leq 16} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d, \zeta_\ell) = 0$$

and the theorem follows.

Case (3) Suppose $p = 2$, so that $B = B_1$ or B_2.

Let $M \in \{2^{5+8}.(3 \times A_5).2, 2^6.S_6(2), (2 \times 2_+^{1+8}){:}\,U_4(2){:}\,2\}$ be a subgroup of Fi_{22} and $H = 3.M$ the preimage of M in G. Let S be a Sylow 2-subgroup of H, $H_1 = O_3(Z(H))$ and $H_2 = \langle [H, H], S \rangle$, where $[H, H]$ is the commutator subgroup of H. Then $H = H_1 \times H_2$ and $H_2 \simeq M$, so that $H \simeq 3 \times M$. Let

$$\Omega = \{3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 23, 24\}$$

Defect d	7	6	4	otherwise
k(1, d)	27	14	2	0
k(6, d) = k(13, d)	27	5	0	0
k(7, d) = k(9, d) = k(14, d) = k(16, d)	9	2	0	0
k(8, d) = k(10, d)	9	2	1	0
k(11, d) = k(15, d)	9	2	0	0
k(12, d, u)	27	14	0	0

Table 16: Values of $\mathrm{k}(N(C(i)), B_0, d, \zeta_\ell)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 10$

and $i \in \Omega$. Then $N_G(C(i))$ is a subgroup of some $H = 3.M$ with $N_{\mathrm{Fi}_{22}}(C(i)) \leq M$, so that we may suppose

$$N_G(C(i)) = 3 \times N_{\mathrm{Fi}_{22}}(C(i)).$$

It follows that for any $i \in \Omega$, $j, \ell \in \{1, 2\}$,

$$\mathrm{k}(N_G(C(i)), B_j, d, \zeta_\ell) = \sum_{u \geq 0} \mathrm{k}(N_{\mathrm{Fi}_{22}}(C(i)), B_0(\mathrm{Fi}_{22}), d, u) \tag{8.1}$$

and $\mathrm{k}(N_G(C(i)), B_j, d, \zeta_\ell)$ can be obtained by knowing $\mathrm{k}(N_{\mathrm{Fi}_{22}}(C(i)), B_0(\mathrm{Fi}_{22}), d, u)$, which is given in the proof of Theorem 6.1. Since $\mathrm{k}(N(C(i)), B_j, d, \zeta_\ell)$ is independent of the choices of j and ℓ, we set $\mathrm{k}(i, d) = \mathrm{k}(N(C(i)), B, d, \zeta_\ell)$ for integers i, d.

Now we consider the 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j)) = 15$, so that $j \in \{10, 11, 12, 13, 15, 18, 19, 20\} \subseteq \Omega$. Thus if $C = C(j)$, then $N(C) \simeq 3 \times N_{\mathrm{Fi}_{22}}(C)$ and $\mathrm{k}(j, d)$ is given by (8.1). It follows by (6.3) that

$$\sum_{j \in \{11,13,15,19\}} \mathrm{k}(N(C(j)), B, d, \zeta_\ell) = \sum_{j \in \{10,12,18,20\}} \mathrm{k}(N(C(j)), B, d, \zeta_\ell) \tag{8.2}$$

for each ℓ.

Next we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 17$, so that $j \in \{1, 14, 16, 17, 21, 22, 23, 24\}$. The values $\mathrm{k}(i, d)$ are given in Table 17.

It follows that

$$\sum_{i \in \{1,17,21,23\}} \mathrm{k}(N(C(i)), B, d, \zeta_\ell) = \sum_{i \in \{14,16,22,24\}} \mathrm{k}(N(C(i)), B, d, \zeta_\ell).$$

Now we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 16$, so that $2 \leq j \leq 9$. The values $\mathrm{k}(i, d)$ are given in Table 18.

It follows that

$$\sum_{i \in \{3,5,7,9\}} \mathrm{k}(N(C(i)), B, d, \zeta_\ell) = \sum_{i \in \{2,4,6,8\}} \mathrm{k}(N(C(i)), B, d, \zeta_\ell).$$

The theorem follows. ∎

Defect d	17	16	15	14	13	12	11	10	9	6	otherwise
k(1, d)	16	12	2	4	3	9	2	1	0	1	0
k(14, d)	16	12	2	4	4	0	0	1	0	0	0
k(16, d)	16	12	10	24	16	18	12	3	3	1	0
k(17 d)	16	12	10	28	13	13	2	3	0	0	0
k(21, d)	16	12	18	10	20	4	0	0	0	0	0
k(22, d)	16	12	18	10	19	13	2	0	0	0	0
k(23, d)	16	12	26	30	32	30	12	6	3	0	0
k(24, d)	16	12	26	34	29	25	2	6	0	0	0

Table 17: Values of $\mathrm{k}(N(C(i)), B, d, \zeta_\ell)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 17$

Defect d	16	15	14	13	12	11	10	9	6	otherwise
k(2, d)	16	8	4	16	9	10	4	2	2	0
k(3, d)	16	8	4	24	25	20	8	6	2	0
k(4, d)	16	8	4	20	23	6	6	0	0	0
k(5, d)	16	8	4	12	7	0	2	0	0	0
k(6, d)	16	8	24	12	23	4	4	0	0	0
k(7, d)	16	8	24	16	25	14	6	2	0	0
k(8, d)	16	8	24	24	41	24	14	6	0	0
k(9, d)	16	8	24	20	39	10	12	0	0	0

Table 18: Values of $\mathrm{k}(N(C(i)), B, d, \zeta_\ell)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 16$

9 Uno's projective conjecture for $6.\mathrm{Fi}_{22}$

Let C be a radical p-chain of Fi_{22} and $N_{6.\mathrm{Fi}_{22}}(C) = 6.N_{\mathrm{Fi}_{22}}(C)$. The character table of $N_{6.\mathrm{Fi}_{22}}(C)$ can either be found in the library of character tables distributed with GAP or computed directly using MAGMA or GAP as in Section 6.

Let $H = N_{6.\mathrm{Fi}_{22}}(C)$ and let η_1 and η_2 be the faithful linear characters of $Z(6.\mathrm{Fi}_{22})$. Denote by $\mathrm{Irr}(H \mid \eta_i)$ the subset of $\mathrm{Irr}(H)$ consisting of characters covering η_i. The tables listing degrees of irreducible characters referenced in the proof of Theorem 9.1 are available electronically [5].

Theorem 9.1 *Let B be a p-block of $G = 6.\mathrm{Fi}_{22}$ with $D(B) \neq O_p(G)$. Then B satisfies Uno's projective conjecture.*

PROOF: We may suppose B has a non-cyclic defect group. Let $N(C) = N_G(C)$ for each $C \in \mathcal{R}(\mathrm{Fi}_{22})$.

Case (1) Suppose $p = 5$, so that $B = B_1$ or B_2. Then

$$\mathrm{k}(N(C(4)), B, d, \eta_\ell, [r]) = \sum_{u \geq 0} \mathrm{k}(N_{\mathrm{Fi}_{22}}(C), B_1(2.\mathrm{Fi}_{22}), d, u, \xi, [r])$$

and the theorem follows by

$$\mathrm{k}(G,B,d,\eta_\ell,[r]) = \mathrm{k}(N(C(4)),B,d,\eta_\ell,[r]) = \begin{cases} 10 & \text{if } d=2 \text{ and } r=1, \\ 10 & \text{if } d=2 \text{ and } r=2, \\ 0 & \text{otherwise.} \end{cases}$$

Case (2) Suppose $p=3$, so that $B=B_1$. Then for each $\ell, j \in \{1,2\}$, the degrees of characters in $\mathrm{Irr}(N(C(i)) \mid \eta_\ell)$ are the same as that of characters in $\mathrm{Irr}(N_{3.\mathrm{Fi}_{22}}(C(i)) \mid \zeta_j)$ and so

$$\mathrm{k}(N(C(i)),B_1,d,\eta_\ell) = \mathrm{k}(N_{3.\mathrm{Fi}_{22}}(C(i)),B_0(3.\mathrm{Fi}_{22}),d,\zeta_j),$$

except when $i \in \{1,2,5,12\}$, in which case the values $\mathrm{k}(i,d) = \mathrm{k}(N(C(i)),B_1,d,\eta_\ell)$ are given in Table 19.

Defect d	7	6	5	4	otherwise
k(1, d)	27	5	0	2	0
k(2, d)	54	15	3	0	0
k(5, d)	54	33	3	0	0
k(12, d)	54	5	0	0	0

Table 19: Values of $\mathrm{k}(N(C(i)),B_1,d,\zeta_\ell)$ when $p=3$

It follows by Tables 15, 16 and 19 that

$$\sum_{i=1}^{14}(-1)^{|C(i)|}\mathrm{k}(N(C(i)),B_0,d,\zeta_\ell) = 0.$$

Case (3) Suppose $p=2$, so that $B=B_1$ or B_2. The proof is similar to that of Theorem 8.1.

Let $M \in \{2^{5+8}.(3\times A_5).2, 2^6.S_6(2), (2\times 2_+^{1+8}){:}\,U_4(2){:}\,2\}$ be a subgroup of Fi_{22} and $H = 6.M$ the preimage of M in G. If $H_1 = O_3(Z(H))$ and H_2 is the subgroup of H generated by both the commutator subgroup $[H,H]$ and a Sylow 2-subgroup of H, then $H = H_1 \times H_2$ and $H_2 \simeq 2.M \leq 2.\mathrm{Fi}_{22}$, so that $H \simeq 3 \times 2.M$. Let

$$\Omega = \{3,4,6,7,8,9,10,11,12,13,15,16,17,18,19,20,23,24\}$$

and $i \in \Omega$. Then $N_G(C(i))$ is a subgroup of $H = 6.M$ for some M, and $N_{\mathrm{Fi}_{22}}(C(i)) \leq M$, so that $N_G(C(i)) = 3 \times N_{2.\mathrm{Fi}_{22}}(C(i))$ and for any $i \in \Omega$, $j,\ell \in \{1,2\}$,

$$\mathrm{k}(N_G(C(i)),B_j,d,\eta_\ell) = \sum_{u\geq 0}\mathrm{k}(N_{2.\mathrm{Fi}_{22}}(C(i)),B_0(2.\mathrm{Fi}_{22}),d,u,\xi), \tag{9.1}$$

where ξ is the restriction of η_ℓ to $O_2(Z(G))$. Thus $\mathrm{k}(N_G(C(i)),B_j,d,\eta_\ell)$ can be obtained by knowing $\mathrm{k}(N_{2.\mathrm{Fi}_{22}}(C(i)),B_0(2.\mathrm{Fi}_{22}),d,u,\xi)$, which is given in the proof of Theorem 7.1.

Since $\mathrm{k}(N(C(i)),B_j,d,\eta_\ell)$ is independent of the choices of j and ℓ, we set $\mathrm{k}(i,d) = \mathrm{k}(N(C(i)),B,d,\eta_\ell)$ for integers i,d.

Consider the 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j)) = 16$, so that $j \in \{10, 11, 12, 13, 15, 18, 19, 20\} \subseteq \Omega$. Thus if $C = C(j)$, then $N(C) \simeq 3 \times N_{2.\mathrm{Fi}_{22}}(C)$ and $\mathrm{k}(j, d)$ is given by (9.1). It follows by (7.1) that

$$\sum_{j \in \{11,13,15,19\}} \mathrm{k}(N(C(j)), B, d, \eta_\ell) = \sum_{j \in \{10,12,18,20\}} \mathrm{k}(N(C(j)), B, d, \eta_\ell)$$

for each ℓ.

Next we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 18$, so that $j \in \{1, 14, 16, 17, 21, 22, 23, 24\}$. The values $\mathrm{k}(i, d)$ are given in Table 20.

Defect d	14	13	12	11	10	8	otherwise
k(1, d)	8	6	8	6	2	4	0
k(14, d)	8	6	2	6	0	0	0
k(16, d)	24	18	20	6	2	4	0
k(17 d)	24	18	10	2	0	0	0
k(21, d)	8	22	2	4	0	0	0
k(22, d)	8	22	8	4	2	0	0
k(23, d)	24	34	20	8	2	0	0
k(24, d)	24	34	10	4	0	0	0

Table 20: Values of $\mathrm{k}(N(C(i)), B, d, \eta_\ell)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 18$

It follows that

$$\sum_{i \in \{1,17,21,23\}} \mathrm{k}(N(C(i)), B, d, \eta_\ell) = \sum_{i \in \{14,16,22,24\}} \mathrm{k}(N(C(i)), B, d, \eta_\ell).$$

Now we consider the radical 2-chains $C(j)$ such that the defect $\mathrm{d}(N(C(j))) = 17$, so that $2 \leq j \leq 9$. The values $\mathrm{k}(i, d)$ are given in Table 21.

Defect d	14	13	12	11	10	7	otherwise
k(2, d)	16	12	20	4	2	2	0
k(3, d)	16	28	28	8	6	2	0
k(4, d)	16	28	8	8	0	0	0
k(5, d)	16	12	4	0	0	0	0
k(6, d)	16	44	4	0	0	0	0
k(7, d)	16	44	20	4	2	0	0
k(8, d)	16	60	28	12	6	0	0
k(9, d)	16	60	8	12	0	0	0

Table 21: Values of $\mathrm{k}(N(C(i)), B, d, \eta_\ell)$ when $p = 2$ and $\mathrm{d}(N(C(i))) = 17$

It follows that

$$\sum_{i\in\{3,5,7,9\}} \mathrm{k}(N(C(i)), B, d, \eta_\ell) = \sum_{i\in\{2,4,6,8\}} \mathrm{k}(N(C(i)), B, d, \eta_\ell).$$

The theorem follows. □

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