The Alperin and Dade conjectures for the Conway simple group Co_1

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Abstract

We classify the radical subgroups and chains of the Conway simple group Co_1 and then verify the Alperin weight conjecture and the Dade final conjecture for this group.

1 Introduction

Applying the local subgroup strategy of [2] and [3], we have previously classified the radical subgroups and radical chains for the sporadic simple groups Fi_{22}, Fi_{23}, Co_2, O′N and Ru, and verified the Alperin and Dade final conjectures for these finite simple groups, (see [2], [3], [4] and [5]). In this paper, we use the strategy to verify the Alperin and Dade conjectures for the Conway simple group Co_1. The challenge is to determine the character tables of the normalizers of some of the radical chains, since they could not be calculated directly from the given representation using either of GAP [14] or MAGMA [7].

Let G be a finite group, p a prime and B a p-block of G. Alperin [1] conjectured that the number of B-weights equals the number of irreducible Brauer characters of B. Dade [12] generalized the Knörr-Robinson version of the Alperin weight conjecture and presented his ordinary conjecture exhibiting the number of ordinary irreducible characters of a fixed defect in B in terms of an alternating sum of related values for p-blocks of some p-local subgroups of G. Dade [13] announced that his final conjecture needs only to be verified for finite non-abelian simple groups; in addition, if a finite group has a trivial outer automorphism group, then the projective conjecture is equivalent to the final conjecture. We verify the Alperin weight conjecture and the Dade final conjecture for Co_1.

The paper is organized as follows. In Section 2, we fix notation and state the two conjectures in detail. In Section 3, we recall our modified local strategy and explain how we applied it to determine the radical subgroups of Co_1. In Section 4, we classify the radical subgroups of Co_1 up to conjugacy and verify the Alperin weight conjecture. In Section 5, we do some cancellations in the alternating sum of Dade's conjecture when $p = 2$, 3 or 5, and then determine radical chains (up to conjugacy) and their

This work was supported in part by the Marsden Fund of New Zealand via grant #9144/3368248.
1991 *Mathematics Subject Classification.* Primary 20C20, 20C34, 20C08

local structures. In Section 6, we verify Dade's ordinary conjecture for Co_1 and in the last section, we verify Dade's projective conjecture for $2.\mathrm{Co}_1$. We give a detailed proof only when the prime p is 3, the proofs for other primes are essentially similar.

2 The Alperin and Dade conjectures

Let R be a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and let $\mathrm{Blk}(G)$ be the set of p-blocks, $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a B-*weight* if $\mathrm{d}(\varphi) = 0$ and $B(\varphi)^G = B$ (in the sense of Brauer), where $\mathrm{d}(\varphi) = \log_p(|G|_p) - \log_p(\varphi(1)_p)$ and $B(\varphi)$ is the block of $N(R)$ containing φ. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights, and $\ell(B)$ the number of irreducible Brauer characters of B. Alperin conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$.

Given a p-subgroup chain

$$C : P_0 < P_1 < \cdots < P_n \tag{2.1}$$

of G, define $|C| = n$, $C_k : P_0 < P_1 < \cdots < P_k$, $C(C) = C_G(P_n)$, and

$$N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \cdots \cap N(P_n). \tag{2.2}$$

The chain C is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$ for $1 \leq k \leq n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G.

Let Z be a cyclic group and $\tilde{G} = Z.G$ a central extension of Z by G, and $C \in \mathcal{R}(G)$. Denote by $N_{\tilde{G}}(C)$ the preimage $\eta^{-1}(N(C))$ of $N(C)$ in $\tilde{G}$, where η is the natural group homomorphism from $\tilde{G}$ onto G with kernel Z. Let ρ be a faithful linear character of Z and $\tilde{B}$ a block of $\tilde{G}$ covering the block $B(\rho)$ of Z containing ρ. Denote by $\mathrm{Irr}(N_{\tilde{G}}(C), \tilde{B}, d, \rho)$ the irreducible characters ψ of $N_{\tilde{G}}(C)$ such that ψ lies over ρ, $\mathrm{d}(\psi) = d$ and $B(\psi)^{\tilde{G}} = \tilde{B}$ and set $\mathrm{k}(N_{\tilde{G}}(C), \tilde{B}, d, \rho) = |\mathrm{Irr}(N_{\tilde{G}}(C), \tilde{B}, d, \rho)|$.

Dade's Projective Conjecture [13]. *If $O_p(G) = 1$ and $\tilde{B}$ is a p-block of $\tilde{G}$ covering $B(\rho)$ with defect group $D(\tilde{B}) \neq O_p(Z)$, then*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N_{\tilde{G}}(C), \tilde{B}, d, \rho) = 0, \tag{2.3}$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

If $Z = 1$ and ρ is the trivial character of Z, then $G = \tilde{G}$ and we set $B = \tilde{B}$, and

$$\mathrm{k}(N_{\tilde{G}}(C), \tilde{B}, d, \rho) = \mathrm{k}(N_G(C), B, d).$$

Hence the Projective Conjecture is reduced to the Ordinary Conjecture.

Dade's Ordinary Conjecture [12]. *If $O_p(G) = 1$ and B is a p-block of G with defect* $\mathrm{d}(B) > 0$, *then*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N(C), B, d) = 0. \tag{2.4}$$

3 A local subgroup strategy

The maximal subgroups of Co_1 were classified by Wilson [20], [21]. Using this classification and its proof, we know that when $p = 2$ or 3, there are 6 maximal subgroups M such that each radical p-subgroup R of Co_1 is radical in one of the subgroups M and further that $N_{\mathrm{Co}_1}(R) = N_M(R)$.

In [2] and [3], a (modified) local strategy was developed to classify the radical subgroups R. We review this method here.

Step (1). We first consider the case where M is a p-local subgroup. Let $Q = O_p(M)$, so that $Q \le R$. Choose a subgroup X of M. Using MAGMA, we explicitly compute the coset action of M on the cosets of X in M; we obtain a group W representing this action, a group homomorphism f from M to W, and the kernel K of f. For a suitable X, we have $K = Q$ and the degree of the action of W on the cosets is much smaller than that of M. We can now directly classify the radical p-subgroup classes of W, and the preimages in M of the radical subgroup classes of W are the radical subgroup classes of M.

Step (2). Now consider the case where M is not p-local. We may be able to find its radical p-subgroup classes directly. Alternatively, we find a subgroup K of M such that $N_K(R) = N_M(R)$ for each radical subgroup R of M. If K is p-local, then we apply Step (1) to K. If K is not p-local, we can replace M by K and repeat Step (2).

Steps (1) and (2) constitute the *modified local strategy.* After applying the strategy, we list subgroups R satisfying $N_M(R) = N_{\mathrm{Co}_1}(R)$, so these are the radical subgroups of Co_1. Possible fusions among the resulting list of radical subgroups can be decided readily by testing whether the subgroups in the list are pairwise G-conjugate.

In our investigations of the conjectures for Co_1 and $2.\mathrm{Co}_1$, we used the minimal degree representation of Co_1 as a permutation group on 98280 points, and a representation of $2.\mathrm{Co}_1$ as a permutation group on 196560 points. The maximal subgroups of Co_1 were constructed using the details supplied in [10] and the black-box algorithms of Wilson [22]. We also made extensive use of the algorithm described in [11] to construct random elements, and the procedures described in [2] and [3] for deciding the conjectures.

The computations reported in this paper were carried out using MAGMA V2.7-2 on a Sun UltraSPARC Enterprise 4000 server.

4 Radical subgroups and weights

Let $\Phi(G,p)$ be a set of representatives for conjugacy classes of radical subgroups of G. For $H, K \le G$, we write $H \le_G K$ if $x^{-1}Hx \le K$; and write $H \in_G \Phi(G,p)$ if $x^{-1}Hx \in \Phi(G,p)$ for some $x \in G$. We follow the notation of [10]. In particular, if p is odd, then $p_+^{1+2\gamma}$ is an extra-special group of order $p^{1+2\gamma}$ with exponent p; if δ is $+$ or $-$, then $2_\delta^{1+2\gamma}$ is an extra-special group of order $2^{1+2\gamma}$ with type δ, where δ is $+$ or $-$, according as the extra-special group is a central product of an even or odd number of quaternion factors. If X and Y are groups, we use $X.Y$ and $X : Y$ to denote an extension and a split extension of X by Y, respectively. Given a positive integer n, we use E_{p^n} or simply p^n to denote the elementary abelian group of order p^n, $\mathbb{Z}_n$ or simply

n to denote the cyclic group of order n, and D_{2n} to denote the dihedral group of order $2n$.

Let G be the simple Conway group Co_1. Then

$$|G| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23,$$

and we may suppose $p \in \{2, 3, 5, 7\}$, since both conjectures hold for a block with a cyclic defect group by Theorem 9.1 of [12].

We denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H and by $\mathrm{d}(H)$ the number $\log_p(|H|_p)$. Given $R \in \Phi(G,p)$, let $C(R) = C_G(R)$ and $N = N_G(R)$. If $B_0 = B_0(G)$ is the principal p-block of G, then (c.f. (4.1) of [2])

$$\mathcal{W}(B_0) = \sum_R |\mathrm{Irr}^0(N/C(R)R)|, \tag{4.1}$$

where R runs over the set $\Phi(G,p)$ such that the p-part $\mathrm{d}(C(R)R/R) = 0$. The character table of $N/C(R)R$ can be calculated by MAGMA, and so we find $|\mathrm{Irr}^0(N/C(R)R)|$. If $d(C(R)R/R) \neq 0$, then we leave the entries of the last column blank in Tables 1-2, since they do not contribute weights for the principal block.

Lemma 4.1 *Let $G = \mathrm{Co}_1$. The non-trivial radical p-subgroups R of G (up to conjugacy) and their local structures are given in Tables 1 and 2 according as p is odd or even, where H^* denotes a subgroup of G such that $H^* \simeq H$ and $H^* \neq_G H$, and Syl_p is a Sylow p-subgroup of G.*

PROOF: We prove the lemma when $p = 3$, the proofs for other primes are either trivial or similar.

Suppose $p = 3$. Let $i \in \{1, \ldots, 6\}$, and let M_i denote a maximal subgroup of $G = \mathrm{Co}_1$ where $M_1 = N(3A) \simeq 3.\mathrm{Suz}{:}\,2$, $M_2 = N(3^2) \simeq 3^2.U_4(3).D_8$, $M_3 = N(3^6) \simeq 3^6{:}\,2M_{12}$, $M_4 = N(3C) \simeq 3_+^{1+4}{:}\,2U_4(2){:}\,2$, $M_5 = N(3^3) \simeq 3^{3+4}{:}\,2(S_4 \times S_4)$ and $M_6 = N(3D) \simeq A_9 \times S_3$. As shown in [21], each 3-local subgroup of G is conjugate to a subgroup of M_i for some i.

The subgroup M_1, M_4 and M_6 are normalizers of some $3A$, $3C$ and $3D$ elements, so we can easily construct them in G. The subgroups $3^2 = O_3(M_2)$, $3^3 = Z(O_3(M_5))$ and $3^6 = O_3(M_3)$ can be constructed as subgroups of M_1. Indeed, using MAGMA we can first explicitly compute the coset action of M_1 on the cosets of a subgroup X of M_1; we obtain a group W representing this action, a group homomorphism η from M_1 to W, and the kernel K of η. For a suitable X, we have $K = O_3(M_1)$ and the degree of the action of W on the cosets is 1782. The commutator group H of W is the group Suz. By Wilson [18, Section 2.2], H contains exactly 3 classes of maximal 3-local subgroups of $3A$-type, $K_1 = 3.U_4(3).2$, $K_2 = 3^5{:}\,M_{11}$ and $K_3 = 3^{2+4}{:}\,2(A_4 \times 2^2).2$, where a 3-local subgroup is of $3A$-type if it has a minimal normal subgroup generated by $3A$-elements. Repeated random selections of elements allow us to obtain maximal 3-local subgroups K_i of $3A$-type. If H_i is the preimage of K_i in M_1, then $3^2 = O_3(H_1)$, $3^6 = O_3(H_2)$ and $3^3 = O_3(H_3)$, so we can construct all of the subgroups M_i.

Let R be a non-trivial radical 3-subgroup of G. Then $N(R)$ is 3-local, so that we may suppose $N(R) \leq M_i$ for some i and hence $R \in \Phi(M_i, 3)$ with $N(R) = N_{M_i}(R)$. We apply the local strategy of [2] or the modified local strategy [3] to each M_i.

R	$C(R)$	N	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
7	$7\times A_7$	$(7{:}\,3\times A_7).2$	
7^*	$7\times L_2(7)$	$(7{:}\,3\times L_2(7)).2$	
7^2	7^2	$7^2{:}\,(3\times 2A_4)$	21
5	$5\times(A_5\times A_5).2$	$(D_{10}\times(A_5\times A_5).2).2$	
5^*	$5\times J_2$	$(5{:}\,2\times J_2){:}\,2$	
5^2	$5^2\times A_5$	$(5^2\times A_5).4.S_3$	
$(5^2)^*$	5^2	$5^2{:}\,2A_5$	1
5^3	5^3	$5^3{:}\,(4\times A_5).2$	8
5_+^{1+2}	5	$5_+^{1+2}{:}\,\mathrm{GL}_2(5)$	4
Sy_5	5	$Sy_5{:}\,4^2$	16
3	$3.\mathrm{Suz}$	$3.\mathrm{Suz}{:}\,2$	
3^*	$3\times A_9$	$S_3\times A_9$	
3^2	$3^2.U_4(3)$	$3^2.U_4(3).D_8$	
3_+^{1+2}	$3\times A_6$	$3_+^{1+2}.(8\times A_6).2$	
3_+^{1+4}	3	$3_+^{1+4}{:}\,2U_4(2){:}\,2$	2
3^6	3^6	$3^6{:}\,2M_{12}$	1
$3^6.3$	3^2	$3^6.3.2(A_4\times 2)$	2
3^{3+4}	3^3	$3^{3+4}.2(S_4\times S_4)$	4
$3_+^{1+4}.3^3$	3	$3_+^{1+4}.3^3.2.(S_4\times 2)$	4
$3^6.3^2$	3^3	$3^6.3^2.2.\mathrm{GL}_2(3)$	4
$3^{3+4}.3$	3	$3^{3+4}.3.2.(2\times S_4)$	4
Sy_3	3	$Sy_3.2^3$	8

Table 1: Non-trivial radical p-subgroups of Co_1 with p odd

(1) Let $H_1 = 3^2.U_4(3){:}\,2^2$, $H_2 = 3^6{:}\,(M_{11}\times 2)$, $H_3 = 3^{3+4}{:}\,2(S_4\times D_8)$ and $H_4 = 3_+^{1+2}.(8\times A_6).2$ be maximal subgroups of $M_1 = 3.\mathrm{Suz}{:}\,2$, and let R be a radical 3-subgroup of M_1. By [18, Sections 2.2 and 2.3], we may suppose $R\in\Phi(H_i,3)$ with $N_{M_1}(R) = N_{H_i}(R)$ for some i.

Let $3^2 = O_3(H_1)$, $3^6 = O_3(H_2)$ and $3^{3+4} = O_3(H_3)$. Then we may take

$$\Phi(H_1,3) = \{3^2, 3^6, 3^{3+4}, S'\},$$

where S' is a Sylow 3-subgroup of M_1. Moreover, $N(R)\neq N_{M_1}(R)\neq N_{H_1}(R)$ for each $R\in\Phi(H_1,3)\backslash\{3^2\}$ and

$$N_{H_1}(R) = \begin{cases} 3^6.A_6{:}\,2^2 & \text{if } R = 3^6,\\ 3^{3+4}.4S_4.2 & \text{if } R = 3^{3+4},\\ S'.(2_-^{1+2}\times 2) & \text{if } R = S'.\end{cases}$$

In addition, $C_{M_1}(3^6) = 3^6$, $C_{M_1}(3^{3+4}) = 3^3$ and $C_{M_1}(S') = 3^3$.

R	$C(R)$	$N(R)$	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
2^2	$2^2 \times G_2(4)$	$(A_4 \times G_2(4)){:}\,2$	
$(2^2)^*$	$2^2 \times G_2(2)$	$S_4 \times G_2(2)$	
D_8	$2 \times G_2(2)$	$D_8 \times G_2(2)$	
2_+^{1+8}	2	$2_+^{1+8}.O_8^+(2)$	1
2^{11}	2^{11}	$2^{11}{:}\,M_{24}$	0
2^{2+12}	2^2	$2^{2+12}{:}\,(A_8 \times S_3)$	1
$2^{2+12}.2$	2	$2^{2+12}.2.A_8$	1
$2_+^{1+8}.2^6$	2	$2_+^{1+8}.2^6.A_8$	1
$2^{11}.2^4$	2	$2^{11}.2^4.A_8$	1
2^{4+12}	2^4	$2^{4+12}.(S_3 \times 3S_6)$	1
$2^{11}.2^6$	2^2	$2^{11}.2^6.(L_3(2) \times S_3)$	1
$2^{4+12}.2$	2^4	$2^{4+12}.2.3.S_6$	1
$2^{2+12}.2^3$	2^2	$2^{2+12}.2^3.(L_3(2) \times S_3)$	1
$2_+^{1+8}.2^6.2^3$	2	$2_+^{1+8}.2^6.2^3.L_3(2)$	1
$2^{11}.2^3.2^4$	2	$2^{11}.2^3.2^4.L_3(2)$	1
$2^{11}.2_+^{1+6}$	2	$2^{11}.2_+^{1+6}.L_3(2)$	1
$2_+^{1+8}.2_+^{1+8}$	2	$2_+^{1+8}.2_+^{1+8}.(S_3 \times S_3 \times S_3)$	1
$2^{2+12}.2^4$	2^2	$2^{2+12}.2^4.(S_3 \times S_3 \times S_3)$	1
$2^{2+12}.2^5$	2	$2^{2+12}.2^5.(S_3 \times S_3)$	1
$2_+^{1+8}.2^6.2^4$	2	$2_+^{1+8}.2^6.2^4.(S_3 \times S_3)$	1
$2_+^{1+8}.2_+^{1+8}.2$	2	$2_+^{1+8}.2_+^{1+8}.2.(S_3 \times S_3)$	1
$2^{11}.2^4.2^4$	2^2	$2^{11}.2^4.2^4.(S_3 \times S_3)$	1
$2^{11}.2^2.2^6$	2^2	$2^{11}.2^2.2^6.(S_3 \times S_3)$	1
$2^{2+12}.2^3.2^2$	2^2	$2^{2+12}.2^3.2^2.(S_3 \times S_3)$	1
$2_+^{1+8}.2^6.2^3.2^2$	2	$2_+^{1+8}.2^6.2^3.2^2.S_3$	1
$2^{2+12}.2^3.2^2.2$	2	$2^{2+12}.2^3.2^2.2.S_3$	1
$2^{11}.2.2^3.2^5$	2	$2^{11}.2.2^3.2^5.S_3$	1
$2_+^{1+8}.2^6.2^4.2$	2	$2_+^{1+8}.2^6.2^4.2.S_3$	1
$2^{11}.2^2.2^3.2^4$	2^2	$2^{11}.2^2.2^3.2^4.S_3$	1
Sy_2	2	Sy_2	1

Table 2: Non-trivial radical 2-subgroups of Co_1

If $3_+^{1+2} = O_3(H_4)$, then we may take

$$\Phi(H_i, 3) = \begin{cases} \{3^6, S'\} & \text{if } i = 2, \\ \{3^{3+4}, S'\} & \text{if } i = 3, \\ \{3_+^{1+2}, S''\} & \text{if } i = 4, \end{cases}$$

where S'' is a Sylow 3-subgroup of H_4. Moreover, for $R \in \Phi(H_i, 3)$, $N_{M_1}(R) = N_{H_i}(R)$ except when $R = S''$, in which case $N_{M_1}(S'') \neq N_{H_4}(S'') = S''.(8 \times 2^2).2$, and in addition, $N(3_+^{1+2}) = N_{M_1}(3_+^{1+2}) = H_4$, $N_{M_1}(S') \simeq S'.(SD_{2^4} \times 2)$, where SD_{2^4} is the semidihedral group of order 2^4.

It follows that
$$\Phi(M_1, 3) = \{3, 3^2, 3_+^{1+2}, 3^6, 3^{3+4}, S'\},$$
$N(3) = N_{M_1}(3) = M_1$ and $N(3_+^{1+2}) = N_{M_1}(3_+^{1+2}) = H_4$.

(2) We may take
$$\Phi(M_2, 3) = \{3^2, 3^6, 3^{3+4}, S'\},$$
and moreover, $N(R) \neq N_{M_2}(R)$ for each $R \in \Phi(M_2, 3)\backslash\{3^2\}$. In addition,
$$N_{M_2}(R) = \begin{cases} 3^6.A_6{:}\, D_8 & \text{if } R = 3^6, \\ 3^{3+4}.2_-^{1+4}.D_{12} & \text{if } R = 3^{3+4}, \\ S'.(SD_{2^4} \times 2) & \text{if } R = S'. \end{cases}$$

(3) We may take
$$\Phi(M_3, 3) = \{3^6, 3^6.3, 3_+^{1+4}.3^3, 3^6.3^2, Sy_3\},$$
and moreover, $N(R) = N_{M_3}(R)$ for each $R \in \Phi(M_3, 3)$, so that we may suppose $\Phi(M_3, 3) \subseteq \Phi(G, 3)$.

(4) If $3_+^{1+4} = O_3(M_4)$, then we may take
$$\Phi(M_4, 3) = \{3_+^{1+4}, 3_+^{1+4}.3^3, 3^{3+4}.3, Sy_3\},$$
and moreover, $N(R) = N_{M_4}(R)$ for each $R \in \Phi(M_4, 3)$, so that we may suppose $\Phi(M_4, 3) \subseteq \Phi(G, 3)$.

(5) If $3^{3+4} = O_3(M_5)$, then we may take
$$\Phi(M_5, 3) = \{3^{3+4}, 3^{3+4}.3, 3^6.3^2, Sy_3\},$$
and moreover, $N(R) = N_{M_5}(R)$ for each $R \in \Phi(M_5, 3)$, so that we may suppose $\Phi(M_5, 3) \subseteq \Phi(G, 3)$.

(6) If $3^* = O_3(M_6)$, then we may take
$$\Phi(M_6, 3) = \{3^*, 3^2, 3^3, 3^4, S''\},$$
and moreover, $N(R) \neq N_{M_5}(R)$ for $R \in \Phi(M_6, 3)\backslash\{3^*\}$. In addition, (c.f. [10], p. 37)
$$N_{M_6}(R) = \begin{cases} S_3 \times (3 \times A_6){:}\, 2 & \text{if } R = 3^2, \\ S_3 \times 3^2{:}\, 2A_4 & \text{if } R = 3^3, \\ S_3 \times 3^3{:}\, S_4 & \text{if } R = 3^4, \\ S''.2^2 & \text{if } R = S''. \end{cases}$$

Thus the radical 3-subgroups are as claimed. The centralizers and normalizers of R can be obtained by MAGMA. $\square$

Lemma 4.2 *Let $G = \mathrm{Co}_1$ and $B_0 = B_0(G)$, and let $\mathrm{Blk}^+(G,p)$ be the set of p-blocks with a non-trivial defect group and $\mathrm{Irr}^+(G)$ the characters of $\mathrm{Irr}(G)$ with positive p-defect. If a defect group $D(B)$ of B is cyclic, then $\mathrm{Irr}(B)$ is given by [15, p. 304-311].*

(a) *If $p = 7$, then $\mathrm{Blk}^+(G,p) = \{B_i \mid 0 \leq i \leq 5\}$ such that $D(B_i) \simeq 7$ for $i \geq 1$, so that $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash (\cup_{i=1}^{5} \mathrm{Irr}(B_i))$. Moreover, $\ell(B_i) = 6$ for $1 \leq i \leq 5$ and $\ell(B_0) = 21$.*

(b) *If $p = 5$, then $\mathrm{Blk}^+(G,p) = \{B_i \mid 0 \leq i \leq 6\}$ such that $D(B_1) \simeq 5^2$ and $D(B_i) \simeq 5$ when $i \geq 2$. In the notation of* [10, p. 184],

$$\mathrm{Irr}(B_1) = \{\chi_7, \chi_9, \chi_{12}, \chi_{24}, \chi_{35}, \chi_{38}, \chi_{44}, \chi_{48}, \chi_{52}, \chi_{54}, \chi_{64}, \chi_{66}, \chi_{81}, \chi_{83}, \chi_{91}, \chi_{99}\}$$

and $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash (\cup_{i=1}^{6} \mathrm{Irr}(B_i))$. Moreover, $\ell(B_i) = 4$ when $i \geq 2$, $\ell(B_1) = 12$ and $\ell(B_0) = 29$.

(c) *If $p = 3$, then $\mathrm{Blk}^+(G,2) = \{B_0, B_1, B_2, B_3\}$ such that $D(B_1) \simeq 3_+^{1+2}$, $D(B_2) \simeq 3^2$ and $D(B_3) =_G 3^*$. In the notation of* [10, p. 184],

$$\mathrm{Irr}(B_1) = \{\chi_{20}, \chi_{27}, \chi_{28}, \chi_{34}, \chi_{35}, \chi_{40}, \chi_{64}, \chi_{67}, \chi_{71}, \chi_{73}, \chi_{83}, \chi_{86}, \chi_{94}, \chi_{98}\},$$

$\mathrm{Irr}(B_2) = \{\chi_{29}, \chi_{38}, \chi_{51}, \chi_{55}, \chi_{62}, \chi_{80}, \chi_{85}, \chi_{89}, \chi_{91}\}$, *and*

$$\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash (\cup_{i=1}^{3} \mathrm{Irr}(B_i)).$$

Moreover, $\ell(B_1) = 7$, $\ell(B_2) = 5$, $\ell(B_3) = 2$ and $\ell(B_0) = 29$.

(d) *If $p = 2$, then $\mathrm{Blk}^+(G,2) = \{B_0, B_1\}$ such that $D(B_1) \simeq D_8$. In the notation of* [10, p. 184],

$$\mathrm{Irr}(B_1) = \{\chi_{50}, \chi_{66}, \chi_{71}, \chi_{93}, \chi_{99}\}$$

and $\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \backslash \mathrm{Irr}(B_1)$. Moreover, $\ell(B_1) = 2$ and $\ell(B_0) = 26$.

PROOF: We prove the lemma when $p = 3$, the proofs for other primes are similar. So we suppose $p = 3$.

If $B \in \mathrm{Blk}(G,p)$ is non-principal with $D = D(B)$, then $\mathrm{Irr}^0(C(D)D/D)$ has a non-trivial character θ and $N(\theta)/C(D)D$ is a p'-group, where $N(\theta)$ is the stabilizer of θ in $N(D)$. By [15, p. 304-311], we may suppose D is non-cyclic. Since Suz has no irreducible character of 3-defect 0, it follows by Lemma 4.1 that $D = 3^2$ or 3_+^{1+2}. Since $|\mathrm{Irr}^0(C(D)D/D)| = 1$, it follows that G has exactly one block with defect D.

Using the method of central characters, $\mathrm{Irr}(B)$ is as above. If $D(B)$ is cyclic, then $\ell(B)$ is given by [15, p. 304-311]. If $B = B_1$, then $D(B) =_G 3_+^{1+2}$ and the non-trivial elements of $D(B)$ consists of $3A$ and $3D$ elements, $C_G(3A) = 3.\mathrm{Suz}$ and $C_G(3D) = 3 \times A_9$. It follows by [16, Theorem 5.4.13] that $\mathrm{k}(B) = \ell(B) + \ell(b_1) + \ell(b_2)$, where $b_1 \in \mathrm{Blk}(3.\mathrm{Suz})$ and $b_2 \in \mathrm{Blk}(3 \times A_9)$ such that each $b_i^G = B$. In addition, $b_2 = B_0(3) \times b_2'$ with $b_2' \in \mathrm{Blk}(A_9)$ and $D(b_2') \simeq 3$, so that $\ell(b_2')$ is the number of b_2'-weights, which is 2 since $N_{A_9}(D(b_2')) = (3 \times A_6).2$.

Let b_1' be the block of Suz contained in b_1. Then $\ell(b_1) = \ell(b_1')$, $D(b_1') \simeq 3^2$, $C_{\mathrm{Suz}}(D(b_1')) = 3^2 \times A_6$ and $N_{\mathrm{Suz}}(D(b_1')) = (3^2{:}\,4 \times A_6).2$. Since each non-trivial element of $D(b_1')$ is of type $3C$ in Suz and $C_{\rm Suz}(3C) = 3^2 \times A_6$, it follows that $\ell(b_1') = {\rm k}(b_1') - 1 = 5$, so that $\ell(b_1) = 5$ and $\ell(B_1) = 14 - 5 - 2 = 7$.

If $B = B_2$, then $D(B) =_G 3^2$ and the non-trivial elements of $D(B)$ consists of $3A$ and $3B$ elements, and $C_G(3B) = 3^2.U_4(3).2$. Thus ${\rm k}(B) = \ell(B) + \ell(b_1) + \ell(b_2)$, where $b_1 \in {\rm Blk}(3.{\rm Suz})$ and $b_2 \in {\rm Blk}(3^2.U_4(3).2)$ with each $b_i^G = B$. But $C_{\rm Suz}(3A) = 3.U_4(3).2$, so a similar proof to above shows that each $\ell(b_i) = 2$, so that $\ell(B) = 9 - 2 - 2 = 5$.

If $\ell_3(G)$ is the number of 3-regular G-conjugacy classes, then $\ell_3(G) = 44$ and $\ell(B_0)$ can be calculated by the following equation due to Brauer:

$$\ell_p(G) = \sum_{B \in {\rm Blk}^+(G,p)} \ell(B) + |{\rm Irr}^0(G)|.$$

This completes the proof. $\square$

Theorem 4.3 *Let $G = {\rm Co}_1$ and let B be a p-block of G with a non-cyclic defect group. Then the number of B-weights is the number of irreducible Brauer characters of B.*

PROOF: If $B = B_0$, then the proof of Theorem 4.3 follows by Lemmas 4.1, 4.2 and (4.1). Suppose $p = 3$ and $B \neq B_0$

If $B = B_1$ and (R, φ) is a B-weight with $R = 3^2$, then we may suppose $3^2 \subseteq D(B_1) = 3_+^{1+2}$, which is impossible, since 3^2 contains a $3B$-element. Thus $R \in \{3, 3^*, 3_+^{1+2}\}$. If $R = 3$, then G has no weight of the form $(3, \varphi)$. If $R = 3^*$, then G has exactly two B_3-weights of the form $(3^*, \varphi)$. Thus $R = 3_+^{1+2}$ and G has 7 B-weights, since $N(3_+^{1+2})/3_+^{1+2}$ has exactly 7 irreducible characters of 3-defect 2.

If $B = B_2$, then the B-weights have the form $(3^2, \varphi)$. Thus G has 5 B-weights, since $N(3^2)/C(3^2) \simeq D_8$ has 5 irreducible characters. $\square$

5 Radical chains of ${\rm Co}_1$

Let $G = {\rm Co}_1$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$. In this section we do some cancellations in the alternating sum of Dade's conjectures. We first list some radical p-chains $C(i)$ and their normalizers for certain integers i, then reduce the proof of Dade's conjecture to the subfamily $\mathcal{R}^0(G)$ of $\mathcal{R}(G)$, where $\mathcal{R}^0(G)$ is the union G-orbits of all $C(i)$. Table 3 is the only one gives a complete list of orbit representatives of radical 7-chains. In Table 6, $4^2.2, 4.2^2.2, S'$ are the radical 2-subgroups of $G_2(2)$ with S' a Sylow subgroup. The other subgroups of the p-chains in Tables 3-6 are given either by Lemma 4.1 or by its proof.

Lemma 5.1 *Let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that*

$$\mathcal{R}^0(G)/G = \begin{cases} \{C(i) : 1 \leq i \leq 6\} & \textit{with } C(i) \textit{ given in Table } 3 \textit{ if } p = 7, \\ \{C(i) : 1 \leq i \leq 16\} & \textit{with } C(i) \textit{ given in Table } 4 \textit{ if } p = 5, \\ \{C(i) : 1 \leq i \leq 26\} & \textit{with } C(i) \textit{ given in Table } 5 \textit{ if } p = 3, \\ \{C(i) : 1 \leq i \leq 20\} & \textit{with } C(i) \textit{ given in Table } 6 \textit{ if } p = 2. \end{cases}$$

Then

$$\sum_{C\in\mathcal{R}(G)/G}(-1)^{|C|}\mathrm{k}(N(C),B_0,d)=\sum_{C\in\mathcal{R}^0(G)/G}(-1)^{|C|}\mathrm{k}(N(C),B_0,d) \tag{5.1}$$

for all integers $d \geq 0$.

C		$N(C)$
$C(1)$	1	Co_1
$C(2)$	$1<7$	$(7{:}\,3\times A_7).2$
$C(3)$	$1<7<7^2$	$7^2{:}\,(3\times 6)$
$C(4)$	$1<7^*$	$(7{:}\,3\times L_2(7)).2$
$C(5)$	$1<7^*<7^2$	$7^2{:}\,(3\times 6)$
$C(6)$	$1<7^2$	$7^2{:}\,(3\times 2A_4)$

Table 3: Radical 7-chains of Co_1

C		$N(C)$
$C(1)$	1	Co_1
$C(2)$	$1<5$	$(D_{10}\times(A_5\times A_5).2).2$
$C(3)$	$1<5<5^2$	$(D_{10}\times 5{:}\,2\times A_5).2$
$C(4)$	$1<5<5^2<5^3$	$5^3{:}\,(4\times 2^2)$
$C(5)$	$1<5<5^3$	$5^3{:}\,(4\times 2).2^2$
$C(6)$	$1<5^*$	$(5{:}\,2\times J_2){:}\,2$
$C(7)$	$1<5^*<5^2$	$(D_{10}\times 5{:}\,2\times A_5).2$
$C(8)$	$1<5^*<5^2<5^3$	$5^3{:}\,(4\times 2^2)$
$C(9)$	$1<5^*<5^3$	$5^3{:}\,(4\times 2).6$
$C(10)$	$1<5^2$	$(5^2\times A_5).4.S_3$
$C(11)$	$1<5^2<5^3$	$5^3{:}\,(4\times 2).6$
$C(12)$	$1<(5^2)^*$	$5^2{:}\,2A_5$
$C(13)$	$1<(5^2)^*<5^2{:}\,5$	$5^2{:}\,5.4$
$C(14)$	$1<5^3$	$5^3{:}\,(4\times A_5).2$
$C(15)$	$1<5_+^{1+2}<Sy_5$	$Sy_5{:}\,4^2$
$C(16)$	$1<5_+^{1+2}$	$5_+^{1+2}{:}\,\mathrm{GL}_2(5)$

Table 4: Some radical 5-chains of Co_1

PROOF: We prove the lemma when $p=3$, the proof for other primes are either trivial or similar.

Suppose $p=3$ and C' is a radical chain such that

$$C' : 1 < P_1' < \ldots < P_m'. \tag{5.2}$$

C		$N(C)$
$C(1)$	1	Co_1
$C(2)$	$1 < 3$	$3.\mathrm{Suz}{:}\,2$
$C(3)$	$1 < 3 < 3^2$	$3^2.U_4(3){:}\,2^2$
$C(4)$	$1 < 3 < 3^2 < 3^6$	$3^6.A_6{:}\,2^2$
$C(5)$	$1 < 3 < 3^6$	$3^6{:}\,(2 \times M_{11})$
$C(6)$	$1 < 3 < 3^2 < 3^{3+4}$	$3^{3+4}{:}\,4S_4.2$
$C(7)$	$1 < 3 < 3^{3+4}$	$3^{3+4}{:}\,2(S_4 \times D_8)$
$C(8)$	$1 < 3 < 3^6 < S'$	$S'.(SD_{2^4} \times 2)$
$C(9)$	$1 < 3 < 3^2 < 3^6 < S'$	$S'.(2_-^{1+2} \times 2)$
$C(10)$	$1 < 3^2$	$3^2{:}\,U_4(3).D_8$
$C(11)$	$1 < 3^2 < 3^6$	$3^6.A_6.D_8$
$C(12)$	$1 < 3^2 < 3^6 < S'$	$S'.(SD_{2^4} \times 2)$
$C(13)$	$1 < 3^2 < 3^{3+4}$	$3^{3+4}{:}\,2_-^{1+4}.D_{12}$
$C(14)$	$1 < 3^6$	$3^6{:}\,2M_{12}$
$C(15)$	$1 < 3_+^{1+4} < 3_+^{1+4}.3^3$	$3_+^{1+4}.3^3.2(S_4 \times 2)$
$C(16)$	$1 < 3_+^{1+4}$	$3_+^{1+4}{:}\,2U_4(2){:}\,2$
$C(17)$	$1 < 3^{3+4} < 3^{3+4}.3$	$3^{3+4}.3.2(S_4 \times 2)$
$C(18)$	$1 < 3^{3+4}$	$3^{3+4}{:}\,2(S_4 \times S_4)$
$C(19)$	$1 < 3^{3+4} < 3^6.3^2$	$3^6.3^2.2(S_4 \times 2)$
$C(20)$	$1 < 3^{3+4} < 3^{3+4}.3 < Sy_3$	$Sy_3.2^3$
$C(21)$	$1 < 3^* < 3^2$	$S_3 \times (3 \times A_6){:}\,2$
$C(22)$	$1 < 3^*$	$S_3 \times A_9$
$C(23)$	$1 < 3^* < 3^3$	$S_3 \times 3^2{:}\,2A_4$
$C(24)$	$1 < 3^* < 3^2 < 3^4$	$S_3 \times 3^3{:}\,D_8$
$C(25)$	$1 < 3^* < 3^4$	$S_3 \times 3^3{:}\,S_4$
$C(26)$	$1 < 3^* < 3^3 < 3 \times 3^2.3$	$S_3 \times 3^2.3.2$

Table 5: Some radical 3-chains of Co_1

Let $C \in \mathcal{R}(G)$ be given by (2.1) with $P_1 \in \Phi(G,3)$.

Case (1). We first consider the radical subgroups of G contained in M_3. Let $R \in \Phi(M_3,3)\backslash\{3^6\}$. Define G-invariant subfamilies $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ of $\mathcal{R}(G)$, such that

$$\begin{aligned} \mathcal{M}^+(R)/G &= \{C' \in \mathcal{R}/G : P_1' = R\}, \\ \mathcal{M}^0(R)/G &= \{C' \in \mathcal{R}/G : P_1' = 3^6, P_2' = R\}. \end{aligned} \tag{5.3}$$

For $C' \in \mathcal{M}^+(R)$ given by (5.2), the chain

$$g(C') : 1 < 3^6 < P_1' = R < P_2' < \ldots < P_m' \tag{5.4}$$

is a chain in $\mathcal{M}^0(R)$ and as shown in the proof (3) of Lemma 4.1, $N(C') = N(g(C'))$. For $B \in \mathrm{Blk}(G)$ and for integer $d \geq 0$,

$$\mathrm{k}(N(C'), B, d) = \mathrm{k}(N(g(C')), B, d). \tag{5.5}$$

In addition, g is a bijection between $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$. So we may suppose

$$C \notin \bigcup_{R \in \Phi(M_3, 3) \setminus \{3^6\}} (\mathcal{M}^+(R) \cup \mathcal{M}^0(R)).$$

C		$N(C)$
$C(1)$	1	Co_1
$C(2)$	$1 < 2^{11}$	$2^{11}{:}\, M_{24}$
$C(3)$	$1 < 2_+^{1+8} < 2^{11}.2^4$	$2^{11}.2^4.A_8$
$C(4)$	$1 < 2_+^{1+8}$	$2_+^{1+8}.O_8^+(2)$
$C(5)$	$1 < 2^2 < 2^2 \times 2^{2+8}$	$(A_4 \times 2^{2+8}{:}\,(A_5 \times 3)){:}\,2$
$C(6)$	$1 < 2^2$	$(A_4 \times G_2(4)){:}\,2$
$C(7)$	$1 < 2^2 < 2^2 \times 2^{4+6}$	$(A_4 \times 2^{4+6}{:}\,(3 \times A_5)){:}\,2$
$C(8)$	$1 < 2^2 < 2^2 \times 2^{4+6} < 2^2 \times 2^{4+6}.2^2$	$A_4 \times 2^{4+6}{:}\,(A_4 \times 3)$
$C(9)$	$1 < 2^{2+12} < 2^{2+12}.2$	$2^{2+12}.(2 \times A_8)$
$C(10)$	$1 < 2^{2+12}$	$2^{2+12}{:}\,(S_3 \times A_8)$
$C(11)$	$1 < 2^{2+12} < 2^{11}.2^6$	$2^{11}.2^6.(L_3(2) \times S_3)$
$C(12)$	$1 < 2^{2+12} < 2^{11}.2^6 < 2^{11}.2^3.2^4$	$2^{11}.2^3.2^4.L_3(2)$
$C(13)$	$1 < 2^{4+12} < 2^{4+12}.2$	$2^{4+12}.2.3S_6$
$C(14)$	$1 < 2^{4+12}$	$2^{4+12}.(S_3 \times 3S_6)$
$C(15)$	$1 < 2^{4+12} < 2_+^{1+8}.2_+^{1+8}$	$2_+^{1+8}.2_+^{1+8}.(S_3 \times S_3 \times S_3)$
$C(16)$	$1 < 2^{4+12} < 2_+^{1+8}.2_+^{1+8} < 2_+^{1+8}.2_+^{1+8}.2$	$2_+^{1+8}.2_+^{1+8}.2.(S_3 \times S_3)$
$C(17)$	$1 < 2^{4+12} < 2^{2+12}.2^4$	$2^{2+12}.2^4.(S_3 \times S_3 \times S_3)$
$C(18)$	$1 < 2^{4+12} < 2^{2+12}.2^4 < 2^{11}.2^4.2^4$	$2^{11}.2^4.2^4.(S_3 \times S_3)$
$C(19)$	$1 < 2^{4+12} < 2^{2+12}.2^4 < 2^{2+12}.2^5 < 2_+^{1+8}.2^6.2^4.2$	$2_+^{1+8}.2^6.2^4.2.S_3$
$C(20)$	$1 < 2^{4+12} < 2^{2+12}.2^4 < 2^{2+12}.2^5$	$2^{2+12}.2^5.(S_3 \times S_3)$
$C(21)$	$1 < (2^2)^* < D_8$	$D_8 \times G_2(2)$
$C(22)$	$1 < (2^2)^*$	$S_4 \times G_2(2)$
$C(23)$	$1 < (2^2)^* < 2^2 \times 4^2.2$	$S_4 \times 4^2.2.S_3$
$C(24)$	$1 < (2^2)^* < 2^2 \times 4^2.2 < D_8 \times 4^2.2$	$D_8 \times 4^2.2.S_3$
$C(25)$	$1 < (2^2)^* < 2^2 \times 4.2^2.2$	$S_4 \times 4.2^2.2.S_3$
$C(26)$	$1 < (2^2)^* < 2^2 \times 4.2^2.2 < D_8 \times 4.2^2.2$	$D_8 \times 4.2^2.2.S_3$
$C(27)$	$1 < (2^2)^* < 2^2 \times 4.2^2.2 < 2^2 \times S' < D_8 \times S'$	$D_8 \times S'$
$C(28)$	$1 < (2^2)^* < 2^2 \times 4.2^2.2 < 2^2 \times S'$	$S_4 \times S'$

Table 6: Some radical 2-chains of Co_1

Thus $P_1 \notin \{3^6.3, 3_+^{1+4}.3^3, 3^6.3^2, Sy_3\}$, and if $P_1 = 3^6$, then $C =_G C(14)$. We may suppose

$$P_1 \in \{3, 3^*, 3^2, 3_+^{1+2}, 3_+^{1+4}, 3^{3+4}, 3^{3+4}.3\} \subseteq \Phi(G,3).$$

Case (2). Let $Q = S' \in \Phi(M_1, 3)$. By the proof (1) of Lemma 4.1, we may suppose $Q \in \Phi(N_{M_1}(3^{3+4}), 3)$, and moreover, $N_{M_1}(Q) = N_{N_{M_1}(3^{3+4})}(Q)$. Define G-invariant subfamilies $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ of $\mathcal{R}(G)$, such that

$$\begin{aligned} \mathcal{L}^+(Q)/G &= \{C' \in \mathcal{R}/G : P_1' = 3, P_2' = Q\}, \\ \mathcal{L}^0(Q)/G &= \{C' \in \mathcal{R}/G : P_1' = 3, P_2' = 3^{3+4}, P_3' = Q\}. \end{aligned} \tag{5.6}$$

A similar proof to Case (1) shows that there exists a bijection g between $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ such that $N(C') = N(g(C'))$ for each $C' \in \mathcal{L}^+(Q)$. Thus we may suppose

$$C \notin (\mathcal{L}^+(Q) \cup \mathcal{L}^0(Q)). \tag{5.7}$$

It follows that if $P_1 = 3$, then we may assume $P_2 \in \Phi(M_1, 3)\backslash\{S'\}$ and if, moreover, $P_2 = 3^{3+4}$, then $C =_G C(7)$.

Let $\mathcal{M}^+(3_+^{1+2})$ and $\mathcal{M}^0(3_+^{1+2})$ be defined by (5.3) with R replaced by 3_+^{1+2} and 3^6 by 3. A similar proof shows that we may suppose

$$C \notin (\mathcal{M}^+(3_+^{1+2}) \cup \mathcal{M}^0(3_+^{1+2})),$$

so we may suppose $P_1 \neq_G 3_+^{1+2}$ and if $P_1 = 3$, then $P_2 \neq_G 3_+^{1+2}$.

Let $C' : 1 < 3 < 3^2 < S'$ and $g(C') : 1 < 3 < 3^2 < 3^{3+4} < S'$. Then $N(C') = N(g(C'))$ and we may delete C' and $g(C')$. It follows that if $P_1 = 3$, then $C \in_G \{C(i) : 2 \leq i \leq 9\}$ and we may suppose

$$P_1 \in \{3^*, 3^2, 3_+^{1+4}, 3^{3+4}, 3^{3+4}.3\} \subseteq \Phi(G,3).$$

Case (3). Let $C' : 1 < 3^2 < S'$ and $g(C') : 1 < 3^2 < 3^{3+4} < S'$. By the proof (2) of Lemma 4.1, $N(C') = N(g(C'))$ and we may delete C' and $g(C')$. Thus if $P_1 = 3^2$, then $C \in_G \{C(10), C(11), C(12), C(13)\}$.

Case (4). Let $C' : 1 < 3_+^{1+4} < Sy_3$ and $g(C') : 1 < 3_+^{1+4} < 3_+^{1+4}.3^3 < Sy_3$. By the proof (4) of Lemma 4.1, $N(C') = N(g(C'))$ and we may delete C' and $g(C')$. Let $\mathcal{M}^+(3^{3+4}.3)$ and $\mathcal{M}^0(3^{3+4}.3)$ be defined as in (5.3) with R replaced by $3^{3+4}.3$, 3^6 by 3_+^{1+4}. Then (5.5) holds. Thus we may suppose $P_1 \neq_G 3^{3+4}.3$ and if $P_1 = 3_+^{1+4}$, then $C \in_G \{C(15), C(16)\}$.

Case (5). Let $C' : 1 < 3^{3+4} < Sy_3$ and $g(C') : 1 < 3^{3+4} < 3^6.3^2 < Sy_3$. By the proof (5) of Lemma 4.1, $N(C') = N(g(C'))$ and we may delete C' and $g(C')$. Thus if $P_1 = 3^{3+4}$, then $C \in \{C(17), C(18), C(19), C(20)\}$.

Case (6). Let $C' : 1 < 3^* < S''$ and $g(C') : 1 < 3^* < 3^4 < S''$. By the proof (6) of Lemma 4.1, $N(C') = N(g(C'))$ and we may delete C' and $g(C')$. Thus if $P_1 = 3^*$, then $C \in \{C(i) : 21 \leq i \leq 26\}$. $\square$

The proof of the following Remark is similar to that of Lemma 5.1, since $N(C') = N(g(C'))$ implies $N_{\tilde{G}}(C') = N_{\tilde{G}}(g(C'))$.

Remark 5.2 *Let $p = 2, 3, 5$ and let $\tilde{G}$ be a covering group $2.\mathrm{Co}_1$ of $G = \mathrm{Co}_1$, ρ a faithful linear character of $Z(\tilde{G})$ and $\tilde{B}$ a block of $\tilde{G}$ covering the block $B(\rho)$ containing ρ. Let $\mathcal{R}^*(G) = \mathcal{R}^0(G)$ except when $p = 2$, in which case $\mathcal{R}^*(G)$ is the G-invariant subfamily of $\mathcal{R}(G)$ such that $\mathcal{R}^*(G)/G = \{C(i) : 1 \le i \le 28\}$, where $C(i)$ are defined in Table 6. If $D(\tilde{B}) \neq O_p(Z(\tilde{G}))$, then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N_{\tilde{G}}(C), \tilde{B}, d, \rho) = \sum_{C \in \mathcal{R}^*(G)/G} (-1)^{|C|} \mathrm{k}(N_{\tilde{G}}(C), \tilde{B}, d, \rho)$$

for all integers $d \ge 0$.

6 The proof of Dade's ordinary conjecture for Co_1

Let $N(C)$ be the normalizer of a radical p-chain. If $N(C)$ is a maximal subgroup of Co_1, then the character table of $N(C)$ can be found in the library of character tables distributed with GAP. If this is not the case, we construct a "useful" description of $N(C)$ and attempt to compute directly its character table using MAGMA.

If $N(C)$ is soluble, we construct a power-conjugate presentation for $N(C)$ and use this presentation to obtain the character table.

If $N(C)$ is insoluble, we construct faithful representations for $N(C)$ and use these as input to the character table construction function. We employ two strategies to obtain faithful representations of $N(C)$.

1. Construct the action of $N(C)$ on the cosets of soluble subgroups of $N(C)$.

2. Construct the actions of $N(C)$ on the cosets of its stabilizers acting on the underlying set of Co_1.

In several cases, however, none of the representations constructed was of sufficiently small degree to allow us to construct the required character table.

In these cases, we directly calculate the character table of $N(C)$ as follows: first calculate the character tables of some subgroups and quotient groups of $N(C)$; next induce or lift these characters to $N(C)$, so the liftings and the irreducible characters from the induction form a partial character table T of $N(C)$; then decompose the remaining inductions or the tensor products of the inductions using the table T, and extend the table T. If we choose suitable subgroups and quotient groups, then the character table can be obtained from the extension of T.

The tables listing degrees of irreducible characters referenced in the proof of Theorem 6.1 are available electronically [6].

Theorem 6.1 *Let B be a p-block of $G = \mathrm{Co}_1$ with a positive defect. Then B satisfies the ordinary conjecture of Dade.*

PROOF: We prove the theorem when $p = 3$, the proofs for other primes are similar. We may suppose B has a non-cyclic defect group.

Suppose $p = 3$, so that by Lemma 4.2 (c), $B \in \{B_0, B_1, B_2\}$. We set $\mathrm{k}(i, d) = \mathrm{k}(N(C(i)), B_0, d)$ for integers i, d.

First, we consider the radical 3-chains $C(j)$ with $\mathrm{d}(N(C(j))) \leq 5$, and so $21 \leq j \leq 26$. Then $N(C(i))$ has only the principal block except when $i = 21$, in which case $N(C(21))$ has exactly two blocks b_0 and b_2 with $b_j^G = B_j$ for $j = 0, 2$, and

$$\mathrm{k}(N(C(21)), B_2, d) = \begin{cases} 9 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

The values $\mathrm{k}(i, d)$ are given in Table 7.

Defect d	5	4	3	otherwise
k(21, d) = k(24, d)	0	45	0	0
k(22, d) = k(25, d)	18	24	0	0
k(23, d) = k(26, d)	0	27	3	0

Table 7: Values of $\mathrm{k}(i, d)$ with $\mathrm{d}(N(C(i))) \leq 5$

It follows that

$$\sum_{i=21}^{26} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d) = 0.$$

Next we consider the chains $C(i)$ such that $\mathrm{d}(N(C(i))) = 8$, so that $2 \leq i \leq 13$. If $i = 2$, then $N(C(2))$ has exactly 3 blocks b_0, b_1 and b_2 such that $b_j^{\mathrm{Co}_1} = B_j$ for each j. In addition, $\mathrm{k}(N(C(2)), B_2, d) = \mathrm{k}(N(C(21)), B_2, d)$ is given by (6.1) and

$$\mathrm{k}(N(C(2)), B_1, d) = \begin{cases} 9 & \text{if } d = 3, \\ 5 & \text{if } d = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{6.2}$$

If $i = 3$, then $N(C(3))$ has exactly 2 blocks b_0 and b_2 such that $b_j^{\mathrm{Co}_1} = B_j$ for $j = 0, 2$, and $\mathrm{k}(N(C(3)), B_2, d) = \mathrm{k}(N(C(21)), B_2, d)$ is given by (6.1). The values $\mathrm{k}(i, d)$ are given in Table 8.

It follows that

$$\sum_{i=2}^{13} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d) = 0.$$

Finally, we consider the chain $C(i)$ with $\mathrm{d}(N(C(i))) = 9$, so that $i \in \{1, 14 \leq i \leq 20\}$. Then $N(C(i))$ has only the principal block except when $i = 1$, in which case $N(C(1)) = G$ has exactly three blocks such that $\mathrm{k}(N(C(1)), B_2, d) = \mathrm{k}(N(C(21)), B_2, d)$ is given by (6.1) and $\mathrm{k}(N(C(1)), B_1, d) = \mathrm{k}(N(C(2)), B_1, d)$ is given by (6.2). This implies the theorem for $B \neq B_0$.

The values $\mathrm{k}(i, d)$ are given in Table 9.

It follows that

$$\sum_{i \in \{1,15,17,19\}} \mathrm{k}(N(C(i)), B_0, d) = \sum_{i \in \{14,16,18,20\}} \mathrm{k}(N(C(i)), B_0, d)$$

and Theorem 6.1 follows. $\square$

Defect d	8	7	6	5	otherwise
k(2, d)	27	33	8	15	0
k(3, d)	18	21	34	18	0
k(4, d)	18	12	34	0	0
k(5, d)	27	24	8	0	0
k(6, d)	18	21	46	18	0
k(7, d) = k(13, d)	27	33	38	15	0
k(8, d) = k(12, d)	27	24	38	0	0
k(9, d)	18	12	46	0	0
k(10, d)	27	33	26	15	0
k(11, d)	27	24	26	0	0

Table 8: Values of $\mathrm{k}(i, d)$ with $\mathrm{d}(N(C(i))) = 8$

Defect d	9	8	7	6	5	3	otherwise
k(1, d)	27	24	9	4	9	1	0
k(14, d)	27	24	3	4	0	0	0
k(15, d)	27	24	27	11	0	0	0
k(16, d)	27	24	21	11	6	1	0
k(17, d)	27	42	21	32	6	0	0
k(18, d)	27	42	9	21	9	0	0
k(19, d)	27	42	3	21	0	0	0
k(20, d)	27	42	27	32	0	0	0

Table 9: Values of $\mathrm{k}(i, d)$ with $\mathrm{d}(N(C(i))) = 9$

7 The proof of Dade's projective conjecture for $2.\mathrm{Co}_1$

Let C be a radical p-chain of Co_1 and $N_{2.\mathrm{Co}_1}(C) = 2.N_{\mathrm{Co}_1}(C)$. The character tables of $N_{2.\mathrm{Co}_1}(C)$ can either be found in the library of character tables distributed with GAP or computed directly using MAGMA as in Section 6, except when $C = C(4)$, in which case $N_{2.\mathrm{Co}_1}(C(4)) \simeq 2.(2_+^{1+8}.O_8^+(2))$ is a maximal subgroup of $2.\mathrm{Co}_1$. The approach outlined in Section 6 to construct character tables does not complete in available resources.

If $H = 2.(2_+^{1+8}.O_8^+(2))$, then $Z = Z(H) \simeq 2^2$. Suppose $Z(2.\mathrm{Co}_1) = \langle z \rangle$ and $Z(2_+^{1+8}.O_8^+(2)) = \langle y \rangle$, so that $Z = Z(H) = \langle y, z \rangle$. Thus $H/Z \simeq 2^8.O_8^+(2)$, $H/\langle y \rangle \simeq 2^9.O_8^+(2)$ and $H/\langle yz \rangle \not\simeq H/\langle z \rangle \simeq 2_+^{1+8}.O_8^+(2)$. For $W \leq Z$, we may regard $\mathrm{Irr}(H/W)$ as a subset of $\mathrm{Irr}(H)$. If $\chi \in \mathrm{Irr}(H)$, then $\ker(\chi) \cap Z \neq 1$, so that $\chi \in \mathrm{Irr}(H/W)$ for some non-trivial subgroup W of Z. It follows that

$$\mathrm{Irr}(H) = \mathrm{Irr}(H/\langle z \rangle) \cup (\mathrm{Irr}(H/\langle y \rangle) \backslash \mathrm{Irr}(H/Z)) \cup (\mathrm{Irr}(H/\langle zy \rangle) \backslash \mathrm{Irr}(H/Z)) \quad \text{(disjoint)}.$$

The character table of $H/\langle z \rangle \simeq 2_+^{1+8}.O_8^+(2)$ can be found in the library of character tables distributed with GAP, and the character tables of $H/\langle y\rangle$, $H/\langle zy\rangle$ and H/Z can be computed using MAGMA as in Section 6. Thus $\mathrm{Irr}(H)$ can be obtained.

Let ξ be the faithful linear character of $2 = Z(2.\mathrm{Co}_1)$ and K a subgroup of $2.\mathrm{Co}_1$ containing $Z(2.\mathrm{Co}_1)$. Denote by $\mathrm{Irr}(K \mid \xi)$ the characters in $\mathrm{Irr}(K)$ covering ξ.

The tables listing degrees of irreducible characters referenced in the proof of Theorem 7.1 are in [6].

Theorem 7.1 *Let B be a p-block of $G = 2.\mathrm{Co}_1$ with $D(B) > O_p(G)$. Then B satisfies the projective conjecture of Dade.*

PROOF: We prove the theorem when $p = 3$, the proofs for other primes are similar.

Suppose $p = 3$. If B is a block of G and H is a subgroup of G containing $Z(G)$, then let $\mathrm{Irr}(H, B, \xi) = \mathrm{Irr}(H \mid \xi) \cap \mathrm{Irr}(B)$, and let $\mathrm{Deg}(H, B, \xi)$ be the set of degrees of characters in $\mathrm{Irr}(H, B, \xi)$.

We may suppose B has a non-cyclic defect group. Thus G has exactly one block B with a non-cyclic defect group and $\mathrm{Irr}(G \mid \xi) \cap \mathrm{Irr}(B) \neq \emptyset$.

Let $\Omega = \{4, 5, 8, 9, 11, 12, 13, 14, 15, 16, 17, 19, 20, 23, 26\}$. Then by MAGMA, for each $i \in \Omega$, $\mathrm{Deg}(N_{2.\mathrm{Co}_1}(C(i)), B, \xi) = \mathrm{Deg}(B_0(N_{\mathrm{Co}_1}(C(i))))$, and

$$\mathrm{k}(N_{2.\mathrm{Co}_1}(C(i)), B, d, \xi) = \mathrm{k}(N_{\mathrm{Co}_1}(C(i)), B_0(\mathrm{Co}_1), d). \qquad (7.1),$$

so that $\mathrm{k}(N_{2.\mathrm{Co}_1}(C(i)), B, d, \xi)$ is given by Tables 7, 8 and 9. For each j we set

$$\mathrm{k}(j, d) = \mathrm{k}(N_{2.\mathrm{Co}_1}(C(j)), B, d, \xi).$$

First, we consider the radical 3-chains $C(j)$ with $\mathrm{d}(N_{2.\mathrm{Co}_1}(C(j))) \leq 5$, and so $21 \leq j \leq 26$. Thus

Defect d	5	4	otherwise
k(21, d) = k(24, d)	0	36	0
k(22, d) = k(25, d)	18	18	0

Table 10: Values of $\mathrm{k}(j, d)$ with $j \notin \Omega$ and $\mathrm{d}(N_{2.\mathrm{Co}_1}(C(j))) \leq 5$

Since $23, 26 \in \Omega$, it follows by (7.1) and Table 7 that $\mathrm{k}(N_{2.\mathrm{Co}_1}(C(23)), B, d, \xi) = \mathrm{k}(N_{2.\mathrm{Co}_1}(C(26)), B, d, \xi)$, so that

$$\sum_{i=21}^{26} (-1)^{|C(i)|} \mathrm{k}(N_{2.\mathrm{Co}_1}(C(i)), B, d, \xi) = 0.$$

Next we consider the chains $C(j)$ such that $\mathrm{d}(N_{2.\mathrm{Co}_1}(C(j))) = 8$, so that $2 \leq j \leq 13$. The values $\mathrm{k}(j, d)$ with $j \notin \Omega$ are given in Table 11.

For $j \in \{4, 5, 8, 9, 10, 11, 12, 13\}$ the values $\mathrm{k}(N_{2.\mathrm{Co}_1}(C(j)), B, d, \xi)$ are given in Table 8. It follows that

$$\sum_{i=2}^{13} (-1)^{|C(i)|} \mathrm{k}(N_{2.\mathrm{Co}_1}(C(i)), B, d, \xi) = 0.$$

Defect d	8	7	6	5	otherwise
k(2, d)	27	24	8	6	0
k(3, d)	18	12	34	9	0
k(6, d)	18	12	46	0	0
k(7, d)	27	24	38	6	0
k(10, d)	27	24	26	6	0

Table 11: Values of $\mathrm{k}(j,d)$ with $j \notin \Omega$ and $\mathrm{d}(N_{2.\mathrm{Co}_1}(C(j))) = 8$

Defect d	9	8	7	6	5	3	otherwise
k(1, d)	27	24	3	4	3	1	0
k(18, d)	27	42	3	21	3	0	0

Table 12: Values of $\mathrm{k}(j,d)$ with $j \notin \Omega$ and $\mathrm{d}(N_{2.\mathrm{Co}_1}(C(j))) = 9$

Finally, we consider the chains $C(j)$ with $\mathrm{d}(N_{2.\mathrm{Co}_1}(C(j))) = 9$, so that $j \in \{1, 14 \leq j \leq 20\}$ and $j \in \Omega$ when $j \neq 1, 18$. Thus the values $\mathrm{k}(N_{2.\mathrm{Co}_1}(C(j)), B, d, \xi)$ are given in Table 9 except when $j \neq 1, 18$, and the values $\mathrm{k}(N_{2.\mathrm{Co}_1}(C(1)), B, d, \xi)$ and $\mathrm{k}(N_{2.\mathrm{Co}_1}(C(18)), B, d, \xi)$ are given in Table 12.

It follows that

$$\sum_{i \in \{1,15,17,19\}} \mathrm{k}(N_{2.\mathrm{Co}_1}(C(i)), B, d, \xi) = \sum_{i \in \{14,16,18,20\}} \mathrm{k}(N_{2.\mathrm{Co}_1}(C(i)), B, d, \xi)$$

and Theorem 7.1 follows. □

References

[1] J.L. Alperin, Weights for finite groups, *in* “The Arcata Conference on Representations of Finite Groups”, Proc. of Symposia in Pure Math. **47** (1987), 369-379.

[2] Jianbei An and E.A. O’Brien, *A local strategy to decide the Alperin and Dade conjectures*, J. Algebra, **206** (1998), 183-207.

[3] Jianbei An and E.A. O’Brien, *The Alperin and Dade conjectures for the simple Fischer group* Fi_{23}, International J. Algebra Comput., **9** (1999), 621-670.

[4] Jianbei An and E.A. O’Brien, *The Alperin and Dade conjectures for the simple Fischer group* Fi_{22}, submitted.

[5] Jianbei An and E.A. O’Brien, *The Alperin and Dade conjectures for the O’Nan and Rudvalis simple groups*, Comm. Algebra, to appear.

[6] Jianbei An and E.A. O'Brien, *Chain normalizers of the simple Conway group* Co_1, www.math.auckland.ac.nz/~obrien/research/conway/appendix.dvi

[7] Wieb Bosma, John Cannon and Catherine Playoust (1997), *The* MAGMA *algebra system I: The user language*, J. Symbolic Comput. **24**, 235–265.

[8] N. Burgoyne and C. Williamson, *On a theorem of Borel and Tits for finite Chevalley groups*, Arch. Math. **27** (1976), 489-491.

[9] R.T. Curtis, *On subgroups of* $\cdot 0$ *II. Local structure*, J. Algebra **63** (1980), 413-434.

[10] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups.* Clarendon Press, Oxford, 1985.

[11] Frank Celler, Charles R. Leedham-Green, Scott H. Murray, Alice C. Niemeyer and E.A. O'Brien, *Generating random elements of a finite group*, Comm. Algebra **23** (1995), 4931–4948.

[12] E.C. Dade, *Counting characters in blocks*, I, Invent. Math. **109** (1992), 187-210.

[13] E.C. Dade, *Counting characters in blocks.* II.9. Representation theory of finite groups (Columbus, OH, 1995), Ohio State Univ. Math. Res. Inst. Publ. **6** (1997), pp. 45-59. de Gruyter, Berlin.

[14] The GAP Team, GAP – *Groups, Algorithms, and Programming, Version* 4. Lehrstuhl D für Mathematik, RWTH Aachen, and School of Mathematical and Computational Sciences, University of St Andrews (2000).

[15] G. Hiss and K. Lux, *Brauer Trees of Sporadic Groups.* Oxford Science Publications, 1989.

[16] H. Nagao and Y. Tsushima, *Representations of Finite Groups.* Academic Press, Inc., Boston MA, 1989.

[17] K. Uno, *Dade's conjecture for tame blocks*, Osaka J. Math. **31** (1994), 747-772.

[18] R.A. Wilson, *The complex Leech latice and maximal subgroups of the Suzuki group*, J. Algebra **84** (1983), 151-188.

[19] R.A. Wilson, *The maximal subgroups of Conway's group* $\cdot 2$, J. Algebra **84** (1983), 107-114.

[20] R.A. Wilson, *The maximal subgroups of Conway's group* Co_1, J. Algebra **85** (1983), 144-165.

[21] R.A. Wilson, *On the* 3*-local subgroups of Conway's group* Co_1, J. Algebra **113** (1988), 261-262.

[22] Robert A. Wilson, *Standard generators for sporadic simple groups*, J. Algebra **184** (1996), 505–515.

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