Local Representation Theory of Finite Groups and Cyclotomic Algebras

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- Feit–Thompson, 1963

  If G is a non abelian simple finite group, then $2 \mid |G|$.

- Cauchy (1789–1857)

  If $\ell \mid |G|$, there are non trivial ℓ–subgroups in G.

- Sylow, 1872

  The maximal ℓ–subgroups of G are all conjugate under G.

Assume $P \subset S$ and $P \subset S'$. There is $g \in G$ such that $S' = S^g$ $(= g^{-1}Sg)$, hence

$$P \subset S \quad \text{and} \quad {}^gP\,(= gPg^{-1}) \subset S\,.$$

This is a *fusion*.

The Frobenius Category

$\mathrm{Frob}_\ell(G)$:

- Objects : the ℓ–subgroups of G,
- $\mathrm{Hom}(P, Q) := \{g \in G \mid ({}^gP \subset Q)\}/C_G(P)\,.$

Note that $\mathrm{Aut}(P) = N_G(P)/C_G(P)\,.$

Alperin's fusion theorem (1967) says essentially that $\mathrm{Frob}_\ell(G)$ is known as soon as the automorphisms of some of its objects are known.

Main question of local group theory

How much is known about G from the knowledge (up to equivalence of categories) of $\mathrm{Frob}_\ell(G)$?

Well, certainly not more than $G/O_{\ell'}(G)$!
(where $O_{\ell'}(G)$ denotes the largest normal subgroup of G of order not divisible by ℓ)

Indeed, $O_{\ell'}(G)$ disappears in the Frobenius category, since, for P an ℓ–subgroup,

$$O_{\ell'}(G) \cap N_G(P) \subseteq C_G(P)\,.$$

But perhaps all of $G/O_{\ell'}(G)$?

Control subgroup

Let H be a subgroup of G. The following conditions are equivalent :

(i) The inclusion $H \hookrightarrow G$ induces an equivalence of categories

$$\mathrm{Frob}_\ell(H) \xrightarrow{\sim} \mathrm{Frob}_\ell(G)\,,$$

(ii) H contains a Sylow ℓ–subgroup of G, and if P is a ℓ–subgroup of H and g is an element of G such that ${}^gP \subseteq H$, then there is $h \in H$ and $z \in C_G(P)$ such that $g = hz$.

If the preceding conditions are satisfied, we say that H controls ℓ–fusion in G, or that H is a control subgroup in G.

The first question may now be reformulated as follows :

If H controls ℓ–fusion in G, does the inclusion $H \hookrightarrow G$ induce an isomorphism

$$H/O_{\ell'}(H) \xrightarrow{\sim} G/O_{\ell'}(G)\,?$$

In other words, do we have

$$G = HO_{\ell'}(G)\,?$$

- Frobenius theorem, 1905

  If a Sylow ℓ–subgroup S of G controls ℓ–fusion in G, then the inclusion induces an isomorphism $S \simeq G/O_{\ell'}(G)$.

- ℓ–solvable groups, ?

  Assume that G is ℓ–solvable. If H controls ℓ-fusion in G, then the inclusion induces an isomorphism $H/O_{\ell'}(H) \simeq G/O_{\ell'}(G)$.

- Z_ℓ^*–theorem (Glauberman, 1966 for $\ell = 2$, Classification for other primes)

  Assume that $H = C_G(P)$ where P is an ℓ–subgroup of G. If H controls ℓ-fusion in G, then the inclusion induces an isomorphism $H/O_{\ell'}(H) \simeq G/O_{\ell'}(G)$.

- But

  Burnside (1852–1927)

  Assume that a Sylow ℓ–subgroup S of G is abelian. Set $H := N_G(S)$. Then H controls ℓ-fusion in G.

Consider the **Monster**, a finite simple group of order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \simeq 8.10^{53} \,.$$

(predicted in 1973 by Fischer and Griess, constructed in 1980 by Griess, proved to be unique by Thompson)

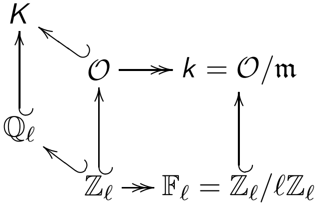
and the normalizer H of one of its Sylow 11–subgroups, a group of order 72600, isomorphic to $(C_{11} \times C_{11}) \rtimes (C_5 \times \mathrm{SL}_2(5))$ (here we denote by C_m the cyclic group of order m).

Here we have $G \neq HO_{11'}(G)$ since G is simple.

Remark : one may think of more elementary examples...

Local Representation Theory

Let K be a finite extension of the field of ℓ–adic numbers $\mathbb{Q}_\ell$ which contains the $|G|$-th roots of unity. Let $\mathcal{O}$ be the ring of integers of K over $\mathbb{Z}_\ell$, with maximal ideal $\mathfrak{m}$ and residue field $k := \mathcal{O}/\mathfrak{m}$.



Block decomposition

$$\begin{array}{ccccc} \mathcal{O}G & = & \bigoplus & B & \text{(indecomposable algebra)} \\ \downarrow & & & & \downarrow \\ kG & = & \bigoplus & kB & \text{(indecomposable algebra)} \end{array}$$

The augmentation map $\mathcal{O}G \to \mathcal{O}$ factorizes through a unique block of $\mathcal{O}G$ called *the principal block* and denoted by $(\mathcal{O}G)_0$.

$$\begin{array}{ccc} \mathcal{O}G & \longrightarrow & (\mathcal{O}G)_0 \\ & \searrow & \downarrow \\ & & \mathcal{O} \end{array}$$

Factorisation and principal block

If H is a subgroup of G, the following assertions are equivalent

(i) $G = HO_{\ell'}(G)$.

(ii) The map Res_H^G induces an isomorphism from $(\mathcal{O}G)_0$ onto $(\mathcal{O}H)_0$.

Let us re-examine the counterexamples to factorisation coming from Burnside's theorem.

Assume that a Sylow ℓ–subgroup S of G is abelian, let $H := N_G(S)$ be its normalizer.

Even if $G \neq H\,O_{\ell'}(G)$, it appears that there is some connection between the (representation theory of) $(\mathcal{O}G)_0$ and $(\mathcal{O}H)_0$.

SOME NUMERICAL MIRACLES

Let us consider the case $G = \mathfrak{A}_5$ and $\ell = 2$. Then we have $H \simeq \mathfrak{A}_4$.

Remark : on a larger screen, we might as well consider the above case of the Monster and of the prime $\ell = 11$.

Table: Character table of $\mathfrak{A}_5$

	(1)	(2)	(3)	(5)	(5')
1	1	1	1	1	1
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0
χ_3	3	-1	0	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$
χ_3'	3	-1	0	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$

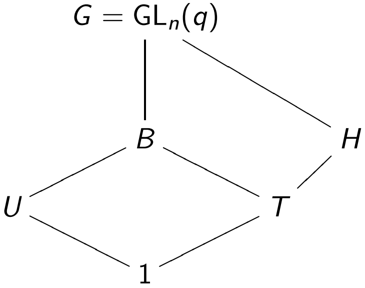
Table: Character table of $(\mathcal{O}\mathfrak{A}_5)_0$

	(1)	(2)	(5)	(5')	(3)
1	1	1	1	1	1
χ_5	5	1	0	0	-1
χ_3	3	-1	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$	0
χ_3'	3	-1	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$	0

Table: Character table of $(\mathcal{O}\mathfrak{A}_4)_0$

	(1)	(2)	(3)	(3')
1	1	1	1	1
$-\alpha_3$	-3	1	0	0
$-\alpha_1$	-1	-1	$(1+\sqrt{-3})/2$	$(1-\sqrt{-3})/2$
$-\alpha_1'$	-1	-1	$(1-\sqrt{-3})/2$	$(1+\sqrt{-3})/2$

A kind of generic counterexample :



$$|T| = (q-1)^n$$

$$H := N_G(T)\,,\ H/T = \mathfrak{S}_n$$

$$|U| = q^{\binom{n}{2}}\,,\ B = U \rtimes T$$

$$\ell \mid q-1\,, \ell > n \ \Rightarrow\ S = T_\ell$$

$$T = S \times T_{\ell'}\,,\ H = N_G(S)$$

We certainly have

$$G \neq HO_{\ell'}(G)\,.$$

Morita equivalences

Definition

A Morita equivalence between A and B is the following datum :

- an object M of ${}_A\mathbf{Mod}_B$ and an object N of ${}_B\mathbf{Mod}_A$,
- two isomorphisms

$$M \otimes_B N \xrightarrow{\sim} A \quad \text{in} \quad {}_A\mathbf{Mod}_A \qquad \text{and} \qquad N \otimes_A M \xrightarrow{\sim} B \quad \text{in} \quad {}_B\mathbf{Mod}_B \,.$$

Given a Morita equivalence, the functors

$$M \otimes_B \cdot \quad \text{and} \quad N \otimes_A \cdot$$

are reciprocal equivalences of categories between ${}_A\mathbf{Mod}$ and ${}_B\mathbf{Mod}$.

Fundamental example

Whenever $n \geq 1$ is an integer, $\mathrm{Mat}_n(A)$ and A are Morita equivalent.

Proof.

Consider the bimodules M and N defined as follows :

- M is the set of $n \times 1$ matrices with coefficients in A, on which $\mathrm{Mat}_n(A)$ acts by left multiplication and A acts by (right) multiplication,
- N is the set of $1 \times n$ matrices with coefficients in A, on which $\mathrm{Mat}_n(A)$ acts by right multiplication and A acts by (left) multiplication.

Then the multiplication of matrices defines isomorphisms

$$M \otimes_A N \xrightarrow{\sim} \mathrm{Mat}_n(A) \quad \text{and} \quad N \otimes_{\mathrm{Mat}_n(A)} M \xrightarrow{\sim} A \,.$$

□

Morita equivalences and local representations

Assume

$$K \hookleftarrow \mathcal{O} \twoheadrightarrow k$$

and that A and B are $\mathcal{O}$–algebras.

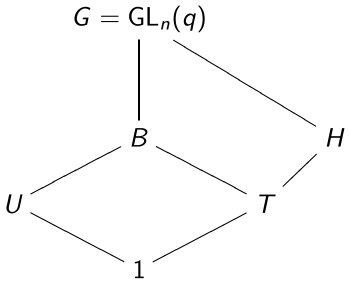
Then a Morita equivalence between A and B induces Morita equivalences

$$KA \equiv KB \quad \text{and} \quad kA \equiv kB\,,$$

via

$$KM \hookleftarrow M \twoheadrightarrow kM \qquad \text{and} \qquad KN \hookleftarrow N \twoheadrightarrow kN$$

On $\mathrm{GL}_n(q)$ again

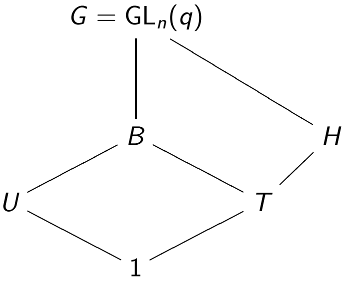


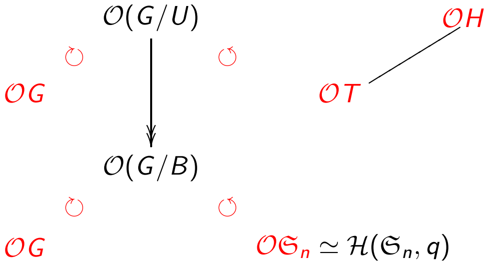
The principal block algebras of G and H respectively are Morita equivalent.

There exist M and N, respectively an $\mathcal{O}G$–module–$\mathcal{O}H$ and an $\mathcal{O}H$–module–$\mathcal{O}G$ with the following properties :

$$M \otimes_{\mathcal{O}H} N \simeq (\mathcal{O}G)_0 \text{ as } \mathcal{O}G\text{–module–}\mathcal{O}G$$

$$N \otimes_{\mathcal{O}G} M \simeq (\mathcal{O}H)_0 \text{ as } \mathcal{O}H\text{–module–}\mathcal{O}H$$





- Viewed as a $\mathcal{O}G$–module–$\mathcal{O}S$, we have $M \simeq \mathcal{O}(G/U)$, *i.e.*, the functor $M \otimes_{\mathcal{O}S} ?$ is the Harish–Chandra induction.
- $M/T = \mathcal{O}(G/B)$ whose commuting algebra is the Hecke algebra $\mathcal{H}(\mathfrak{S}_n, q)$.

Rickard equivalences

Definition

A Rickard equivalence between A and B is the following datum :

- an object M of $\mathcal{C}^b({}_A\mathbf{Mod}_B)$ and an object N of $\mathcal{C}^b({}_B\mathbf{Mod}_A)$,
- two isomorphisms

$$M \otimes_B N \xrightarrow{\sim} A \text{ in } \mathcal{C}^b({}_A\mathbf{Mod}_A) \quad \text{and} \quad N \otimes_A M \xrightarrow{\sim} B \text{ in } \mathcal{C}^b({}_B\mathbf{Mod}_B)\,.$$

Given a Rickard equivalence, the functors

$$M \otimes_B \cdot \quad \text{and} \quad N \otimes_A \cdot$$

are reciprocal equivalences of suitable categories.

Back to the principal 2–block of $\mathfrak{A}_5$

- View $(\mathcal{O}\mathfrak{A}_5)_0$ as a $\mathcal{O}\mathfrak{A}_5$–module–$\mathcal{O}\mathfrak{A}_4$.
- Let I be the kernel of the augmentation map : $(\mathcal{O}\mathfrak{A}_5)_0 \to \mathcal{O}$.
- Let P denote a projective cover of I and consider

$$\begin{array}{ccc} P & & \\ \downarrow\!\!\downarrow & \searrow & \\ I & \longrightarrow & (\mathcal{O}\mathfrak{A}_5)_0 \end{array}$$

- We set

$$M := 0 \to P \to (\mathcal{O}\mathfrak{A}_5)_0 \to 0$$

  - a complex of $\mathcal{O}\mathfrak{A}_5$–modules–$\mathcal{O}\mathfrak{A}_4$,
  - $(\mathcal{O}\mathfrak{A}_5)_0$ in degree 0 and C in degree -1.
- and $N := M^*$.

Proposition

The pair of complexes (M, N) induces a Rickard equivalence between $(\mathcal{O}\mathfrak{A}_5)_0$ and $(\mathcal{O}\mathfrak{A}_4)_0$.

Assume that a Sylow ℓ–subgroup S of G is abelian, let $H := N_G(S)$ be its normalizer.

- (ASC) :

  The algebras $(\mathcal{O}G)_0$ and $(\mathcal{O}H)_0$ are Rickard equivalent.

- (Strong ASC) :

  They are Rickard equivalent in a way which is compatible with the equivalence of Frobenius categories

Which means : There is a G–equivariant collection of derived equivalences

$$\{\, \mathcal{E}(P) \,:\, \mathcal{D}^b((\mathcal{O}C_G(P))_0) \xrightarrow{\sim} \mathcal{D}^b((\mathcal{O}C_H(P))_0) \,\}_{P \subseteq S}$$

compatible with Brauer morphisms.

Known to be true :
Sylow cyclic (Rickard), G ℓ–solvable, $G = \mathfrak{S}_n$ (Chuang–Rouquier), $G = \mathrm{SL}_2(\ell^n)$ (Okuyama), a bunch of sporadic simple groups (the Japanese school),...

What about the nonabelian Sylow case ?

The fact that the derived category of $(\mathcal{O}G)_0$ is determined by $\mathrm{Frob}_\ell(G)$ is definitely false :

There are groups G and a subgroup H such that

- the inclusion $H \subset G$ induces an equivalence $\mathrm{Frob}_\ell(H) \xrightarrow{\sim} \mathrm{Frob}_\ell(G)$,
- and yet $\mathcal{D}^b((\mathcal{O}H)_0)$ and $\mathcal{D}^b((\mathcal{O}G)_0)$ are not equivalent.

But there seem to be lots of numerical similarities between $(\mathcal{O}H)_0$ and $(\mathcal{O}G)_0$.

Case of finite reductive groups

$\mathbf{G}$ is a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius–like endomorphism F. The group $G := \mathbf{G}^F$ is a finite reductive group.

Example

$$\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_q) \ , \ F : (a_{i,j}) \mapsto (a_{i,j}^q) \ , \ G = \mathrm{GL}_n(q)$$

- Polynomial order — There is a polynomial in $\mathbb{Z}[x]$

$$|\mathbb{G}|(x) = x^N \prod_d \Phi_d(x)^{a(d)}$$

such that $|\mathbb{G}|(q) = |G|$.

Example

$$|\mathrm{GL}_n|(x) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} (x^d - 1) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} \Phi_d(x)^{[n/d]}$$

- Admissible subgroups — The tori of G are the subgroups of the shape $\mathbf{T}^F$ where $\mathbf{T}$ is an F–stable torus (*i.e.*, isomorphic to some $\overline{\mathbb{F}}^\times \times \cdots \times \overline{\mathbb{F}}^\times$ in $\mathbf{G}$).

  The Levi subgroups of G are the subgroups of the shape $\mathbf{L}^F$ where $\mathbf{L}$ is a centralizer of an F–stable torus in $\mathbf{G}$.

  Examples

  The split maximal torus $T_1 = (\mathbb{F}_q^\times)^n$ of order $(q-1)^n$

  The Coxeter maximal torus $T_c = \mathrm{GL}_1(\mathbb{F}_{q^n})$ of order $q^n - 1$

  Levi subgroups have shape $\mathrm{GL}_{n_1}(q^{a_1}) \times \cdots \times \mathrm{GL}_{n_s}(q^{a_s})$

- Cauchy theorem — The (polynomial) order of an admissible subgroup divides the (polynomial) order of the group.

The generic Sylow theorems

For $\Phi_d(x)$ a cyclotomic polynomial, a $\Phi_d(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi_d(x)$. Hence such a group is a torus.

Sylow theorems

1. Maximal $\Phi_d(x)$–subgroups ("Sylow $\Phi_d(x)$–subgroups") of G have as (polynomial) order the contribution of $\Phi_d(x)$ to the (polynomial) order of G :
$$|S_d| = |\mathbf{S}_d^F| = \Phi_d(q)^{a(d)} .$$
Notation : Set $\mathbf{L}_d := C_{\mathbf{G}}(\mathbf{S}_d)$ and $\mathbf{N}_d := N_{\mathbf{G}}(\mathbf{S}_d) = N_{\mathbf{G}}(\mathbf{L}_d)$
2. Sylow $\Phi_d(x)$–subgroups are all conjugate by G.
3. The (polynomial) index $|G : N_d|$ is congruent to 1 modulo $\Phi_d(x)$.
4. $W_d := N_d/L_d$ is a true finite group, a complex reflection group in its action on $\mathbb{C} \otimes Y(\mathbf{S}_d)$. $=$ This is the d–cyclotomic Weyl group of the finite reductive group G.

Example

Recall that

$$|\mathrm{GL}_n(q)| = q^{\binom{n}{2}} \prod_{d=1}^{d=n} \Phi_d(q)^{[n/d]}$$

For each d $(1 \leq d \leq n)$, $\mathrm{GL}_n(q)$ contains a subtorus of (polynomial) order $\Phi_d(x)^{[\frac{n}{d}]}$

Assume $n = md + r$ with $r < d$. Then

$$L_d = \mathrm{GL}_1(q^d)^m \times \mathrm{GL}_r(q)$$

$$W_d = \boldsymbol{\mu}_d \wr \mathfrak{S}_m$$

Generic and ordinary Sylow subgroups

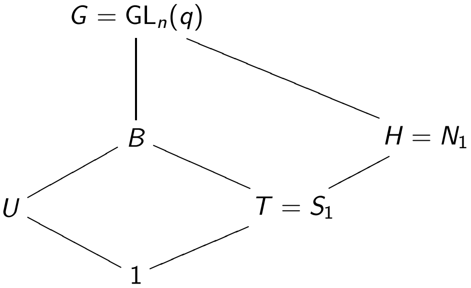
Let ℓ be a prime number which does not divide $|W|$.

- If ℓ divides $|G$, there is a unique integer d such that ℓ divides $\Phi_d(q)$.
- Then the Sylow ℓ–subgroups of G are nothing but the Sylow ℓ–subgroups S_ℓ of $S_d = \mathbf{S}_d^F$ ($\mathbf{S}_d$ a Sylow $\Phi_d(x)$–subgroup of $\mathbf{G}$).
- We have

$$N_G(S_\ell) = N_d \text{ and } C_G(S_\ell) = L_d \,.$$

hence

$$N_G(S_\ell)/C_G(S_\ell) = W_d \,.$$

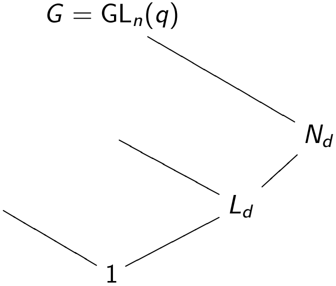


$$|S_1| = (q-1)^n$$

$$W_1 = \mathfrak{S}_n$$

$$\ell \mid q-1\,, \ell > n \;\Rightarrow\; S = T_\ell$$

$$T = S \times T_{\ell'}\,,\; H = N_G(S)$$



$$|S_d| = \Phi_d(q)^{a(d)}$$

$$L_d / C_d = W_d$$

$$\ell \mid \Phi_d(q)\,, \ell > n \;\Rightarrow\; S = (S_d)_\ell$$

$$S_d = (S_d)_\ell \times (S_d)_{\ell'}$$

Complex reflection groups

A finite reflection group (abbreviated frg) on K is a finite subgroup of $\mathrm{GL}_K(V)$ (V a finite dimensional K–vector space) generated by *reflections*, *i.e.*, linear maps represented by

$$\begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- A finite reflection group on $\mathbb{R}$ is called a Coxeter group.
- A finite reflection group on $\mathbb{Q}$ is called a Weyl group.

Main characterisation

Theorem (Shephard–Todd, Chevalley–Serre)

Let G be a finite subgroup of $\mathrm{GL}(V)$ (V an r–dimensional vector space over a characteristic zero field K). Let S denote the symmetric algebra of V, isomorphic to the polynomial ring $K[v_1, v_2, \dots, v_r]$.
The following assertions are equivalent.

1. G is generated by reflections.
2. The ring $R := S^G$ of G–fixed polynomials is a polynomial ring $K[u_1, u_2, \dots, u_r]$ in r homogeneous algebraically independant elements.
3. S is a free R–module.

Then

- If $d_i := \deg(u_i)$, the family $(d_1, \dots, d_r)$ is called the family of <u>invariant degrees</u> of G,
- and we have

$$|G| = d_1 d_2 \cdots d_r \,.$$

Examples

- For $G = \mathfrak{S}_r$, which acts naturally on $V = \mathbb{C}^r = \bigoplus \mathbb{C} v_i$, one may choose

$$\begin{cases} u_1 = v_1 + \cdots + v_r \\ u_2 = v_1 v_2 + v_1 v_3 + \cdots + v_{r-1} v_r \\ \vdots \quad\quad \vdots \\ u_r = v_1 v_2 \cdots v_r \end{cases}$$

- For $G = \langle e^{2\pi i/d} \rangle$, cyclic group of order d acting by multiplication on $V = \mathbb{C}$, we have

$$S = K[x] \quad \text{and} \quad R = K[x^d] .$$

- Consider again the action of $\mathfrak{S}_r$ on $V = \mathbb{C}^r = \bigoplus \mathbb{C} v_i$. Fix $d \geq 2$. For each coordinate consider the reflection $v_i \mapsto \zeta_d v_i$. We obtain the wreath product $C_d \wr \mathfrak{S}_r$, generated by reflections.
  This group is called $G(d, 1, r)$.
  For each divisor e of d, there is a normal reflection subgroup $G(d, e, r)$ of $G(d, 1, r)$ of index e.

- Let $G \leq \mathrm{SL}_2(\mathbb{C})$ be finite and $g \in G$. Let ζ be an eigenvalue of g. Then $\zeta^{-1} g$ is a reflection.
  - So, if $G = \langle g_1, \dots, g_r \rangle$, the group $\langle \zeta_1^{-1} g_1, \dots, \zeta_r^{-1} g_r \rangle$ is an frg.
  - Note that for G irreducible, we have $G/Z(G) \in \{D_n, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5\}$.
  - For example, the group

$$G := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \frac{\sqrt{-3}}{3} \begin{pmatrix} -\zeta_3 & \zeta_3^2 \\ 2\zeta_3^2 & 1 \end{pmatrix} \right\rangle \leq \mathrm{GL}_2(\mathbb{Q}(\zeta_3)),$$

    with $\zeta_3 := \exp(2\pi i/3)$, is a frg of order 72, denoted G_5, isomorphic to $\mathrm{SL}_2(3) \times C_3$.
  - We may choose

$$u_1 := v_1^6 + 20 v_1^3 v_2^3 - 8 v_2^6, \quad u_2 := 3 v_1^3 v_2^9 + 3 v_1^6 v_2^6 + v_1^9 v_2^3 + v_2^{12},$$

    with degrees $d_1 = 6, d_2 = 12$ (note that $d_1 d_2 = 72 = |G|$).
- If $g \in \mathrm{SL}_3(\mathbb{C})$ is an involution, then $-g$ is a reflection. Note that $\mathfrak{A}_5$, $\mathrm{PSL}_2(7)$ and $3.\mathfrak{A}_6$ have faithful 3-dimensional representations and are generated by involutions.

Classification

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
   - There is one infinite series $G(de, e, r)$ (d ,e and r integers),
   - ...and 34 exceptional groups G_4 , G_5 , . . . , G_{37}.
2. The group $G(de, e, r)$ (d ,e and r integers) consists of all $r \times r$ monomial matrices with entries in μ_{de} such that the product of entries belongs to μ_d.
3. We have

$$
\begin{aligned}
&G(d,1,r) \simeq C_d \wr \mathfrak{S}_r \\
&G(e,e,2) = D_{2e} \quad \text{(dihedral group of order } 2e\text{)} \\
&G(2,2,r) = W(\mathrm{D}_r) \\
&G_{23} = H_3 \;,\; G_{28} = F_4 \;,\; G_{30} = H_4 \\
&G_{35,36,37} = E_{6,7,8} \,.
\end{aligned}
$$

Braid groups

Let $\mathcal{A}$ be the arrangement of reflecting hyperplanes for the crg G.
Set

$$V^{\text{reg}} := V - \bigcup_{H \in \mathcal{A}} H .$$

The covering $V^{\text{reg}} \twoheadrightarrow V^{\text{reg}}/G$ is Galois, hence induces a short exact sequence

$$1 \longrightarrow \Pi_1(V^{\text{reg}}, x_0) \longrightarrow \Pi_1(V^{\text{reg}}/G, x_0) \longrightarrow G \longrightarrow 1$$

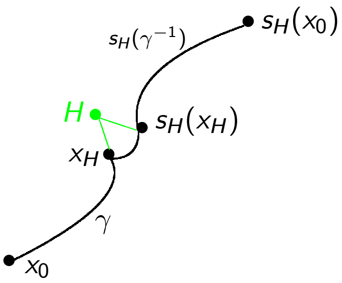
$$\Pi_1(V^{\text{reg}}, x_0) = P_G \quad \text{(Pure braid group)}$$

$$\Pi_1(V^{\text{reg}}/G, x_0) = B_G \quad \text{(Braid group)}$$

Braid reflections

Let γ be a path in V^{reg} from x_0 to x_H.

We define : $\sigma_{H,\gamma} := s_H(\gamma^{-1}) \cdot \mathbf{s}_{H,x_H} \cdot \gamma$



Definition

We call *braid reflections* the elements $\mathbf{s}_{H,\gamma} \in B$ defined by the paths $\sigma_{H,\gamma}$.

The following properties are immediate.

- $\mathbf{s}_{H,\gamma}$ and $\mathbf{s}_{H,\gamma'}$ are conjugate in P.
- $\mathbf{s}_{H,\gamma}^{e_H}$ is a loop in V^{reg} :

The variety V (resp. V/G) is connected, the hyperplanes are irreducible divisors (irreducible closed subvarieties of codimension one), and the braid reflections are "generators of the monodromy" around the irreducible divisors. Then

Theorem

1. The braid group B_G is generated by the braid reflections $(\mathbf{s}_{H,\gamma})$ (for all H and all γ).
2. The pure braid group P_G is generated by the elements $(\mathbf{s}_{H,\gamma}^{e_H})$

Artin–like presentations

An *Artin–like* presentation is

$$\langle \mathbf{s} \in \mathbf{S} \mid \{\mathbf{v}_i = \mathbf{w}_i\}_{i \in I} \rangle$$

where

- $\mathbf{S}$ is a finite set of distinguished braid reflections,
- I is a finite set of relations which are multi–homogeneous,

  i.e., such that (for each i) $\mathbf{v}_i$ and $\mathbf{w}_i$ are positive words in elements of $\mathbf{S}$

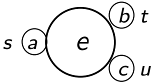
Theorem (Bessis)

Let $G \subset \mathrm{GL}(V)$ be a complex reflection group. Let $d_1 \leq d_2 \leq \cdots \leq d_r$ be the family of its invariant degrees.

1. The following integers are equal (denoted by Γ_G) :
   - The minimal number of reflections needed to generate G
   - The minimal number of braid reflections needed to generate B_G
   - $\lceil (N_r + N_h)/d_r \rceil$ ($N_r :=$ number of reflections, $N_h :=$ number of hyperplanes)
2. Either $\Gamma_G = r$ or $\Gamma_G = r + 1$, and the group B_G has an Artin–like presentation by Γ_G braid reflections.

The braid diagrams

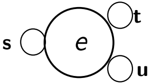
Let $\mathcal{D}$ be a diagram like



$\mathcal{D}$ represents the relations

$$\underbrace{stustu\cdots}_{e \text{ factors}} = \underbrace{tustus\cdots}_{e \text{ factors}} = \underbrace{ustust\cdots}_{e \text{ factors}} \quad \text{and} \quad s^a = t^b = u^c = 1$$

We denote by $\mathcal{D}_{\mathrm{br}}$ and call *braid diagram* the diagram

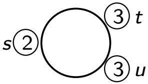


which represents the relations

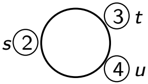
$$\underbrace{stustu\cdots}_{e \text{ factors}} = \underbrace{tustus\cdots}_{e \text{ factors}} = \underbrace{ustust\cdots}_{e \text{ factors}}$$

Note that

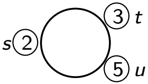
G_7 :



G_{11} :



G_{19} :



have same braid diagram.

For each irreducible complex irreducible group G,

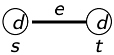
there is a diagram $\mathcal{D}$,

whose set of nodes $\mathcal{N}(\mathcal{D})$ is identified with a set of distinguished reflections in G,

such that

Theorem

For each $s \in \mathcal{N}(\mathcal{D})$, there exists a braid reflection $\mathbf{s} \in B_G$ above s such that the set $\{\mathbf{s}\}_{s \in \mathcal{N}(\mathcal{D})}$, together with the braid relations of $\mathcal{D}_{\mathrm{br}}$, is a presentation of B_G.

- The groups G_n for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type 

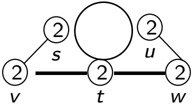
, corresponding to the presentation

$$s^d = t^d = 1 \text{ and } \underbrace{ststs\cdots}_{e \text{ factors}} = \underbrace{tstst\cdots}_{e \text{ factors}}$$

- The group G_{18} has diagram 

corresponding to the presentation

$$s^5 = t^3 = 1 \text{ and } stst = tsts\,.$$

- The group G_{31} has diagram 

corresponding to the presentation

$$s^2 = t^2 = u^2 = v^2 = w^2 = 1\,,$$
$$uv = vu\,,\ sw = ws\,,\ vw = wv\,,\quad sut = uts = tsu\,,$$
$$svs = vsv\,,\ tvt = vtv\,,\ twt = wtw\,, wuw = uwu\,.$$

Back to finite reductive groups : the Sylow ℓ–subgroups and their normalizers

- ℓ a prime number, prime to q, $\ell \mid |G|$, $\ell \nmid |W|$
  $\Longrightarrow$ There exists one d $(a(d) > 0)$ such that $\ell \mid \Phi_d(q)$, and the Sylow ℓ–subgroup S_ℓ of S_d is a Sylow of G.
- $L_d = C_G(S_\ell)$ and $N_d = N_\ell = N_G(S_\ell)$:

$$\begin{array}{c} N_\ell \\ \Big| \Big\} W_d \\ L_d \\ \Big| \\ 1 \end{array}$$

Since the "local" block is

$$(\mathbb{Z}_\ell N_\ell)_0 \simeq \mathbb{Z}_\ell[S_\ell \rtimes W_d]$$

our conjecture reduces to

Conjecture

$$\mathcal{D}^b((\mathbb{Z}_\ell G)_0) \simeq \mathcal{D}^b(\mathbb{Z}_\ell[S_\ell \rtimes W_d])$$

Role of Deligne–Lusztig varieties

- Let $\mathbf{P}$ be a parabolic subgroup with Levi subgroup $\mathbf{L}_d$, and with unipotent radical $\mathbf{U}$.

  Note that $\mathbf{P}$ is never rational if $d \neq 1$.

- The Deligne–Lusztig variety is

$$\mathcal{V}_{\mathbf{P}} := {}_{G}\circlearrowright \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g\mathbf{U} \cap F(g\mathbf{U}) \neq \emptyset\} \circlearrowleft {}_{L_d}$$

  hence defines an object

  $R\Gamma_c(\mathcal{V}_{\mathbf{P}}, \mathbb{Z}_\ell) \in \mathcal{D}^b({}_{\mathbb{Z}_\ell G}\,\mathbf{mod}\,{}_{\mathbb{Z}_\ell L_d})$ hence $R\Gamma_c(\mathcal{V}_{\mathbf{P}}, \mathbb{Z}_\ell)_0 \in \mathcal{D}^b({}_{\mathbb{Z}_\ell G}\,\mathbf{mod}\,{}_{\mathbb{Z}_\ell S_\ell})$

Conjecture

There is a choice of $\mathbf{U}$ such that

1. $R\Gamma_c(\mathcal{V}_{\mathbf{P}}, \mathbb{Z}_\ell)_0$ is a Rickard complex between $(\mathbb{Z}_\ell G)_0$ and its commuting algebra $\mathcal{C}(\mathbf{U})$.
2. $\mathcal{C}(\mathbf{U}) \simeq (\mathbb{Z}_\ell N_\ell)_0$

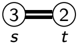
The case where $d = 1$

If $d = 1$,

- $\mathbf{S}_d = \mathbf{T} = \mathbf{L}_d$ and $W_d = W$
- $\mathcal{V}_{\mathbf{B}} = G/U$ and $\mathrm{R}\Gamma_c(\mathcal{V}_{\mathbf{P}}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(G/U)$
- $$\mathbb{Z}_\ell G \circlearrowright \mathbb{Z}_\ell(G/U) \circlearrowleft \mathcal{C}(U)$$
- If $\mathcal{H}(W,q)$ denotes the usual Hecke algebra of the Weyl group W (an algebra over $\mathbb{Z}[q,q^{-1}]$), we have
  1. $\mathcal{C}(U) \simeq \mathbb{Z}_\ell T . \mathbb{Z}_\ell \mathcal{H}(W,q)$
  2. $\overline{\mathbb{Q}}_\ell \mathcal{H}(W,q) \simeq \overline{\mathbb{Q}}_\ell W$

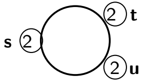
- A d–cyclotomic Hecke algebra for W_d is in particular
  - an image of the group algebra of the braid group B_{W_d},
  - a deformation in one parameter q of the group algebra of W_d,
  - which specializes to that group algebra when q becomes $e^{2\pi i/d}$
- Examples :
  - The ordinary Hecke algebra $\mathcal{H}(W)$ is 1–cyclotomic,
  - Case where $G = \mathrm{GL}_6$, $d = 3$:

$$W_3 = B_2(3) = \boldsymbol{\mu}_3 \wr \mathfrak{S}_2 \quad \longleftrightarrow$$



$$\mathcal{H}(W_3) = \left\langle S, T \; ; \left\{ \begin{array}{l} STST = TSTS \\ (S-1)(S-q)(S-q^2) = 0 \\ (T-q^3)(T+1) = 0 \end{array} \right\} \right\rangle$$

- For $G = \mathrm{O}_8(q)$, $W = D_4$, $d = 4$,

$W_4 = G(4,2,2) \quad \longleftrightarrow$ 

$$\mathcal{H}(W_4) = \left\langle\, S, T, U \,;\, \left\{ \begin{array}{l} STU = TUS = UST \\ (S - q^2)(S-1) = 0 \end{array} \right\} \right\rangle$$

The unipotent part

- Extend the scalars to $\overline{\mathbb{Q}}_\ell =: K \quad \Rightarrow \quad$ Get into a semisimple situation
  - $\mathrm{R}\Gamma_c(\mathcal{V}(\mathbf{U}), \mathbb{Z}_\ell)$ becomes

$$H_c^\bullet(\mathcal{V}(\mathbf{U}), K) := \bigoplus_i H_c^i(\mathcal{V}(\mathbf{U}), K)$$

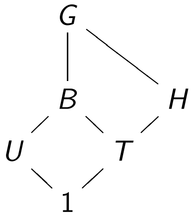
- Replace $\mathcal{V}(\mathbf{U})$ by $\quad \mathcal{V}(\mathbf{U})^{\mathrm{un}} := \mathcal{V}(\mathbf{U})/L_d \quad \Rightarrow$

  Only unipotent characters of G are involved

Semisimplified unipotent

1. The different $H_c^i(\mathcal{V}(\mathbf{U})^{\mathrm{un}}, K)$ are disjoint as KG–modules,
2. $\mathcal{H}(\mathbf{U}) := \mathrm{End}_{KG}\, H_c^\bullet(\mathcal{V}(\mathbf{U})^{\mathrm{un}}, K) \simeq KW_d$

Again the particular case $d = 1$ ...



and

$$\mathcal{V}^{\mathrm{un}}(\mathbf{U}) = G/B\,,$$

so

$$\mathcal{H}(\mathbf{U}) = K\mathcal{H}(W,q)\,, \quad \text{hence } \mathcal{H}(\mathbf{U}) \simeq KW\,.$$

... suggests what happens in general :

Conjecture

The commuting algebra $\mathcal{H}(\mathbf{U}) := \mathrm{End}_{KG}\, H_c^{\bullet}(\mathcal{V}(\mathbf{U})^{\mathrm{un}}, K)$ is a kind of "Hecke algebra" for the reflection group W_d.

d–cyclotomic Hecke algebras

- A d–cyclotomic Hecke algebra for W_d is in particular
  - an image of the group algebra of the braid group B_{W_d},
  - a deformation in one parameter q of the group algebra of W_d,
  - which specializes to that group algebra when q becomes $e^{2\pi i/d}$
- Examples :
  - The ordinary Hecke algebra $\mathcal{H}(W)$ is 1–cyclotomic,
  - Case where $G = \mathrm{GL}_6$, $d = 3$:

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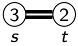
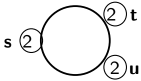


$$\mathcal{H}(W_3) = \left\langle S, T \,;\, \left\{ \begin{array}{l} STST = TSTS \\ (S-1)(S-q)(S-q^2) = 0 \\ (T-q^3)(T+1) = 0 \end{array} \right\} \right\rangle$$

- For $G = \mathrm{O}_8(q)$, $W = D_4$, $d = 4$,

$W_4 = G(4, 2, 2) \quad \longleftrightarrow$ 

$$\mathcal{H}(W_4) = \left\langle S, T, U\,;\, \left\{ \begin{array}{l} STU = TUS = UST \\ (S - q^2)(S - 1) = 0 \end{array} \right\} \right\rangle$$

Case where d is regular

- $\mathbf{L}_d$ is a torus $\iff$ d is a regular number for W
- The set of tori $\mathbf{L}_d$ is a single orbit of rational maximal tori under G, hence corresponds to a conjugacy class of W.
- For w in that class, we have $W_d \simeq C_W(w)$.
- The choice of $\mathbf{U}$ corresponds to the choice of an element w in that class.
- We then have

$$\mathcal{V}(\mathbf{U}_w)^{\mathrm{un}} = \mathbf{X}_w := \{\mathbf{B} \in \mathcal{B} \mid \mathbf{B} \overset{w}{\to} F(\mathbf{B})\}$$

  - $\mathcal{B}$ is the variety of all Borel subgroups of $\mathbf{G}$
  - The orbits of $\mathbf{G}$ on $\mathcal{B} \times \mathcal{B}$ are canonically in bijection with W and we write $\mathbf{B} \overset{w}{\to} \mathbf{B}'$ if the orbit of $(\mathbf{B}, \mathbf{B}')$ corresponds to w.

Relevance of the braid groups

Notation

- $V := \mathbb{C} \otimes Y(\mathbf{T})$ acted on by W,
  $\mathcal{A} :=$ set of reflecting hyperplanes of W
- $V^{\text{reg}} := V - \bigcup_{H \in \mathcal{A}} H$
- $B_W := \Pi_1(V^{\text{reg}}/W, x_0)$
- If

$$W =< S \mid \underbrace{ststs\ldots}_{m_{s,t}\text{ factors}} = \underbrace{tstst\ldots}_{m_{s,t}\text{ factors}}, s^2 = t^2 = 1 >$$

  then

$$B_W =< \mathbf{S} \mid \underbrace{\mathbf{ststs}\ldots}_{m_{s,t}\text{ factors}} = \underbrace{\mathbf{tstst}\ldots}_{m_{s,t}\text{ factors}} >$$

- $\boldsymbol{\pi} := t \mapsto e^{2i\pi t} x_0 \quad \Longrightarrow \quad \boldsymbol{\pi} \in ZB_W$

$$\boldsymbol{\pi} = \mathbf{w}_0^2 = \mathbf{c}^h$$

($\mathbf{c}$ Coxeter element, h Coxeter number).

A theorem of Deligne

- $$\begin{array}{ccc} & \mathcal{O}(w) := \{(\mathbf{B},\mathbf{B}') \mid \mathbf{B} \xrightarrow{w} \mathbf{B}'\} & \\ \swarrow & & \searrow \\ \mathcal{B} & & \mathcal{B} \end{array}$$
- If $l(ww') = l(w) + l(w')$, then $\mathcal{O}(ww') = \mathcal{O}(w) \times_{\mathcal{B}} \mathcal{O}(w')$

Theorem (Deligne)

Whenever $b \in B_W^+$ there is a well defined scheme $\mathcal{O}(b)$ over $\mathcal{B} \times \mathcal{B}$ such that $\mathcal{O}(w) = \mathcal{O}(\mathbf{w})$ and

$$\mathcal{O}(bb') = \mathcal{O}(b) \times_{\mathcal{B}} \mathcal{O}(b')$$

We set $\mathbf{X}_b := \mathcal{O}(b) \cap \mathrm{Graph}(F)\,,$ thus

For $b = \mathbf{w}_1\mathbf{w}_2 \cdots \mathbf{w}_n$ we have

$$\mathbf{X}_b^{(F)} = \{\, (\mathbf{B}_0, \mathbf{B}_1, \dots, \mathbf{B}_n) \mid \mathbf{B}_0 \xrightarrow{w_1} \mathbf{B}_1 \xrightarrow{w_2} \cdots \xrightarrow{w_n} \mathbf{B}_n \text{ and } \mathbf{B}_n = F(\mathbf{B}_0) \,\}$$

The variety $\mathbf{X}_\pi$

$$\mathbf{X}_\pi = \{ (\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2) \mid \mathbf{B}_0 \xrightarrow{w_0} \mathbf{B}_1 \xrightarrow{w_0} \mathbf{B}_2 \text{ and } \mathbf{B}_2 = F(\mathbf{B}_0 \}$$
$$= \{ (\mathbf{B}_0, \mathbf{B}_1, \dots, \mathbf{B}_h) \mid \mathbf{B}_0 \xrightarrow{c} \mathbf{B}_1 \xrightarrow{c} \cdots \xrightarrow{c} \mathbf{B}_h \text{ and } \mathbf{B}_h = F(\mathbf{B}_0) \}$$

The (opposite) monoid B_W^+ acts on $\mathbf{X}_\pi$: For $\mathbf{w} \in B_W^{\mathrm{red}}$, we have

$$\pi = \mathbf{w}b = b\mathbf{w} \quad \text{where } b = \mathbf{w}_1 \cdots \mathbf{w}_n$$
$$D_\mathbf{w} : (\mathbf{B}, \mathbf{B}_0, \dots, \mathbf{B}_n = F(\mathbf{B})) \mapsto (\mathbf{B}_0, \dots, \mathbf{B}_n = F(\mathbf{B}), F(\mathbf{B}_0))$$
$$(\mathbf{B}, \mathbf{B}_0, \dots, \mathbf{B}_n = F(\mathbf{B}), F(\mathbf{B}_0))$$

Hence B_W acts on $H_c^\bullet(\mathbf{X}_\pi)$

- Proposition : The action of B_W on $H_c^\bullet(\mathbf{X}_\pi)$ factorizes through the (ordinary) Hecke algebra $\mathcal{H}(W)$.
- Conjecture :

$$\mathrm{End}_{KG}\, H_c^\bullet(\mathbf{X}_\pi) = \mathcal{H}(W)$$

Relevance of roots of π

Proposition

d regular for $W \iff$ there exists $\mathbf{w} \in B_W^+$ such that $\mathbf{w}^d = \boldsymbol{\pi}$.

Application

1. $\mathbf{X}_{\mathbf{w}}^{(F)}$ embeds into $\mathbf{X}_{\boldsymbol{\pi}}^{(F^d)}$:

$$\mathbf{X}_{\mathbf{w}}^{(F)} \hookrightarrow \mathbf{X}_{\boldsymbol{\pi}}^{(F^d)}$$
$$\mathbf{B} \mapsto (\mathbf{B}, F(\mathbf{B}), \dots, F^d(\mathbf{B}))$$

2. Its image is

$$\{\, \mathbf{x} \in \mathbf{X}_{\boldsymbol{\pi}}^{(F^d)} \mid D_{\mathbf{w}}(\mathbf{x}) = F(\mathbf{x}) \,\}$$

3. $C_{B_W^+}(\mathbf{w})$ acts on $\mathbf{X}_{\mathbf{w}}^{(F)}$.

Belief

A good choice for $\mathbf{U}_w$ is : $\mathbf{w}$ a d–th root of π.

Theorem (David Bessis)

There is a natural isomorphism

$$B_{C_W(w)} \xrightarrow{\sim} C_{B_W}(\mathbf{w})$$

From which follow :

Theorem

The braid group $B_{C_W(w)}$ of the complex reflections group $C_W(w)$ acts on $H_c^\bullet(\mathbf{X}_\mathbf{w})$.

Conjecture

The braid group $B_{C_W(w)}$ acts on $H_c^\bullet(\mathbf{X}_\mathbf{w})$ through a d–cyclotomic Hecke algebra $\mathcal{H}_W(w)$.

Let us summarize

1. $\ell \rightsquigarrow d$, d regular, *i.e.*, $L_d = T_w$, $\mathbf{w}^d = \boldsymbol{\pi}$, $\mathcal{V}(\mathbf{U}_w)/L_d = \mathbf{X_w}$
2. $\mathrm{End}_{KG}\, H_c^\bullet(\mathbf{X}_w) \simeq \mathcal{H}_W(w)$
3. $\mathbb{Z}_\ell \mathcal{H}_W(w) \xrightarrow{\sim} \mathbb{Z}_\ell C_W(w)$
4. $\mathrm{End}_{\mathbb{Z}_\ell G}\, \mathrm{R}\Gamma_c(\mathcal{V}(\mathbf{U}_w), \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell (T_w)_\ell \cdot \mathrm{End}_{\mathbb{Z}_\ell G}\, \mathrm{R}\Gamma_c(\mathbf{X}_w, \mathbb{Z}_\ell) \simeq (\mathbb{Z}_\ell N_\ell)_0$

What is really proven today

- Everything
  - if $d = 1$ (Puig),
  - for $G = \mathrm{GL}_2(q)$ (Rouquier), $\mathrm{SL}_2(q)$ (cf. a book by Bonnafé to appear)
  - for $G = \mathrm{GL}_n(q)$ and $d = n$ (Bonnafé–Rouquier)
- About : $\mathrm{End}_{KG}\, H_c^\bullet(\mathbf{X}_w) \simeq \mathcal{H}_W(w)$?
  - All $\mathcal{H}_W(w)$ are known, all cases (Malle)
  - Assertion $\mathrm{End}_{KG} H_c^\bullet(\mathbf{X}_w) \simeq \mathcal{H}_W(w)$ known for
    - ⋆ $d = h$ (Lusztig),
    - ⋆ $d = 2$ (Lusztig, Digne–Michel),
    - ⋆ small rank GL,
    - ⋆ $d = 4$ for $D_4(q)$ (Digne–Michel).