Local Representation Theory of Finite Groups and Cyclotomic Algebras

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Feit–Thompson, 1963

If $G$ is a non abelian simple finite group, then $2 \mid |G|$.

Cauchy (1789–1857)

If $\ell \mid |G|$, there are non trivial $\ell$–subgroups in $G$.

Sylow, 1872

The maximal $\ell$–subgroups of $G$ are all conjugate under $G$. 
Assume \( P \subset S \) and \( P \subset S' \). There is \( g \in G \) such that \( S' = S^g \) \((= g^{-1}Sg)\), hence

\[
P \subset S \quad \text{and} \quad gP \ (= gPg^{-1}) \subset S.
\]

This is a fusion.

The Frobenius Category

\[\text{Frob}_\ell(G) : \]

- **Objects**: the \( \ell \)-subgroups of \( G \),
- **Hom**\((P, Q) := \{g \in G \mid (gP \subset Q)\}/C_G(P)\).

Note that \( \text{Aut}(P) = N_G(P)/C_G(P) \).

Alperin’s fusion theorem (1967) says essentially that \( \text{Frob}_\ell(G) \) is known as soon as the automorphisms of some of its objects are known.
Main question of local group theory

How much is known about $G$ from the knowledge (up to equivalence of categories) of $\text{Frob}_\ell(G)$?

Well, certainly not more than $G/O_{\ell'}(G)$!

(Where $O_{\ell'}(G)$ denotes the largest normal subgroup of $G$ of order not divisible by $\ell$)

Indeed, $O_{\ell'}(G)$ disappears in the Frobenius category, since, for $P$ an $\ell$–subgroup,

$$O_{\ell'}(G) \cap N_G(P) \subseteq C_G(P).$$

But perhaps all of $G/O_{\ell'}(G)$?
Control subgroup

Let \( H \) be a subgroup of \( G \). The following conditions are equivalent:

(i) The inclusion \( H \hookrightarrow G \) induces an equivalence of categories

\[
\text{Frob}_\ell(H) \simeq \text{Frob}_\ell(G),
\]

(ii) \( H \) contains a Sylow \( \ell \)-subgroup of \( G \), and if \( P \) is a \( \ell \)-subgroup of \( H \) and \( g \) is an element of \( G \) such that \( gP \subseteq H \), then there is \( h \in H \) and \( z \in C_G(P) \) such that \( g = hz \).

If the preceding conditions are satisfied, we say that \( H \) controls \( \ell \)-fusion in \( G \), or that \( H \) is a control subgroup in \( G \).
The first question may now be reformulated as follows:

If $H$ controls $\ell$–fusion in $G$, does the inclusion $H \hookrightarrow G$ induce an isomorphism

$$H/O_{\ell'}(H) \sim G/O_{\ell'}(G)?$$

In other words, do we have

$$G = HO_{\ell'}(G)?$$
• Frobenius theorem, 1905

If a Sylow $\ell$–subgroup $S$ of $G$ controls $\ell$–fusion in $G$, then the inclusion induces an isomorphism $S \simeq G/O_{\ell'}(G)$.

• $\ell$–solvable groups, ?

Assume that $G$ is $\ell$–solvable. If $H$ controls $\ell$-fusion in $G$, then the inclusion induces an isomorphism $H/O_{\ell'}(H) \simeq G/O_{\ell'}(G)$.

• $Z_{\ell}^*$–theorem (Glauberman, 1966 for $\ell = 2$, Classification for other primes)

Assume that $H = C_G(P)$ where $P$ is an $\ell$–subgroup of $G$. If $H$ controls $\ell$-fusion in $G$, then the inclusion induces an isomorphism $H/O_{\ell'}(H) \simeq G/O_{\ell'}(G)$. 
But

Burnside (1852–1927)

Assume that a Sylow $\ell$–subgroup $S$ of $G$ is abelian. Set $H := N_G(S)$. Then $H$ controls $\ell$-fusion in $G$. 
Consider the **Monster**, a finite simple group of order

\[ 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \cong 8.10^{53}. \]

(predicted in 1973 by Fischer and Griess, constructed in 1980 by Griess, proved to be unique by Thompson)

and the normalizer \( H \) of one of its Sylow 11–subgroups, a group of order 72600, isomorphic to \( (C_{11} \times C_{11}) \rtimes (C_5 \times SL_2(5)) \) (here we denote by \( C_m \) the cyclic group of order \( m \)).

Here we have \( G \neq HO_{11'}(G) \) since \( G \) is simple.

Remark : one may think of more elementary examples...
Let $K$ be a finite extension of the field of $\ell$–adic numbers $\mathbb{Q}_\ell$ which contains the $|G|$-th roots of unity. Let $\mathcal{O}$ be the ring of integers of $K$ over $\mathbb{Z}_\ell$, with maximal ideal $m$ and residue field $k := \mathcal{O}/m$. 

\[
K \xrightarrow{\quad} \mathcal{O} \xrightarrow{\quad} k = \mathcal{O}/m \\
\mathbb{Q}_\ell \xleftarrow{\quad} \mathbb{Z}_\ell \xrightarrow{F_\ell = \mathbb{Z}_\ell/\ell\mathbb{Z}_\ell}
Block decomposition

\[ \mathcal{O}G = \bigoplus B \quad \text{(indecomposable algebra)} \]

\[ \downarrow \]

\[ kG = \bigoplus kB \quad \text{(indecomposable algebra)} \]

The augmentation map \( \mathcal{O}G \to \mathcal{O} \) factorizes through a unique block of \( \mathcal{O}G \) called the principal block and denoted by \( (\mathcal{O}G)_0 \).

\[ \mathcal{O}G \to (\mathcal{O}G)_0 \]

\[ \downarrow \]

\[ \mathcal{O} \]
Factorisation and principal block

If \( H \) is a subgroup of \( G \), the following assertions are equivalent

(i) \( G = HO_{\ell'}(G) \).
(ii) The map \( \text{Res}^G_H \) induces an isomorphism from \((\mathcal{O}G)_0\) onto \((\mathcal{O}H)_0\).

Let us re-examine the counterexamples to factorisation coming from Burnside’s theorem.

Assume that a Sylow \( \ell \)-subgroup \( S \) of \( G \) is abelian, let \( H := N_G(S) \) be its normalizer.

Even if \( G \neq H O_{\ell'}(G) \), it appears that there is some connection between the (representation theory of) \((\mathcal{O}G)_0\) and \((\mathcal{O}H)_0\).
SOME NUMERICAL MIRACLES

Let us consider the case $G = \mathfrak{A}_5$ and $\ell = 2$. Then we have $H \simeq \mathfrak{A}_4$.

Remark: on a larger screen, we might as well consider the above case of the Monster and of the prime $\ell = 11$.

Table: Character table of $\mathfrak{A}_5$

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<th>$(1)$</th>
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### Table: Character table of $(O\mathfrak{A}_5)_0$

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### Table: Character table of $(O\mathfrak{A}_4)_0$

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A kind of generic counterexample:

\[ G = \text{GL}_n(q) \]

We certainly have

\[ G \neq \text{HO}_{\ell'}(G). \]
Morita equivalences

Definition

A Morita equivalence between $A$ and $B$ is the following datum:

- an object $M$ of $A \text{Mod}_B$ and an object $N$ of $B \text{Mod}_A$,
- two isomorphisms

$$M \otimes_B N \simto A \text{ in } A \text{Mod}_A \quad \text{and} \quad N \otimes_A M \simto B \text{ in } B \text{Mod}_B.$$ 

Given a Morita equivalence, the functors

$$M \otimes_B \cdot \quad \text{and} \quad N \otimes_A \cdot$$

are reciprocal equivalences of categories between $A \text{Mod}$ and $B \text{Mod}$. 

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**Fundamental example**

Whenever $n \geq 1$ is an integer, $\text{Mat}_n(A)$ and $A$ are Morita equivalent.

**Proof.**

Consider the bimodules $M$ and $N$ defined as follows:

- $M$ is the set of $n \times 1$ matrices with coefficients in $A$, on which $\text{Mat}_n(A)$ acts by left multiplication and $A$ acts by (right) multiplication,
- $N$ is the set of $1 \times n$ matrices with coefficients in $A$, on which $\text{Mat}_n(A)$ acts by right multiplication and $A$ acts by (left) multiplication.

Then the multiplication of matrices defines isomorphisms

$$M \otimes_A N \cong \text{Mat}_n(A) \quad \text{and} \quad N \otimes_{\text{Mat}_n(A)} M \cong A.$$
Assume

\[ K \mathrel{\overset{\mathcal{O}}{\longrightarrow}} k \]

and that \( A \) and \( B \) are \( \mathcal{O} \)-algebras.

Then a Morita equivalence between \( A \) and \( B \) induces Morita equivalences

\[ KA \equiv KB \quad \text{and} \quad kA \equiv kB, \]

via

\[ KM \mathrel{\overset{M}{\longrightarrow}} kM \quad \text{and} \quad KN \mathrel{\overset{N}{\longrightarrow}} kN \]
On $\text{GL}_n(q)$ again

The principal block algebras of $G$ and $H$ respectively are Morita equivalent.

There exist $M$ and $N$, respectively an $\mathcal{O}_G$–module–$\mathcal{O}_H$ and an $\mathcal{O}_H$–module–$\mathcal{O}_G$ with the following properties:

$$M \otimes_{\mathcal{O}_H} N \simeq (\mathcal{O}_G)_0 \text{ as } \mathcal{O}_G\text{–module–} \mathcal{O}_G$$

$$N \otimes_{\mathcal{O}_G} M \simeq (\mathcal{O}_H)_0 \text{ as } \mathcal{O}_H\text{–module–} \mathcal{O}_H$$
$G = \text{GL}_n(q)$

Viewed as a $OG$–module–$OS$, we have $M \simeq O(G/U)$, i.e., the functor $M \otimes OS$ is the Harish–Chandra induction.

$M/T = O(G/B)$ whose commuting algebra is the Hecke algebra $\mathcal{H}(\mathfrak{S}_n, q)$. 
Definition

A Rickard equivalence between $A$ and $B$ is the following datum:

- an object $M$ of $\mathcal{C}^b(A \text{Mod}_B)$ and an object $N$ of $\mathcal{C}^b(B \text{Mod}_A)$,
- two isomorphisms

$$M \otimes_B N \simto A \text{ in } \mathcal{C}^b(A \text{Mod}_A) \quad \text{and} \quad N \otimes_A M \simto B \text{ in } \mathcal{C}^b(B \text{Mod}_B).$$

Given a Rickard equivalence, the functors

$$M \otimes_B \cdot \quad \text{and} \quad N \otimes_A \cdot$$

are reciprocal equivalences of suitable categories.
Back to the principal 2–block of $\mathfrak{A}_5$

- View $(\mathfrak{O} \mathfrak{A}_5)_0$ as a $\mathfrak{O} \mathfrak{A}_5$–module–$\mathfrak{O} \mathfrak{A}_4$.
- Let $I$ be the kernel of the augmentation map : $(\mathfrak{O} \mathfrak{A}_5)_0 \to \mathfrak{O}$.
- Let $P$ denote a projective cover of $I$ and consider $P \rightarrowtail I \twoheadrightarrow (\mathfrak{O} \mathfrak{A}_5)_0$

We set 

$$M := 0 \rightarrow P \rightarrow (\mathfrak{O} \mathfrak{A}_5)_0 \rightarrow 0$$

- a complex of $\mathfrak{O} \mathfrak{A}_5$–modules–$\mathfrak{O} \mathfrak{A}_4$,
- $(\mathfrak{O} \mathfrak{A}_5)_0$ in degree 0 and $C$ in degree $-1$.

and $N := M^*$.

**Proposition**

The pair of complexes $(M, N)$ induces a Rickard equivalence between $(\mathfrak{O} \mathfrak{A}_5)_0$ and $(\mathfrak{O} \mathfrak{A}_4)_0$. 
Assume that a Sylow $\ell$–subgroup $S$ of $G$ is abelian, let $H := N_G(S)$ be its normalizer.

- **(ASC)**: The algebras $(\mathcal{O}_G)_0$ and $(\mathcal{O}_H)_0$ are Rickard equivalent.

- **(Strong ASC)**: They are Rickard equivalent in a way which is compatible with the equivalence of Frobenius categories.

Which means: There is a $G$–equivariant collection of derived equivalences

$$\{ \mathcal{E}(P) : D^b((\mathcal{O}C_G(P))_0) \overset{\sim}{\rightarrow} D^b((\mathcal{O}C_H(P))_0) \}_{P \subseteq S}$$

compatible with Brauer morphisms.
Known to be true:
Sylow cyclic (Rickard), $G$ $\ell$–solvable, $G = \mathfrak{S}_n$ (Chuang–Rouquier), $G = \text{SL}_2(\ell^n)$ (Okuyama), a bunch of sporadic simple groups (the Japanese school),…

What about the nonabelian Sylow case?

The fact that the derived category of $(\mathcal{O} G)_0$ is determined by $	ext{Frob}_\ell(G)$ is definitely false:

There are groups $G$ and a subgroup $H$ such that

- the inclusion $H \subset G$ induces an equivalence $\text{Frob}_\ell(H) \sim \text{Frob}_\ell(G)$,
- and yet $\mathcal{D}^b((\mathcal{O} H)_0)$ and $\mathcal{D}^b((\mathcal{O} G)_0)$ are not equivalent.

But there seem to be lots of numerical similarities between $(\mathcal{O} H)_0$ and $(\mathcal{O} G)_0$. 
Case of finite reductive groups

$G$ is a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group $W$, endowed with a Frobenius–like endomorphism $F$. The group $G := G^F$ is a finite reductive group.

**Example**

$$G = \text{GL}_n(\overline{\mathbb{F}}_q), \quad F : (a_{i,j}) \mapsto (a_{i,j}^q), \quad G = \text{GL}_n(q)$$

**Polynomial order** — There is a polynomial in $\mathbb{Z}[x]$

$$|G|(x) = x^N \prod_{d} \Phi_d(x)^{a(d)}$$

such that $|G|(q) = |G|$. 

**Example**

$$|\text{GL}_n|(x) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} (x^d - 1) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} \Phi_d(x)^{[n/d]}$$
Admissible subgroups — The tori of $G$ are the subgroups of the shape $T^F$ where $T$ is an $F$–stable torus (i.e., isomorphic to some $\mathbb{F}^\times \times \cdots \times \mathbb{F}^\times$ in $G$).

The Levi subgroups of $G$ are the subgroups of the shape $L^F$ where $L$ is a centralizer of an $F$–stable torus in $G$.

Examples

The split maximal torus $T_1 = (\mathbb{F}_q^\times)^n$ of order $(q - 1)^n$.

The Coxeter maximal torus $T_c = \text{GL}_1(\mathbb{F}_q^n)$ of order $q^n - 1$.

Levi subgroups have shape $\text{GL}_{n_1}(q^{a_1}) \times \cdots \times \text{GL}_{n_s}(q^{a_s})$.

Cauchy theorem — The (polynomial) order of an admissible subgroup divides the (polynomial) order of the group.
The generic Sylow theorems

For $\Phi_d(x)$ a cyclotomic polynomial, a $\Phi_d(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi_d(x)$. Hence such a group is a torus.

### Sylow theorems

1. Maximal $\Phi_d(x)$–subgroups (“Sylow $\Phi_d(x)$–subgroups”) of $G$ have as (polynomial) order the contribution of $\Phi_d(x)$ to the (polynomial) order of $G$:
   $$|S_d| = |S_d^F| = \Phi_d(q)^{a(d)}.$$

   Notation: Set $L_d := C_G(S_d)$ and $N_d := N_G(S_d) = N_G(L_d)$

2. Sylow $\Phi_d(x)$–subgroups are all conjugate by $G$.

3. The (polynomial) index $|G : N_d|$ is congruent to 1 modulo $\Phi_d(x)$.

4. $W_d := N_d/L_d$ is a true finite group, a complex reflection group in its action on $\mathbb{C} \otimes Y(S_d)$. This is the $d$–cyclotomic Weyl group of the finite reductive group $G$. 

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Example

Recall that

\[ |GL_n(q)| = q(n) \prod_{d=1}^{d=n} \Phi_d(q)^{[n/d]} \]

For each \( d \) \((1 \leq d \leq n)\), \( GL_n(q) \) contains a subtorus of (polynomial) order \( \Phi_d(x)^{\left\lfloor \frac{n}{d} \right\rfloor} \)

Assume \( n = md + r \) with \( r < d \). Then

\[ L_d = GL_1(q^d)^m \times GL_r(q) \]

\[ W_d = \mu_d \wr S_m \]
Let \( \ell \) be a prime number which does not divide \( |W| \).

- If \( \ell \) divides \( |G| \), there is a unique integer \( d \) such that \( \ell \) divides \( \Phi_d(q) \).
- Then the Sylow \( \ell \)-subgroups of \( G \) are nothing but the Sylow \( \ell \)-subgroups \( S_\ell \) of \( S_d = \mathbf{S}_d^F \) (\( S_d \) a Sylow \( \Phi_d(x) \)-subgroup of \( G \)).
- We have
  \[
  N_G(S_\ell) = N_d \quad \text{and} \quad C_G(S_\ell) = L_d.
  \]
  hence
  \[
  N_G(S_\ell)/C_G(S_\ell) = W_d.
  \]
\[ G = \text{GL}_n(q) \]

\[ H = N_1 \]

\[ T = S_1 \]

\[ B \]

\[ U \]

\[ S_1 = (q - 1)^n \]

\[ W_1 = S_n \]

\[ \ell \mid q - 1, \ell > n \Rightarrow S = T_\ell \]

\[ T = S \times T_{\ell'}, H = N_G(S) \]

\[ |S_d| = \Phi_d(q)^{a(d)} \]

\[ L_d / C_d = W_d \]

\[ \ell \mid \Phi_d(q), \ell > n \Rightarrow S = (S_d)_\ell \]

\[ S_d = (S_d)_\ell \times (S_d)_{\ell'} \]
A finite reflection group (abbreviated frg) on $K$ is a finite subgroup of $\text{GL}_K(V)$ ($V$ a finite dimensional $K$–vector space) generated by reflections, i.e., linear maps represented by

$$
\begin{pmatrix}
\zeta & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
$$

- A finite reflection group on $\mathbb{R}$ is called a Coxeter group.
- A finite reflection group on $\mathbb{Q}$ is called a Weyl group.
Main characterisation

Theorem (Shephard–Todd, Chevalley–Serre)

Let $G$ be a finite subgroup of $\text{GL}(V)$ ($V$ an $r$–dimensional vector space over a characteristic zero field $K$). Let $S$ denote the symmetric algebra of $V$, isomorphic to the polynomial ring $K[v_1, v_2, \ldots, v_r]$. The following assertions are equivalent.

1. $G$ is generated by reflections.
2. The ring $R := S^G$ of $G$–fixed polynomials is a polynomial ring $K[u_1, u_2, \ldots, u_r]$ in $r$ homogeneous algebraically independant elements.
3. $S$ is a free $R$–module.

Then
- If $d_i := \text{deg}(u_i)$, the family $(d_1, \ldots, d_r)$ is called the family of invariant degrees of $G$,
- and we have

$$|G| = d_1 d_2 \cdots d_r.$$
Examples

- For $G = \mathfrak{S}_r$, which acts naturally on $V = \mathbb{C}^r = \bigoplus \mathbb{C}v_i$, one may choose

\[
\begin{align*}
    u_1 &= v_1 + \cdots + v_r \\
    u_2 &= v_1 v_2 + v_1 v_3 + \cdots + v_{r-1} v_r \\
    \vdots & \quad \vdots \\
    u_r &= v_1 v_2 \cdots v_r
\end{align*}
\]

- For $G = \langle e^{2\pi i/d} \rangle$, cyclic group of order $d$ acting by multiplication on $V = \mathbb{C}$, we have

\[
S = K[x] \quad \text{and} \quad R = K[x^d].
\]

- Consider again the action of $\mathfrak{S}_r$ on $V = \mathbb{C}^r = \bigoplus \mathbb{C}v_i$. Fix $d \geq 2$. For each coordinate consider the reflection $v_i \mapsto \zeta_d v_i$. We obtain the wreath product $C_d \wr \mathfrak{S}_r$, generated by reflections. This group is called $G(d, 1, r)$. For each divisor $e$ of $d$, there is a normal reflection subgroup $G(d, e, r)$ of $G(d, 1, r)$ of index $e$. 
Let $G \leq \text{SL}_2(\mathbb{C})$ be finite and $g \in G$. Let $\zeta$ be an eigenvalue of $g$. Then $\zeta^{-1}g$ is a reflection.

- So, if $G = \langle g_1, \ldots, g_r \rangle$, the group $\langle \zeta_1^{-1}g_1, \ldots, \zeta_r^{-1}g_r \rangle$ is an frg.
- Note that for $G$ irreducible, we have $G/Z(G) \in \{D_n, A_4, S_4, A_5\}$.
- For example, the group
  \[ G := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \frac{\sqrt{-3}}{3} \begin{pmatrix} -\zeta_3 & \zeta_3^2 \\ 2\zeta_3^2 & 1 \end{pmatrix} \right\rangle \leq \text{GL}_2(\mathbb{Q}(\zeta_3)), \]
  with $\zeta_3 := \exp(2\pi i/3)$, is a frg of order 72, denoted $G_5$, isomorphic to $\text{SL}_2(3) \times C_3$.
- We may choose
  \[ u_1 := v_1^6 + 20v_1^3v_2^3 - 8v_2^6, \quad u_2 := 3v_1^3v_2^9 + 3v_1^6v_2^6 + v_1^9v_2^3 + v_2^{12}, \]
  with degrees $d_1 = 6, d_2 = 12$ (note that $d_1d_2 = 72 = |G|$).

If $g \in \text{SL}_3(\mathbb{C})$ is an involution, then $-g$ is a reflection. Note that $A_5$, $\text{PSL}_2(7)$ and $3.A_6$ have faithful 3-dimensional representations and are generated by involutions.
Classification

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
   - There is one infinite series $G(de, e, r)$ ($d, e$ and $r$ integers),
   - ...and 34 exceptional groups $G_4, G_5, \ldots, G_{37}$.

2. The group $G(de, e, r)$ ($d, e$ and $r$ integers) consists of all $r \times r$ monomial matrices with entries in $\mu_{de}$ such that the product of entries belongs to $\mu_d$.

3. We have

   \[
   G(d, 1, r) \simeq C_d \wr S_r \\
   G(e, e, 2) = D_{2e} \quad \text{(dihedral group of order $2e$)} \\
   G(2, 2, r) = W(D_r) \\
   G_{23} = H_3, \ G_{28} = F_4, \ G_{30} = H_4 \\
   G_{35,36,37} = E_{6,7,8}.
   \]
Let $\mathcal{A}$ be the arrangement of reflecting hyperplanes for the crg $G$. Set
\[ V^{\text{reg}} := V - \bigcup_{H \in \mathcal{A}} H. \]

The covering $\xrightarrow{} V^{\text{reg}}/G$ is Galois, hence induces a short exact sequence
\[
1 \xrightarrow{} \Pi_1(V^{\text{reg}}, x_0) \xrightarrow{} \Pi_1(V^{\text{reg}}/G, x_0) \xrightarrow{} G \xrightarrow{} 1
\]
with $\mathbb{P}_G$ (Pure braid group) and $\mathbb{B}_G$ (Braid group).
Braid reflections

Let $\gamma$ be a path in $V^{\text{reg}}$ from $x_0$ to $x_H$.

We define: $\sigma_{H,\gamma} := s_H(\gamma^{-1}) \cdot s_{H,x_H} \cdot \gamma$

Definition

We call braid reflections the elements $s_{H,\gamma} \in B$ defined by the paths $\sigma_{H,\gamma}$. 
The following properties are immediate.

- \(s_{H, \gamma}\) and \(s_{H, \gamma'}\) are conjugate in \(P\).
- \(s_{eH, \gamma}^{eH}\) is a loop in \(V^{\text{reg}}\):

The variety \(V\) (resp. \(V/G\)) is connected, the hyperplanes are irreducible divisors (irreducible closed subvarieties of codimension one), and the braid reflections are “generators of the monodromy” around the irreducible divisors. Then

**Theorem**

1. The braid group \(B_G\) is generated by the braid reflections \((s_{H, \gamma})\) (for all \(H\) and all \(\gamma\)).
2. The pure braid group \(P_G\) is generated by the elements \((s_{eH, \gamma}^{eH})\).
Artin–like presentations

An Artin–like presentation is

\[ \langle s \in S \mid \{v_i = w_i\}_{i \in I} \rangle \]

where

- \( S \) is a finite set of distinguished braid reflections,
- \( I \) is a finite set of relations which are multi–homogeneous, i.e., such that (for each \( i \)) \( v_i \) and \( w_i \) are positive words in elements of \( S \)

Theorem (Bessis)

Let \( G \subset GL(V) \) be a complex reflection group. Let \( d_1 \leq d_2 \leq \cdots \leq d_r \) be the family of its invariant degrees.

1. The following integers are equal (denoted by \( \Gamma_G \)):
   - The minimal number of reflections needed to generate \( G \)
   - The minimal number of braid reflections needed to generate \( B_G \)
   - \( \lceil (N_r + N_h)/d_r \rceil \) \((N_r := \text{number of reflections}, N_h := \text{number of hyperplanes})\)

2. Either \( \Gamma_G = r \) or \( \Gamma_G = r + 1 \), and the group \( B_G \) has an Artin–like presentation by \( \Gamma_G \) braid reflections.
The braid diagrams

Let $\mathcal{D}$ be a diagram like $s \circ e \circ b t \circ c u$. $\mathcal{D}$ represents the relations

$stustu \cdots = tustus \cdots = ustust \cdots$  
and  $s^a = t^b = u^c = 1$

We denote by $\mathcal{D}_{br}$ and call braid diagram the diagram $s \circ e \circ u$ which represents the relations

$stustu \cdots = tustus \cdots = ustust \cdots$

Note that

$G_7 : s^2 \circ 3 t \circ 3 u$
$G_{11} : s^2 \circ 3 t \circ 4 u$
$G_{19} : s^2 \circ 3 t \circ 5 u$

have same braid diagram.
For each irreducible complex irreducible group $G$, there is a diagram $\mathcal{D}$, whose set of nodes $\mathcal{N}(\mathcal{D})$ is identified with a set of distinguished reflections in $G$, such that

**Theorem**

For each $s \in \mathcal{N}(\mathcal{D})$, there exists a braid reflection $s \in B_G$ above $s$ such that the set $\{s\} \subset \mathcal{N}(\mathcal{D})$, together with the braid relations of $\mathcal{D}_{br}$, is a presentation of $B_G$.

- The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\xymatrix{ s \ar@{-}[r]^{e} & t }$, corresponding to the presentation

  \[ s^d = t^d = 1 \text{ and } \underbrace{ststs\cdots}_{\text{e factors}} = \underbrace{tstst\cdots}_{\text{e factors}} \]
The group $G_{18}$ has diagram $\begin{tikzpicture}
\node (s) at (0,0) {$s$};
\node (t) at (1,0) {$t$};
\draw (s) -- (t);
\end{tikzpicture}$ corresponding to the presentation
\[ s^5 = t^3 = 1 \text{ and } stst = tsts. \]

The group $G_{31}$ has diagram $\begin{tikzpicture}
\node (s) at (0,0) {$s$};
\node (t) at (1,0) {$t$};
\node (u) at (2,0) {$u$};
\node (v) at (0,1) {$v$};
\node (w) at (2,1) {$w$};
\draw (s) -- (v) -- (u) -- (w) -- (t) -- (v);
\end{tikzpicture}$ corresponding to the presentation
\[ s^2 = t^2 = u^2 = v^2 = w^2 = 1, \]
\[ uv = vu, \; sw = ws, \; vw = wv, \; sut = uts = tsu, \]
\[ svs = vsv, \; tvt = vtv, \; twt = wtw, \; wuw = uwu. \]
Back to finite reductive groups: the Sylow \( \ell \)-subgroups and their normalizers

- \( \ell \) a prime number, prime to \( q \), \( \ell \mid |G| \), \( \ell \nmid |W| \)
- \( \implies \) There exists one \( d \) (\( a(d) > 0 \)) such that \( \ell \mid \Phi_d(q) \), and the Sylow \( \ell \)-subgroup \( S_\ell \) of \( S_d \) is a Sylow of \( G \).
- \( L_d = C_G(S_\ell) \) and \( N_d = N_\ell = N_G(S_\ell) : N_\ell \)

Since the “local” block is

\[
(Z_\ell N_\ell)_0 \cong Z_\ell[S_\ell \rtimes W_d]
\]

our conjecture reduces to

**Conjecture**

\[
D^b((Z_\ell G)_0) \cong D^b(Z_\ell[S_\ell \rtimes W_d])
\]
Let $P$ be a parabolic subgroup with Levi subgroup $L_d$, and with unipotent radical $U$.

Note that $P$ is never rational if $d \neq 1$.

The Deligne–Lusztig variety is

$$V_P := \{ gU \in G/U \mid gU \cap F(gU) \neq \emptyset \} \circ L_d$$

hence defines an object

$$R\Gamma_c(V_P, \mathbb{Z}_\ell) \in D^b(\mathbb{Z}_\ell G \mod \mathbb{Z}_\ell L_d)$$

Conjecture

There is a choice of $U$ such that

1. $R\Gamma_c(V_P, \mathbb{Z}_\ell) \circ \circ \circ 0$ is a Rickard complex between $(\mathbb{Z}_\ell G)_0$ and its commuting algebra $C(U)$.

2. $C(U) \simeq (\mathbb{Z}_\ell N_\ell)_0$
The case where $d = 1$

If $d = 1$,

- $S_d = T = L_d$ and $W_d = W$
- $\mathcal{V}_B = G/U$ and $R\Gamma_c(\mathcal{V}_P, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(G/U)$

\[
\mathbb{Z}_\ell G \otimes \mathbb{Z}_\ell(G/U) \otimes C(U)
\]

If $\mathcal{H}(W, q)$ denotes the usual Hecke algebra of the Weyl group $W$ (an algebra over $\mathbb{Z}[q, q^{-1}]$), we have

1. $C(U) \simeq \mathbb{Z}_\ell T.\mathbb{Z}_\ell \mathcal{H}(W, q)$
2. $\overline{\mathbb{Q}}_\ell \mathcal{H}(W, q) \simeq \overline{\mathbb{Q}}_\ell W$
A $d$–cyclotomic Hecke algebra for $W_d$ is in particular

- an image of the group algebra of the braid group $B_{W_d}$,
- a deformation in one parameter $q$ of the group algebra of $W_d$,
- which specializes to that group algebra when $q$ becomes $e^{2\pi i/d}$

Examples:

- The ordinary Hecke algebra $\mathcal{H}(W)$ is 1–cyclotomic,
- Case where $G = \text{GL}_6$, $d = 3$:

$$W_3 = B_2(3) = \mu_3 \wr S_2 \longleftrightarrow \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0.5,0) {$s$};
\node at (0.5,1) {$t$};
\end{tikzpicture}
\end{array}$$

$$\mathcal{H}(W_3) = \left\langle S, T \right| \begin{cases}
STST = TSTS \\
(S - 1)(S - q)(S - q^2) = 0 \\
(T - q^3)(T + 1) = 0
\end{cases} \right\rangle$$
For $G = O_8(q)$, $W = D_4$, $d = 4$,

$W_4 = G(4, 2, 2) \iff \begin{array}{c}
\circ \ s \ 2 \\
\circ \ t \ 2 \\
\circ \ u \ 2 
\end{array}$

$\mathcal{H}(W_4) = \left\langle S, T, U ; \begin{cases} 
STU = TUS = UST \\
(S - q^2)(S - 1) = 0
\end{cases} \right\rangle$
The unipotent part

- Extend the scalars to $\overline{\mathbb{Q}}_\ell =: K \Rightarrow$ Get into a semisimple situation
  - $R\Gamma_c(\mathcal{V}(U), \mathbb{Z}_\ell)$ becomes
    \[ H^\bullet_c(\mathcal{V}(U), K) := \bigoplus_i H^i_c(\mathcal{V}(U), K) \]
  
- Replace $\mathcal{V}(U)$ by $\mathcal{V}(U)^{un} := \mathcal{V}(U)/L_d \Rightarrow$
  Only unipotent characters of $G$ are involved

Semisimplified unipotent

1. The different $H^i_c(\mathcal{V}(U)^{un}, K)$ are disjoint as $KG$–modules,

2. $\mathcal{H}(U) := \text{End}_{KG} H^\bullet_c(\mathcal{V}(U)^{un}, K) \simeq KW_d$
Again the particular case \( d = 1 \) ... 

\[
\begin{array}{c}
G \\
\downarrow \\
B \\
\downarrow \\
H \\
\downarrow \\
U \\
\downarrow \\
T \\
\downarrow \\
1 \\
\end{array}
\]

and

\[ \mathcal{V}^\text{un}(U) = G/B, \]

so

\[ \mathcal{H}(U) = KH(W, q), \quad \text{hence } \mathcal{H}(U) \cong KW. \]

... suggests what happens in general:

**Conjecture**

The commuting algebra \( \mathcal{H}(U) := \text{End}_{KG} H^\bullet_c(\mathcal{V}(U)^\text{un}, K) \) is a kind of “Hecke algebra” for the reflection group \( W_d \).
A $d$–cyclotomic Hecke algebra for $W_d$ is in particular

- an image of the group algebra of the braid group $B_{W_d}$,
- a deformation in one parameter $q$ of the group algebra of $W_d$,
- which specializes to that group algebra when $q$ becomes $e^{2\pi i/d}$

Examples:

- The ordinary Hecke algebra $\mathcal{H}(W)$ is 1–cyclotomic,
- Case where $G = \text{GL}_6$, $d = 3$:

  $W_3 = B_2(3) = \mu_3 \wr S_2 \leftrightarrow \begin{array}{c}
\begin{array}{ccc}
\mu_3 & \cong & \langle s, t; \quad \begin{cases} STST = TSTS \\
(S - 1)(S - q)(S - q^2) = 0 \end{cases} \\
(T - q^3)(T + 1) = 0 \end{cases}
\end{array}
\end{array}$
For $G = O_8(q)$, $W = D_4$, $d = 4$,

$W_4 = G(4, 2, 2) \xleftrightarrow{s} W_4^t = G(4, 2, 2)$

\[
\mathcal{H}(W_4) = \left\langle S, T, U \mid \begin{cases}
STU = TUS = UST \\
(S - q^2)(S - 1) = 0
\end{cases} \right\rangle
\]
Case where $d$ is regular

- $L_d$ is a torus $\iff d$ is a regular number for $W$
- The set of tori $L_d$ is a single orbit of rational maximal tori under $G$, hence corresponds to a conjugacy class of $W$.
- For $w$ in that class, we have $W_d \simeq C_W(w)$.
- The choice of $U$ corresponds to the choice of an element $w$ in that class.
- We then have

$$\mathcal{V}(U_w)^{un} = X_w := \{ B \in \mathcal{B} \mid B^w \rightarrow F(B) \}$$

- $\mathcal{B}$ is the variety of all Borel subgroups of $G$
- The orbits of $G$ on $\mathcal{B} \times \mathcal{B}$ are canonically in bijection with $W$ and we write $B^w \rightarrow B'$ if the orbit of $(B, B')$ corresponds to $w$. 
Relevance of the braid groups

Notation

- $V := \mathbb{C} \otimes Y(T)$ acted on by $W$,
  $A := \text{set of reflecting hyperplanes of } W$
- $V^{\text{reg}} := V - \bigcup_{H \in A} H$
- $B_W := \Pi_1(V^{\text{reg}}/W, x_0)$
- If
  $$W = \langle S \mid \underbrace{ststs \ldots}_{m_{s,t} \text{ factors}} = \underbrace{tstst \ldots}_{m_{s,t} \text{ factors}}, s^2 = t^2 = 1 \rangle$$
  then
  $$B_W = \langle S \mid \underbrace{ststs \ldots}_{m_{s,t} \text{ factors}} = \underbrace{tstst \ldots}_{m_{s,t} \text{ factors}} \rangle$$
- $\pi := t \mapsto e^{2i\pi t} x_0 \implies \pi \in ZB_W$
  $$\pi = w_0^2 = c^h$$
  ($c$ Coxeter element, $h$ Coxeter number).
A theorem of Deligne

- \( \mathcal{O}(w) := \{(B, B') | B \xrightarrow{w} B'\} \)

- If \( l(ww') = l(w) + l(w') \), then \( \mathcal{O}(ww') = \mathcal{O}(w) \times_B \mathcal{O}(w') \)

**Theorem (Deligne)**

Whenever \( b \in B_{\mathcal{W}}^+ \) there is a well defined scheme \( \mathcal{O}(b) \) over \( B \times B \) such that \( \mathcal{O}(w) = \mathcal{O}(w) \) and

\[
\mathcal{O}(bb') = \mathcal{O}(b) \times_B \mathcal{O}(b')
\]

We set \( X_b := \mathcal{O}(b) \cap \text{Graph}(F) \), thus

For \( b = w_1w_2 \cdots w_n \) we have

\[
X_b^{(F)} = \{(B_0, B_1, \ldots, B_n) | B_0 \xrightarrow{w_1} B_1 \xrightarrow{w_2} \cdots \xrightarrow{w_n} B_n \text{ and } B_n = F(B_0)\}
\]
The variety $X_{\pi}$

$$X_{\pi} = \{ (B_0, B_1, B_2) \mid B_0 \xrightarrow{w_0} B_1 \xrightarrow{w_0} B_2 \text{ and } B_2 = F(B_0) \}$$

$$= \{ (B_0, B_1, \ldots, B_h) \mid B_0 \xrightarrow{c} B_1 \xrightarrow{c} \cdots \xrightarrow{c} B_h \text{ and } B_h = F(B_0) \}$$

The (opposite) monoid $B_W^+$ acts on $X_{\pi}$: For $w \in B_W^{\text{red}}$, we have

$$\pi = wb = bw$$

where $b = w_1 \cdots w_n$

$$D_w : (B, B_0, \ldots, B_n = F(B)) \mapsto (B_0, \ldots, B_n = F(B), F(B_0))$$

$$(B, B_0, \ldots, B_n = F(B), F(B_0))$$

Hence $B_W$ acts on $H_c^\bullet(X_{\pi})$

*Proposition*: The action of $B_W$ on $H_c^\bullet(X_{\pi})$ factorizes through the (ordinary) Hecke algebra $H(W)$.

*Conjecture*:

$$\text{End}_{KG} H_c^\bullet(X_{\pi}) = H(W)$$
Proposition

d regular for \( W \) \iff there exists \( w \in B^+_W \) such that \( w^d = \pi \).

Application

1. \( X_w^{(F)} \) embeds into \( X_{\pi}^{(F^d)} \):

\[
X_w^{(F)} \hookrightarrow X_{\pi}^{(F^d)}
\]

\[
B \mapsto (B, F(B), \ldots, F^d(B))
\]

2. Its image is

\[
\{ x \in X_{\pi}^{(F^d)} \mid D_w(x) = F(x) \}
\]

3. \( C_{B^+_W}(w) \) acts on \( X_w^{(F)} \).
Belief
A good choice for $U_w$ is: $w$ a $d$–th root of $\pi$.

Theorem (David Bessis)
There is a natural isomorphism

$$B_{C_W(w)} \xrightarrow{\sim} C_{B_W}(w)$$

From which follow:

Theorem
The braid group $B_{C_W(w)}$ of the complex reflections group $C_W(w)$ acts on $H^\bullet_c(X_w)$.

Conjecture
The braid group $B_{C_W(w)}$ acts on $H^\bullet_c(X_w)$ through a $d$–cyclotomic Hecke algebra $H_W(w)$. 
Let us summarize

1. \( \ell \rightsquigarrow d, \) \( d \) regular, \( i.e., \) \( L_d = T_w, \) \( w^d = \pi, \) \( \mathcal{V}(U_w)/L_d = X_w \)

2. \( \text{End}_{KG} H^\bullet_c(X_w) \simeq \mathcal{H}_W(w) \)

3. \( \mathbb{Z}_\ell \mathcal{H}_W(w) \xrightarrow{\sim} \mathbb{Z}_\ell C_W(w) \)

4. \( \text{End}_{\mathbb{Z}_\ell G} R\Gamma_c(\mathcal{V}(U_w), \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell (T_w)_\ell \cdot \text{End}_{\mathbb{Z}_\ell G} R\Gamma_c(X_w, \mathbb{Z}_\ell) \simeq (\mathbb{Z}_\ell N_\ell)_0 \)
What is really proven today

- Everything
  - if $d = 1$ (Puig),
  - for $G = \text{GL}_2(q)$ (Rouquier), $\text{SL}_2(q)$ (cf. a book by Bonnafé to appear)
  - for $G = \text{GL}_n(q)$ and $d = n$ (Bonnafé–Rouquier)

About: $\text{End}_{K^G} H^\bullet_c(X_w) \simeq \mathcal{H}_W(w)$?

- All $\mathcal{H}_W(w)$ are known, all cases (Malle)
- Assertion $\text{End}_{K^G} H^\bullet_c(X_w) \simeq \mathcal{H}_W(w)$ known for
  - $d = h$ (Lusztig),
  - $d = 2$ (Lusztig, Digne–Michel),
  - small rank GL,
  - $d = 4$ for $D_4(q)$ (Digne–Michel).