Polyhedral groups

These are the cyclic groups, the dihedral groups and the rotation groups of the Platonic solids. We shall compute their character tables.

But first, some isomorphisms.

The rotation group of a regular tetrahedron is isomorphic to $\text{Alt}(4)$ — acting on the four faces of the tetrahedron.

The rotation group of a cube (or its dual, the octahedron) is isomorphic to $\text{Sym}(4)$ — acting on the four lines through antipodal vertices.

The rotation group of a regular icosahedron (or its dual, the dodecahedron) is isomorphic to $\text{Alt}(5)$ — acting on the five cubes inscribed in the dodecahedron.
Presentations

The polyhedral groups have presentations of the form

\[ ((a, b, c)) = \langle r, s, t \mid r^a s^b t^c = r s t = 1 \rangle. \]

The groups are finite if and only if \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1 \), in which case its order is

\[ 2 / \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \right). \]

- \(((1, n, n))\) the cyclic group of order \( n \)
- \(((2, 2, n))\) the dihedral group of order \( 2n \)
- \(((2, 3, 3))\) the tetrahedral group, of order 12
- \(((2, 3, 4))\) the octahedral group, of order 24
- \(((2, 3, 5))\) the icosahedral group, of order 60

The tetrahedral group \( \text{Alt}(4) \)

\[
\begin{array}{c|cccc}
\text{Class} & (1) & (12)(34) & (123) & (143) \\
\text{Size} & 1 & 3 & 4 & 4 \\
\text{Order} & 1 & 2 & 3 & 3 \\
\chi_1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & 1 & \omega & \omega^2 \\
\chi_3 & 1 & 1 & \omega^2 & \omega \\
\chi_4 & 3 & -1 & 0 & 0 \\
\end{array}
\]

where \( \omega^3 = 1 \).

If \( r = (12)(34) \), \( s = (123) \) and \( t = (143) \), then \( r^2 = s^3 = t^3 = r s t = 1 \).
The octahedral group $\text{Sym}(4)$

$$
\begin{array}{|c|ccccc|}
\hline
\text{Class} & 1 & t^2 & r & s & t \\
\text{Size} & 1 & 3 & 6 & 8 & 6 \\
\text{Order} & 1 & 2 & 2 & 3 & 4 \\
\hline
\chi_1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & 1 & -1 & 1 & -1 \\
\chi_3 & 2 & 2 & 0 & -1 & 0 \\
\chi_4 & 3 & -1 & -1 & 0 & 1 \\
\chi_5 & 3 & -1 & 1 & 0 & -1 \\
\hline
\end{array}
$$

$$r = (12), \quad s = (134), \quad t = (1432), \quad t^2 = (13)(24)$$

The icosahedral group $\text{Alt}(5)$

$$
\begin{array}{|c|ccccc|}
\hline
\text{Class} & 1 & r & s & t & t^2 \\
\text{Size} & 1 & 15 & 20 & 12 & 12 \\
\text{Order} & 1 & 2 & 3 & 5 & 5 \\
\hline
\chi_1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 3 & -1 & 0 & (1+\sqrt{5})/2 & (1-\sqrt{5})/2 \\
\chi_3 & 3 & -1 & 0 & (1-\sqrt{5})/2 & (1+\sqrt{5})/2 \\
\chi_4 & 4 & 0 & 1 & -1 & -1 \\
\chi_5 & 5 & 1 & -1 & 0 & 0 \\
\hline
\end{array}
$$

$$r = (12)(34), \quad s = (254), \quad t = (12345), \quad t^2 = (13524)$$
The binary polyhedral groups

These are the finite subgroups of $\text{SU}(2)$, which is also the group of quaternions of norm 1. The group $\text{SU}(2)$ is a double cover of $\text{SO}(3)$ and the binary polyhedral groups are the inverse images of the polyhedral groups.

They have presentations of the form

$$\langle\langle a, b, c \rangle\rangle = \langle r, s, t \mid r^a = s^b = t^c = rst \rangle.$$ 

- the cyclic group of order $2n$
- $\langle\langle 2,2,n \rangle\rangle$ the binary dihedral group of order $4n$
- $\langle\langle 2,3,3 \rangle\rangle$ the binary tetrahedral group, of order 24
- $\langle\langle 2,3,4 \rangle\rangle$ the binary octahedral group, of order 48
- $\langle\langle 2,3,5 \rangle\rangle$ the binary icosahedral group, of order 120

In all cases $z = rst$ is a central element of order 2.

The quaternion group $Q_8$

The quaternion group is the binary dihedral group

$$\langle\langle 2,2,2 \rangle\rangle = \langle r, s, t \mid r^2 = s^2 = t^2 = rst \rangle$$

<table>
<thead>
<tr>
<th>Class</th>
<th>1</th>
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$\rightarrow$ $\chi_5$
The McKay graphs

The characters of $G$ are $\chi_1, \chi_2, \ldots, \chi_r$. Given any character $\chi$ let $\Gamma(\chi)$ be the graph with $r$ vertices and $m_{ij}$ directed edges from the $i$ th to the $j$ th vertex, where

$$\chi\chi_i = \sum_{j=1}^{r} m_{ij}\chi_j.$$

Taking $G$ to be the quaternion group and $\chi$ the character of degree 2, we find that the graph is

```
   1 1
  1 2
  1 1
```

where the vertices are labelled with the degrees of the characters.

The binary tetrahedral group $\simeq \text{SL}(2,3)$

<table>
<thead>
<tr>
<th>Class</th>
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In $\text{SL}(2,3)$ we have $r \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $s \rightarrow \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$, $t \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

The homomorphism onto $\text{Alt}(4)$ is given by

$$r \rightarrow (12)(34), \quad s \rightarrow (123), \quad t \rightarrow (143).$$
The graph of the binary tetrahedral group

Let’s play the same game with $\text{SL}(2,3)$, where $\chi$ is the real character of degree 2, $\Gamma(\chi)$ is the graph with 7 vertices and there are $m_{ij}$ directed edges from the $i$ th to the $j$ th vertex, where

$$\chi \chi_i = \sum_{j=1}^{7} m_{ij} \chi_j.$$
Coxeter–Dynkin diagrams of type $A$, $D$ and $E$ and the binary polyhedral groups

The graph for the binary octahedral group is

```
1 2 3 4 3 2 1
```

The graphs associated with the binary polyhedral groups are the affine Dynkin diagrams of types

- $\tilde{A}_n$ for the cyclic group of order $n + 1$
- $\tilde{D}_n$ for the binary dihedral group of order $4n - 8$
- $\tilde{E}_6$ for the binary tetrahedral group
- $\tilde{E}_7$ for the binary octahedral group
- $\tilde{E}_8$ for the binary icosahedral group

This is the McKay correspondence.

The binary icosahedral group

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<th>$r$</th>
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The homomorphism onto $\text{Alt}(5)$ is given by

$r \to (12)(34)$, $s \to (254)$, $t \to (12345)$
The group SU(2)

Let \( S^3 \) be the unit sphere in the division algebra \( \mathbb{H} \) of quaternions. We regard \( \mathbb{H} \) as a left vector space of dimension 2 over \( \mathbb{C} \). The elements of \( S^3 \) act on \( \mathbb{H} \) by multiplication on the right. This establishes an isomorphism between \( S^3 \) and the special unitary group \( SU(2) \) of all matrices

\[
\begin{pmatrix}
\alpha & \beta \\
-\bar{\beta} & \bar{\alpha}
\end{pmatrix}
\text{ where } |\alpha|^2 + |\beta|^2 = 1.
\]

Furthermore, there is an action of this matrix on polynomials in \( X \) and \( Y \) such that

\[
X \to \alpha X + \beta Y \quad \text{and} \quad Y \to -\bar{\beta}X + \bar{\alpha}Y.
\]

Representations of SU(2)

Let \( \mathcal{H}_m \) be the space of homogeneous polynomials of degree \( m \) in \( X \) and \( Y \). This is an irreducible representation of \( SU(2) \) of degree \( m + 1 \). Let \( \chi_m \) be the character of \( \mathcal{H}_m \). Every element of \( SU(2) \) is conjugate to a matrix of the form

\[
A = \begin{pmatrix}
q & 0 \\
0 & q^{-1}
\end{pmatrix}
\text{ where } q = e^{i\theta}
\]

and so

\[
\chi_m(A) = q^m + q^{m-2} + \cdots + q^{2-m} + q^{-m} = \frac{\sin((m+1)\theta)}{\sin\theta}.
\]

Thus, from the addition formula for \( \sin \theta \) we have

\[
\chi_m\chi_1 = \chi_{m-1} + \chi_{m+1}.
\]
The McKay correspondence

For each finite subgroup of SU(2) and the appropriate character $\chi$ of degree 2 obtained from the representation $H_1$, the matrix $M$ is the adjacency matrix of the McKay graph. The graphs which occur are precisely those whose adjacency matrix has maximum eigenvalue equal to 2.

The corresponding Cartan matrix is $C = 2I - M$. This describes the root system of a reflection group (or Lie algebra, or ...) of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$.

The columns of the character table are eigenvectors of $C$ and the eigenvalues are $2 - \chi(x_i)$.

References

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*Notes on Coxeter transformations and the McKay correspondence*  