Character Theory of Finite Groups

NZ Mathematics Research Institute
Summer Workshop

Day 4: The McKay Correspondence

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Nelson, 7–13 January 2018

Polyhedral groups

These are the *cyclic* groups, the *dihedral* groups and the rotation groups of the *Platonic solids*. We shall compute their character tables.

But first, some isomorphisms.

The rotation group of a *regular tetrahedron* is isomorphic to $\mathrm{Alt}(4)$
— acting on the four faces of the tetrahedron.

The rotation group of a *cube* (or its dual, the *octahedron*) is isomorphic to $\mathrm{Sym}(4)$
— acting on the four lines through antipodal vertices.

The rotation group of a *regular icosahedron* (or its dual, the *dodecahedron*) is isomorphic to $\mathrm{Alt}(5)$
— acting on the five cubes inscribed in the dodecahedron.

Presentations

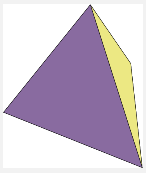
The polyhedral groups have presentations of the form

$$((a,b,c)) = \langle r,s,t \mid r^a = s^b = t^c = rst = 1 \rangle.$$

The groups are finite if and only if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} > 1$, in which case its order is $2\Big/\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1\right)$.

- $((1,n,n))$ the cyclic group of order n
- $((2,2,n))$ the dihedral group of order $2n$
- $((2,3,3))$ the tetrahedral group, of order 12
- $((2,3,4))$ the octahedral group, of order 24
- $((2,3,5))$ the icosahedral group, of order 60

The tetrahedral group $\mathrm{Alt}(4)$

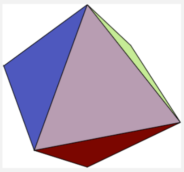


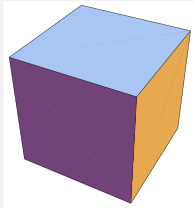
Class	1	$\overset{r}{(12)(34)}$	$\overset{s}{(123)}$	$\overset{t}{(143)}$
Size	1	3	4	4
Order	1	2	3	3
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

where $\omega^3 = 1$.

If $r = (12)(34)$, $s = (123)$ and $t = (143)$, then $r^2 = s^3 = t^3 = rst = 1$.

The octahedral group $\mathrm{Sym}(4)$

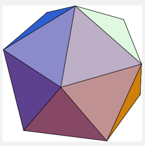


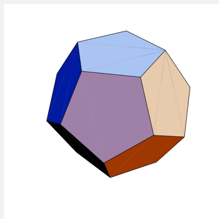


Class	1	t^2	r	s	t
Size	1	3	6	8	6
Order	1	2	2	3	4
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	2	2	0	-1	0
χ_4	3	-1	-1	0	1
χ_5	3	-1	1	0	-1

$$r = (12), \quad s = (134), \quad t = (1432), \quad t^2 = (13)(24)$$

The icosahedral group $\mathrm{Alt}(5)$





Class	1	r	s	t	t^2
Size	1	15	20	12	12
Order	1	2	3	5	5
χ_1	1	1	1	1	1
χ_2	3	-1	0	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$
χ_3	3	-1	0	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$$r = (12)(34), \quad s = (254), \quad t = (12345), \quad t^2 = (13524)$$

The binary polyhedral groups

These are the finite subgroups of $\mathrm{SU}(2)$, which is also the group of quaternions of norm 1. The group $\mathrm{SU}(2)$ is a double cover of $\mathrm{SO}(3)$ and the *binary polyhedral groups* are the inverse images of the polyhedral groups.

They have presentations of the form

$$\langle\langle a,b,c\rangle\rangle = \langle r,s,t \mid r^a = s^b = t^c = rst\rangle.$$

- — the cyclic group of order $2n$
- $\langle\langle 2,2,n\rangle\rangle$ the binary dihedral group of order $4n$
- $\langle\langle 2,3,3\rangle\rangle$ the binary tetrahedral group, of order 24
- $\langle\langle 2,3,4\rangle\rangle$ the binary octahedral group, of order 48
- $\langle\langle 2,3,5\rangle\rangle$ the binary icosahedral group, of order 120

In all cases $z = rst$ is a central element of order 2.

The quaternion group Q_8

The quaternion group is the binary dihedral group

$$\langle\langle 2,2,2\rangle\rangle = \langle r,s,t \mid r^2 = s^2 = t^2 = rst\rangle$$

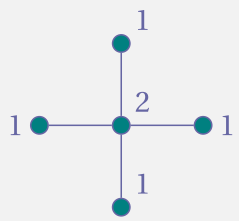
Class	1	z	r	s	t
Size	1	1	2	2	2
Order	1	2	4	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
$\to$ χ_5	2	-2	0	0	0

The McKay graphs

The characters of G are $\chi_1, \chi_2, \ldots, \chi_r$. Given any character χ let $\Gamma(\chi)$ be the graph with r vertices and m_{ij} directed edges from the i th to the j th vertex, where

$$\chi\chi_i = \sum_{j=1}^{r} m_{ij}\chi_j.$$

Taking G to be the quaternion group and χ the character of degree 2, we find that the graph is



where the vertices are labelled with the *degrees* of the characters.

The binary tetrahedral group $\simeq \mathrm{SL}(2,3)$

Class	1	z	s^2	t^2	r	s	t
Size	1	1	4	4	6	4	4
Order	1	2	3	3	4	6	6
χ_1	1	1	1	1	1	1	1
χ_2	1	1	ω^2	ω	1	ω	ω^2
χ_3	1	1	ω	ω^2	1	ω^2	ω
$\to$ χ_4	2	-2	-1	-1	0	1	1
χ_5	2	-2	$-\omega^2$	$-\omega$	0	ω^2	ω
χ_6	2	-2	$-\omega$	$-\omega^2$	0	ω^2	ω
χ_7	3	3	0	0	-1	0	0

In $\mathrm{SL}(2,3)$ we have $r \leftrightarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $s \leftrightarrow \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$, $t \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

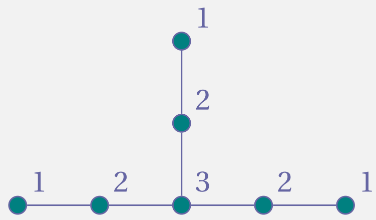
The homomorphism onto $\mathrm{Alt}(4)$ is given by

$$r \to (12)(34), \quad s \to (123), \quad t \to (143).$$

The graph of the binary tetrahedral group

Let's play the same game with $\mathrm{SL}(2,3)$, where χ is the real character of degree 2, $\Gamma(\chi)$ is the graph with 7 vertices and there are m_{ij} directed edges from the i th to the j th vertex, where

$$\chi\chi_i = \sum_{j=1}^{7} m_{ij}\chi_j.$$



The binary octahedral group

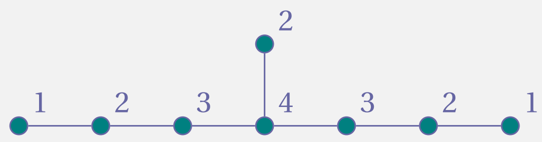
Class	1	z	sz	t^2	r	s	t	tz
Size	1	1	8	6	12	8	6	6
Order	1	2	3	4	4	6	8	8
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	1	-1	-1
χ_3	2	2	-1	2	0	-1	0	0
$\rightarrow \chi_4$	2	-2	-1	0	0	1	$\sqrt{2}$	$-\sqrt{2}$
$\rightarrow \chi_5$	2	-2	-1	0	0	1	$-\sqrt{2}$	$\sqrt{2}$
χ_6	3	3	0	-1	1	0	-1	-1
χ_7	3	3	0	-1	-1	0	1	1
χ_8	4	-4	1	0	0	-1	0	0

The homomorphism onto $\mathrm{Sym}(4)$ is given by

$$r \to (12), \quad s \to (134), \quad t \to (1432)$$

Coxeter–Dynkin diagrams of type A, D and E and the binary polyhedral groups

The graph for the binary octahedral group is



The graphs associated with the binary polyhedral groups are the *affine Dynkin diagrams* of types

- $\widetilde{A}_n$ for the cyclic group of order $n+1$
- $\widetilde{D}_n$ for the binary dihedral group of order $4n-8$
- $\widetilde{E}_6$ for the binary tetrahedral group
- $\widetilde{E}_7$ for the binary octahedral group
- $\widetilde{E}_8$ for the binary icosahedral group

This is the *McKay correspondence*.

The binary icosahedral group

Class	1	z	sz	r	t^2	t^4	s	t^3	t
Size	1	1	20	30	12	12	20	12	12
Order	1	2	3	4	5	5	6	10	10
χ_1	1	1	1	1	1	1	1	1	1
$\to \chi_2$	2	-2	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	1	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
$\to \chi_3$	2	-2	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	1	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
χ_4	3	3	0	-1	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	0	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
χ_5	3	3	0	-1	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$
χ_6	4	4	1	0	-1	-1	1	-1	-1
χ_7	4	-4	1	0	-1	-1	-1	1	1
χ_8	5	5	-1	1	0	0	-1	0	0
χ_9	6	-6	0	0	1	1	0	-1	-1

The homomorphism onto $\mathrm{Alt}(5)$ is given by

$$r \to (12)(34), \quad s \to (254), \quad t \to (12345)$$

The group $\mathrm{SU}(2)$

Let $\mathbb{S}^3$ be the unit sphere in the division algebra $\mathbb{H}$ of quaternions.

We regard $\mathbb{H}$ as a *left* vector space of dimension 2 over $\mathbb{C}$. The elements of $\mathbb{S}^3$ act on $\mathbb{H}$ by multiplication on the *right*. This establishes an isomorphism between $\mathbb{S}^3$ and the *special unitary group* $\mathrm{SU}(2)$ of all matrices

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \quad \text{where } |\alpha|^2 + |\beta|^2 = 1.$$

Furthermore, there is an action of this matrix on polynomials in X and Y such that

$$X \to \alpha X + \beta Y \quad \text{and} \quad Y \to -\overline{\beta} X + \overline{\alpha} Y.$$

Representations of $\mathrm{SU}(2)$

Let $\mathscr{H}_m$ be the space of homogeneous polynomials of degree m in X and Y. This is an irreducible representation of $\mathrm{SU}(2)$ of degree $m+1$. Let χ_m be the character of $\mathscr{H}_m$. Every element of $\mathrm{SU}(2)$ is conjugate to a matrix of the form

$$A = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \quad \text{where } q = e^{i\theta}$$

and so

$$\chi_m(A) = q^m + q^{m-2} + \cdots + q^{2-m} + q^{-m} = \frac{\sin((m+1)\theta)}{\sin\theta}.$$

Thus, from the addition formula for $\sin\theta$ we have

$$\chi_m \chi_1 = \chi_{m-1} + \chi_{m+1}.$$

The McKay correspondence

For each finite subgroup of $\mathrm{SU}(2)$ and the appropriate character χ of degree 2 obtained from the representation $\mathscr{H}_1$, the matrix M is the adjacency matrix of the McKay graph. The graphs which occur are precisely those whose adjacency matrix has maximum eigenvalue equal to 2.

The corresponding Cartan matrix is $C = 2I - M$. This describes the root system of a reflection group (or Lie algebra, or . . .) of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$.

The columns of the character table are eigenvectors of C and the eigenvalues are $2 - \chi(x_i)$.

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