Character Theory of Finite Groups

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Summer Workshop

Day 3: Induced characters

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Induced class functions

Let H be a subgroup of G and φ be a class function on H. Extend φ to $\mathring{\varphi} : G \to \mathbb{C}$ by defining $\mathring{\varphi}(x) = 0$ for $x \notin H$ and $\mathring{\varphi}(x) = \varphi(x)$ for $x \in H$.

Definition

The class function on G *induced* by φ is defined by

$$\operatorname{ind}_H^G \varphi(x) = \frac{1}{|H|} \sum_{u \in G} \mathring{\varphi}(u^{-1}xu).$$

Let T be a (left) *transversal* of H in G; that is, a set of representatives for the left cosets xH. Then the formula for $\operatorname{ind}_H^G \varphi(x)$ can be written

$$\operatorname{ind}_H^G \varphi(x) = \sum_{t \in T} \mathring{\varphi}(t^{-1}xt).$$

because $\mathring{\varphi}$ is constant on H-conjugacy classes.

Frobenius reciprocity

Let φ be a class function on H, let ψ be a class function on G where $H \leq G$ and let $\operatorname{res}_H \psi$ denote the restriction of ψ to H. Then

$$\langle \operatorname{ind}_H^G \varphi \mid \psi \rangle_G = \langle \varphi \mid \operatorname{res}_H \psi \rangle_H.$$

Proof.

From the definitions of ind_H^G and $\langle - \mid - \rangle$ we have

$$\begin{aligned}
\langle \operatorname{ind}_H^G \varphi \mid \psi \rangle_G &= \frac{1}{|G|\,|H|} \sum_{u,x \in G} \mathring{\varphi}(u^{-1}xu)\overline{\psi(x)} \\
&= \frac{1}{|G|\,|H|} \sum_{u,x \in G} \mathring{\varphi}(x)\overline{\psi(u^{-1}xu)} = \frac{1}{|G|\,|H|} \sum_{\substack{u \in G \\ x \in H}} \varphi(x)\overline{\psi(x)} \\
&= \frac{1}{|H|} \sum_{x \in H} \varphi(x)\overline{\psi(x)} = \langle \varphi \mid \operatorname{res}_H \psi \rangle_H
\end{aligned}$$

□

When is a class function a character?

If χ is a character of H, $\operatorname{ind}_H^G \chi$ is a class function. Is it a character?

First of all, if $\varphi : G \to \mathbb{C}$ is a class function and $\chi_1, \chi_2, \ldots, \chi_r$ are the irreducible characters of G, it follows from the first orthogonality relations that

$$\varphi = \sum_{i=1}^{r} \langle \varphi \mid \chi_i \rangle \chi_i.$$

Thus φ is a character if and only if $\langle \varphi \mid \chi_i \rangle$ is a nonnegative integer for $1 \leq i \leq r$.

Furthermore, φ is irreducible if and only if, in addition, $\langle \varphi \mid \varphi \rangle = 1$.

If $\langle \varphi \mid \chi_i \rangle$ is an *integer* for $1 \leq i \leq r$, then φ is called a *generalised* character of G.

Induced characters

Theorem

Suppose H is a subgroup of G.

- *If φ is a character of H, then $\operatorname{ind}_H^G \varphi$ is a character of G.*
- *If φ is a generalised character of H, then $\operatorname{ind}_H^G \varphi$ is a generalised character of G.*

Proof.

If φ is a character of H and χ is an irreducible character of G, then $\langle \operatorname{ind}_H^G \varphi \mid \chi \rangle_G = \langle \varphi \mid \operatorname{res}_H \chi \rangle_H \in \mathbb{N}$ and therefore $\operatorname{ind}_H^G \varphi$ is a character.

If $\varphi = \chi - \xi$, where χ and ξ are characters of H, then $\operatorname{ind}_H^G \varphi = \operatorname{ind}_H^G \chi - \operatorname{ind}_H^G \xi$, hence $\operatorname{ind}_H^G \varphi$ is a generalised character. □

Properties of the induction map

Let H be a subgroup of G, let φ be a class function on H and let ψ be a class function on G. Then

1. If $H \le K$, then $\operatorname{ind}_H^G \varphi = \operatorname{ind}_K^G(\operatorname{ind}_H^K \varphi)$.
2. $\operatorname{ind}_H^G(\varphi \operatorname{res}_H \psi) = (\operatorname{ind}_H^G \varphi)\psi$

Proof.

1. Let χ be an irreducible character of G. Then

$$\langle \operatorname{ind}_K^G(\operatorname{ind}_H^K \varphi) \mid \chi \rangle = \langle \operatorname{ind}_H^K \varphi \mid \operatorname{res}_K \chi \rangle = \langle \varphi \mid \operatorname{res}_H \chi \rangle = \langle \operatorname{ind}_H^G \varphi \mid \chi \rangle.$$

2. For $x \in G$,

$$\operatorname{ind}_H^G(\varphi \operatorname{res}_H \psi)(x) = \frac{1}{|H|} \sum_{u \in G} \mathring{\varphi}(u^{-1}xu)\mathring{\psi}(u^{-1}xu)$$

$$= \frac{1}{|H|} \sum_{u \in G} \mathring{\varphi}(u^{-1}xu)\psi(u^{-1}xu) = \frac{\psi(x)}{|H|} \sum_{u \in G} \mathring{\varphi}(u^{-1}xu) = \psi(x)(\operatorname{ind}_H^G \varphi)(x). \quad \square$$

Permutation actions

Suppose that G acts on the set Γ. Choose $\alpha \in \Gamma$.

The *stabiliser* of α is the subgroup $G_\alpha = \{x \in G \mid x\alpha = \alpha\}$.

The *orbit* of α is the subset $G\alpha = \{x\alpha \mid x \in G\}$.

There is a *bijection* between the left cosets of G_α and the orbit $G\alpha$, given by

$$xG_\alpha \leftrightarrow x\alpha.$$

Therefore, $|G : G_\alpha| = |G\alpha|$, where $|G : G_\alpha|$ is the number of (left) cosets of G_α in G.

Permutation characters

Suppose that G acts *transitively* on Γ (i.e., G has a single orbit on Γ) and let H be the stabiliser of some $\alpha \in \Gamma$.

For $u, x \in G$, we have $\mathring{1}_H(u^{-1}xu) = 1$ if and only if $xu\alpha = u\alpha$. Thus $(\mathrm{ind}_H^G 1_H)(x) = |H|^{-1} \sum_{u \in G} \mathring{1}_H(u^{-1}xu) = |\mathrm{Fix}_\Gamma(x)|$.

Therefore $\mathrm{ind}_H^G 1_H$ is the *permutation character* corresponding to Γ.

More generally, suppose that G acts on Γ with orbits $\Gamma_1, \ldots, \Gamma_k$ and for $1 \le i \le k$ let H_i be the stabiliser of a point in Γ_i. The permutation character of Γ is $\pi = \mathrm{ind}_{H_1}^G 1_{H_1} + \cdots + \mathrm{ind}_{H_k}^G 1_{H_k}$.

Then $\langle \pi \mid 1_G \rangle_G = \langle \mathrm{ind}_{H_1}^G 1_{H_1} \mid 1_G \rangle_G + \cdots + \langle \mathrm{ind}_{H_k}^G 1_{H_k} \mid 1_G \rangle_G$.

But $\langle \mathrm{ind}_{H_i}^G 1_{H_i} \mid 1_G \rangle_G = \langle 1_H \mid 1_H \rangle_H = 1$ for all i and so

$\langle \pi \mid 1_G \rangle_G = k$, *the number of orbits of G on Γ*

The Mackey formulas

- H and K are subgroups of G
- S is a set of representatives for the double cosets KxH
- φ is a class function on H
- ψ is a class function on K
- φ^s is the class function $\varphi^s : sHs^{-1} \to \mathbb{C} : x \mapsto \varphi(s^{-1}xs)$
- $H_s = sHs^{-1} \cap K$

$$\operatorname{res}_K \operatorname{ind}_H^G \varphi = \sum_{s \in S} \operatorname{ind}_{H_s}^K \operatorname{res}_{H_s} \varphi^s$$

$$\langle \operatorname{ind}_H^G \varphi \mid \operatorname{ind}_K^G \psi \rangle_G = \sum_{s \in S} \langle \operatorname{res}_{H_s} \varphi^s \mid \operatorname{res}_{H_s} \psi \rangle_{H_s}$$

$$\operatorname{ind}_H^G \varphi \cdot \operatorname{ind}_K^G \psi = \sum_{s \in S} \operatorname{ind}_{H_s} (\operatorname{res}_{H_s} \varphi^s \cdot \operatorname{res}_{H_s} \psi)$$

The second formula follows from the first by Frobenius reciprocity.

Proof

K acts on the cosets sH by left multiplication. The stabiliser in K of sH is $H_s = sHs^{-1} \cap K$ and the union of the orbit of K is KsH.

For $s \in S$ let T_s be a set of representatives for the left cosets of H_s in K. Then $\{tsH \mid s \in S, t \in T_s\}$ is the set of left cosets of H in G. Thus, for $x \in K$,

$$\operatorname{ind}_H^G \varphi(x) = \sum_{s \in S, t \in T_s} \mathring{\varphi}(s^{-1}t^{-1}xts) = \sum_{s \in S, t \in T_s} \mathring{\varphi}^s(t^{-1}xt)$$

But $\mathring{\varphi}^s(y) = 0$ for $y \notin H_s$, therefore

$$\sum_{t \in T_s} \mathring{\varphi}^s(t^{-1}xt) = \operatorname{ind}_{H_s}^K \operatorname{res}_{H_s} \varphi^s(x)$$

and hence $\operatorname{res}_K \operatorname{ind}_H^G \varphi = \sum_{s \in S} \operatorname{ind}_{H_s}^K \operatorname{res}_{H_s} \varphi^s$. □

Double transitivity

Suppose that G acts transitively on Γ and that π is the corresponding permutation character. Then $\pi = \mathrm{ind}_H^G 1_H$, where H is the stabiliser of a point in Γ.

From the second Mackey formula with $H = K$ and $\varphi = 1_H$ we find that $\langle \pi \mid \pi \rangle$ is the number of double cosets HsH.

The cosets of H are in bijection with the points of Γ and the double cosets are in bijection with the orbits of H on Γ.

In particular, $\langle \pi \mid \pi \rangle = 2$ if and only if H has two orbits on Γ, which must be $\{\alpha\}$ and $\Gamma \setminus \{\alpha\}$. This is equivalent to G acting transitively on the set of ordered pairs (α, β) where $\alpha \neq \beta$.

That is, $\langle \pi \mid \pi \rangle = 2$ if and only if G acts *doubly transitively* on Γ.

In this case $\pi = 1_G + \chi$, where χ is irreducible.

The existence of Frobenius kernels

Suppose that $H \le G$ and that $H \cap xHx^{-1} = 1$ for all $x \in G \setminus H$. If $1 < H < G$, then G is called a *Frobenius group* and H is a *Frobenius complement*. A normal subgroup $N \lhd G$ is called a *Frobenius kernel* if $H \cap N = 1$ and $G = HN$.

Theorem (Frobenius)

Every Frobenius group has a Frobenius kernel.

Currently there is no known proof of this result which doesn't use some character theory. (However, in 2013 Terry Tao produced a proof that uses only representations of the *centre* of the group algebra.)

Let $S = \bigcup_{x \in G} (xHx^{-1} \setminus \{1\})$ and put $N = G \setminus S$. The number of conjugates of H in G is the index $|G : N_G(H)|$ of the normaliser $N_G(H)$ of H and therefore $|S| = |G : H|(|H| - 1)$ because our hypotheses imply $N_G(H) = H$. Thus $|N| = |G : H|$. We must show that N is a subgroup.

Proof. Constructing irreducible characters of G

Begin with the irreducible characters $\varphi_0 = 1_H$, φ_1, ..., φ_m of H and define $\psi_i = \varphi_i(1)1_H - \varphi_i$ for $1 \le i \le m$. Then

$$\langle \psi_i \mid \psi_j \rangle = \varphi_i(1)\varphi_j(1) + \delta_{ij}, \ \ \langle \psi_i \mid 1_H \rangle = \varphi_i(1) \ \text{ and } \ \psi_i(1) = 0. \qquad (\star)$$

If $s \notin H$, then $H_s = sHs^{-1} \cap H = 1$ and since $\psi_i(1) = 0$, the second Mackey formula reduces to

$$\langle \mathrm{ind}_H^G \psi_i \mid \mathrm{ind}_H^G \psi_j \rangle_G = \langle \psi_i \mid \psi_j \rangle_H \qquad (\star\star)$$

Furthermore, by Frobenius reciprocity, $\langle \mathrm{ind}_H^G \psi_i \mid 1_G \rangle = \langle \psi_i \mid 1_H \rangle = \varphi_i(1)$ and therefore $\mathrm{ind}_H^G \psi_i = \varphi_i(1)1_G - \chi_i$ for some generalised character χ_i such that $\langle \chi_i \mid 1_G \rangle = 0$.

But now, from $(\star)$ and $(\star\star)$ we have $\langle \chi_i \mid \chi_j \rangle = \delta_{ij}$. Also, from $\mathrm{ind}_H^G \psi_i(1) = 0$ we have $\chi_i(1) = \varphi_i(1) > 0$ and therefore χ_i is an *irreducible* character.

Proof that N is a normal subgroup

Let $N_0 = \bigcap_{i=1}^m \ker \chi_i$. Using our assumption that $H \cap sHs^{-1} = 1$ for all $s \notin H$, the first Mackey formula implies

$$\varphi_i(1)1_H - \mathrm{res}_H \chi_i = \mathrm{res}_H \mathrm{ind}_H^G \psi_i = \mathrm{res}_H \psi_i = \varphi_i(1)1_H - \varphi_i$$

and therefore $\mathrm{res}_H \chi_i = \varphi_i$ for all $i = 1, \ldots, m$.

Suppose that $h \in H \cap N_0$. Then $\varphi_i(h) = \varphi_i(1)$ for $1 \le i \le m$ and we also have $\varphi_0(h) = 1 = \varphi_0(1)$. Thus $\mathrm{reg}_H(h) = \sum_{i=0}^m \varphi_i(1)^2 = |H| = \mathrm{reg}_H(1)$. But $\mathrm{reg}_H(h) = 0$ if $h \neq 1$ and therefore $h = 1$ and hence $H \cap N_0 = 1$.

If $x \in N$ and $x \neq 1$, no conjugate of x is in H and so $\mathrm{ind}_H^G \psi_i(x) = 0$. Thus $\chi_i(x) = \varphi_i(1)$ for all i whence $x \in N_0$, and therefore $N \subseteq N_0$. On the other hand, $H \cap N_0 = 1$ and so $|N_0| \le |G|/|H| = |N|$. It follows that $N = N_0$. $\square$

Further reading

David M. Goldschmidt.
Group characters, symmetric functions, and the Hecke algebra.
American Mathematical Society, Providence, RI, 1993.