Character Theory of Finite Groups

NZ Mathematics Research Institute
Summer Workshop

Day 1: Essentials

Don Taylor

The University of Sydney

Nelson, 7–13 January 2018

Origins

Suppose that G is a finite group and for every element $g \in G$ we have an indeterminate x_g.

What are the factors of the *group determinant*

$$\det\left(x_{gh^{-1}}\right)_{g,h} \;?$$

This was a question posed by Dedekind. Frobenius discovered character theory (in 1896) when he set out to answer it.

Characters of abelian groups had been used in number theory but Frobenius developed the theory for nonabelian groups.

Representations • Matrices • Characters

A *linear representation* of a group G is a homomorphism $\rho : G \to \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the group of all invertible linear transformations of the vector space V, which we assume to have finite dimension over the field $\mathbb{C}$ of complex numbers.

The dimension n of V is called the *degree* of ρ.

If $(e_i)_{1\le i\le n}$ is a basis of V, there are functions $a_{ij} : G \to \mathbb{C}$ such that

$$\rho(x)e_j = \sum_i a_{ij}(x)e_i.$$

The matrices $A(x) = \big(a_{ij}(x)\big)$ define a homomorphism $A : G \to \mathrm{GL}(n,\mathbb{C})$ from G to the group of all invertible $n \times n$ matrices over $\mathbb{C}$.

The *character* of ρ is the function $G \to \mathbb{C}$ that maps $x \in G$ to $\mathrm{Tr}(\rho(x))$, the *trace* of $\rho(x)$, namely the sum of the diagonal elements of $A(x)$.

Early applications

Frobenius first defined characters as solutions to certain equations.

A year later he established the connection with matrix representations. He then used character theory to establish structural properties of finite groups.

Another early success of character theory was Burnside's 1904 proof that groups of order $p^a q^b$ (p and q primes) are soluble.

My aim is to introduce you to enough character theory today so that we can prove this result tomorrow.

Outline

- Maschke's theorem
- Irreduciblity
- Schur's lemma
- Orthogonality relations
- The number of irreducible characters
- The character table of a finite group
- Examples

Some background reading

- Charles W. Curtis.
  Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer, volume 15 of *History of Mathematics*.
  American Mathematical Society, 1999.
- Walter Feit.
  Characters of finite groups.
  W. A. Benjamin, 1967.
- I. Martin Isaacs.
  Character theory of finite groups.
  Academic Press, 1976.
- Jean-Pierre Serre.
  Linear representations of finite groups.
  Springer-Verlag, 1977.

Similarity

Let ρ and ρ' be two linear representations of G with spaces V and V' and let A and A' be the corresponding matrix representations.

We say that ρ and ρ' are *isomorphic* if there is a linear isomorphism $\pi : V \to V'$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\rho(x)} & V \\ \pi \downarrow & & \downarrow \pi \\ V' & \xrightarrow[\rho'(x)]{} & V' \end{array}$$

commutes for all $x \in G$; that is, $\pi\rho(x) = \rho'(x)\pi$.

This is equivalent to the condition $A'(x) = PA(x)P^{-1}$, where P is the matrix of π. That is, linear representations are isomorphic if and only if the corresponding matrix representations are *similar*.

Isomorphic representations have the same character.

Example: representations of degree 1

A representation of *degree 1* of a group G is a homomorphism $\rho : G \to \mathbb{C}^\times$ and in this case ρ is its own character.

For $x \in G$, the image $\rho(x)$ is a root of unity and hence $|\rho(x)| = 1$.

For every group G there is the *principal representation* 1_G defined by $1_G(x) = 1$ for all $x \in G$.

Example (Cyclic groups)

If G is a cyclic group generated by an element x of order d and if $\zeta = \exp(2\pi i/d)$ is a primitive d th root of unity, then for $0 \le k < d$ there is a representation (character) χ_k of degree 1 such that

$$\chi_k(x) = \zeta^k.$$

These representations are pairwise non-isomorphic.

Example: the regular representation

Let $\mathbb{C}[G]$ be the vector space of dimension $|G|$ with basis $(e_x)_{x\in G}$.

For $x \in G$, let $R_G(x)$ be the automorphism of $\mathbb{C}[G]$ such that

$$R_G(x)e_y = e_{xy}.$$

This is the *regular representation* of G. Its degree is $|G|$.

Let reg_G be the *character* of R_G. For $x \in G$ we have

$$\mathrm{reg}_G(x) = \begin{cases} |G| & x = 1 \\ 0 & x \neq 1 \end{cases}$$

Example: permutation representations

Suppose that G acts on a finite set Γ of size n. That is, for each $x \in G$ there is a permutation $\alpha \mapsto x\alpha$ of Γ such that

$$1\alpha = \alpha \quad \text{and} \quad x(y\alpha) = (xy)\alpha \quad \text{for all } x, y \in G,\ \alpha \in \Gamma.$$

Let V be the vector space with basis $(e_\alpha)_{\alpha\in\Gamma}$

For $x \in G$, let $\rho(x)$ be the automorphism of V such that

$$\rho(x)e_\alpha = e_{x\alpha}.$$

The resulting linear representation of G is the *permutation representation* associated with Γ.

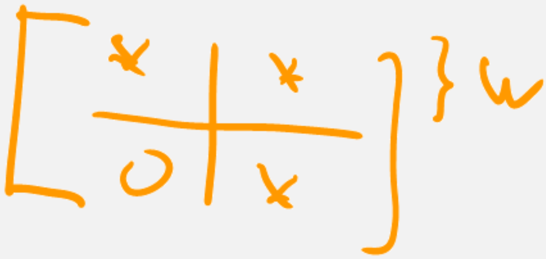
The value of the character χ of ρ at x is the number of fixed points of x; that is, $\chi(x) = |\mathrm{Fix}_\Gamma(x)|$.

Subrepresentations

Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G and let W be a subspace of V.

Suppose that W is G-*invariant*; that is, for $w \in W$, $\rho(x)w \in W$ for all $x \in G$.

The restriction $\rho|_W(x)$ of $\rho(x)$ to W defines a linear representation $\rho|_W : G \to \mathrm{GL}(W)$ of G in W called a *subrepresentation* of V.



Maschke's Theorem

Theorem (Maschke)

Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G in V and let W be a subspace of V invariant under G. Then there is a complement W° of W in V which is also G-invariant.

Proof

Let W' be an arbitrary complement of W in V and let p be the corresponding projection of V onto W. Form the 'average' of the conjugates of p by G:

$$p^\circ = \frac{1}{|G|}\sum_{x\in G}\rho(x)p\rho(x)^{-1}.$$

Since p maps V into W and $\rho(x)$ fixes W we see that p° maps V into W.

On the other hand, if $w\in W$ then $\rho(x)^{-1}w\in W$, whence

$$p\rho(x)^{-1}w=\rho(x)^{-1}w, \text{ i.e. } \rho(x)p\rho(x)^{-1}w=w \quad \text{and so} \quad p^\circ w=w.$$

Thus p° is a projection onto W and we define $W^\circ=\ker p^\circ$. Then W° is a complement to W. Furthermore

$$\rho(y)p^\circ\rho(y)^{-1}=\frac{1}{|G|}\sum_x\rho(y)\rho(x)p\rho(x)^{-1}\rho(y)^{-1}=\frac{1}{|G|}\sum_x\rho(yx)p\rho(yx)^{-1}=p^\circ.$$

If $w\in W^\circ$, then $p^\circ w=0$ and so $p^\circ\rho(y)w=\rho(y)p^\circ w=0$, whence $\rho(y)w\in W^\circ$. This shows that W° is G-invariant, as required. □

W $\parallel$

$\dim V = \dim \ker p^\circ + \dim \operatorname{im} p^\circ$

Irreducible representations and characters

A linear representation $\rho:G\to\mathrm{GL}(V)$ of G is is *irreducible* (or *simple*) if $V\neq 0$ and the only G-invariant subspaces of V are 0 and V.

In view of Maschke's theorem this is equivalent to saying that V is not the direct sum of two proper subrepresentations.

Theorem

Every representation is a direct sum of irreducible representations.

Proof.

Induction and Maschke's theorem. □

A *character* is irreducible if it is not the sum of two other characters.

$$\rho(x) \sim \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$$

Elementary properties of characters

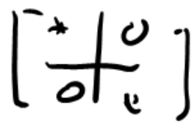
If χ is the character of a representation ρ of G of degree n, then

- $\chi(1) = n$
- $\chi(x^{-1}) = \overline{\chi(x)}$
- $\chi(xyx^{-1}) = \chi(y)$; *i.e. χ is constant on conjugacy classes*

For $x \in G$, $\chi(x)$ is the sum of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\rho(x)$. The λ_i are roots of unity and hence $\lambda_i^{-1} = \overline{\lambda_i}$.

If A and B are matrices, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be linear representations of G and let χ_1 and χ_2 be their characters. Then



- *the character of the direct sum $V_1 \oplus V_2$ is $\chi_1 + \chi_2$;*
- *the character of the tensor product $V_1 \otimes V_2$ is $\chi_1\chi_2$.*

Schur's lemma

f intertwines ρ_1 and ρ_2

Theorem (Schur's lemma)

Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be irreducible representations of G and let $f : V_1 \to V_2$ be a linear transformation such that $\rho_2(x) f = f \rho_1(x)$ for all $x \in G$.

1. *If ρ_1 and ρ_2 are not isomorphic, $f = 0$.*
2. *If $V_1 = V_2$ and $\rho_1 = \rho_2$, then f is a scalar multiple of the identity.*

Proof

$\rho_1 : G \to \mathrm{GL}(V_1)$, $\rho_2 : G \to \mathrm{GL}(V_2)$, $f : V_1 \to V_2$, $\rho_2(x) f = f \rho_1(x)$

❶ If $w \in \ker f$, then $f \rho_1(x) w = \rho_2(x) f w = 0$ and so $\rho_1(x) w \in \ker f$ for all $x \in G$, which proves that $\ker f$ is G-invariant.

Similarly if $v \in \operatorname{im} f$, then $v = f w$ for some $w \in V_1$, whence $\rho_2(x) v = \rho_2(x) f w = f \rho_1(x) w \in \operatorname{im} f$ and so $\operatorname{im} f$ is also G-invariant.

But V_1 and V_2 are irreducible, hence $f \neq 0$ implies $\ker f = 0$ and $\operatorname{im} f = V_2$, which means that f is an isomorphism.

❷ Let λ be an eigenvalue of f and put $f' = f - \lambda I$, where I is the identity transformation. Then $\ker f' \neq 0$.

On the other hand, $\rho_1(x) f' = f' \rho_1(x)$ and it follows from ❶ that $f' = 0$. That is, $f = \lambda I$. □

Schur relations

Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be irreducible representations of G and let $h : V_1 \to V_2$ be a linear transformation. Put

$$h^\circ = \frac{1}{|G|} \sum_{x \in G} \rho_2(x) h \rho_1(x)^{-1}.$$

❶ *If ρ_1 and ρ_2 are not isomorphic, then $h^\circ = 0$.*

❷ *If $V_1 = V_2$ and $\rho_1 = \rho_2$, then $h^\circ = \dfrac{\mathrm{Tr}(h)}{n} I$, where $n = \dim V_1$.*

Proof.

For all $y \in G$, $\rho_2(y) h^\circ = h^\circ \rho_1(y)$. Thus from Schur's lemma we have $h^\circ = 0$ in case ❶ and in case ❷ $h^\circ = \lambda I$ for some λ. On taking the trace

$$\lambda n = \mathrm{Tr}(h^\circ) = \frac{1}{|G|} \sum_x \mathrm{Tr}(\rho_1(x) h \rho_1(x)^{-1}) = \mathrm{Tr}(h).$$ □

Schur relations: matrix formulation

Let $A(x) = \big(a_{ij}(x)\big)$ *be the matrix representing* $\rho_2(x)$ *and let* $B(x) = \big(b_{ij}(x)\big)$ *be the matrix representing* $\rho_1(x)$*. (Recall that we assume that* ρ_1 *and* ρ_2 *are irreducible.)*

- *If* A *is not similar to* B*, then for all* i, j, s, t

$$\sum_{x \in G} a_{is}(x) b_{tj}(x^{-1}) = 0$$

- *If* $A = B$*, then for all* i, j, s, t

$$\frac{1}{|G|} \sum_{x \in G} a_{is}(x) a_{tj}(x^{-1}) = \frac{1}{n} \delta_{ij} \delta_{st}$$

Proof.

Apply the previous result with h given by the matrix (h_{ij}), where $h_{st} = 1$ and all other entries are 0. □

First orthogonality relations

If φ and ψ are two complex valued functions on G, put

$$\langle \varphi \mid \psi \rangle = \frac{1}{|G|} \sum_{t \in G} \varphi(t) \overline{\psi(t)}.$$

This is a *positive definite hermitian inner product*: it is linear in φ, semilinear in ψ and $\langle \varphi \mid \varphi \rangle > 0$ for $\varphi \neq 0$.

Theorem (First orthogonality relations)

1. *If* χ *is the character of an irreducible representation, then* $\langle \chi \mid \chi \rangle = 1$*.*
2. *If* χ *and* χ' *are the characters of two irreducible non-isomorphic representations, then* $\langle \chi \mid \chi' \rangle = 0$*.*

Proof.

Apply the Schur relations to the representations affording χ and χ'. □

A decomposition theorem

Theorem

Let V be a representation of G with character φ and suppose that V is a direct sum of irreducible representations:

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

If W is an irreducible representation of G with character χ, the number of W_i isomorphic to W is equal to $\langle \varphi \mid \chi \rangle$.

Proof.

We have $\varphi = \chi_1 + \chi_2 + \cdots + \chi_k$, where χ_i is the character of W_i and thus $\langle \varphi \mid \chi \rangle = \langle \chi_1 \mid \chi \rangle + \langle \chi_2 \mid \chi \rangle + \cdots + \langle \chi_k \mid \chi \rangle$.

But $\langle \chi_i \mid \chi \rangle$ is 1 or 0 according to whether or not W_i is isomorphic to W. □

Consequences

- If $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ with all W_i irreducible, the number of W_i isomorphic to an irreducible representation W does not depend on the decomposition chosen.
- Two representations with the same character are isomorphic.
- If $\chi_1, \chi_2, \ldots, \chi_r$ are the distinct irreducible characters of G and if $\varphi = m_1\chi_1 + m_2\chi_2 + \cdots + m_r\chi_r$ where the m_i are non-negative integers, then $\langle \varphi \mid \chi_i \rangle = m_i$ and

$$\langle \varphi \mid \varphi \rangle = \sum_{i=1}^{r} m_i^2.$$

- $\langle \varphi \mid \varphi \rangle$ is a positive integer and $\langle \varphi \mid \varphi \rangle = 1$ if and only if φ is irreducible.

The regular character

Let $\chi_1, \chi_2, \ldots, \chi_r$ be the irreducible characters of G, $n_i = \chi_i(1)$ and let reg_G be the character of the regular representation of G.

Theorem

1. $\mathrm{reg}_G = n_1\chi_1 + n_2\chi_2 + \cdots + n_r\chi_r$. *That is, every irreducible representation of G is contained in the regular representation with multiplicity equal to its degree.*
2. *The degrees n_i satisfy the relation $\sum_{i=1}^r n_i^2 = \langle \mathrm{reg}_G \mid \mathrm{reg}_G \rangle = |G|$.*
3. *If $x \in G$ and $x \neq 1$, then $\sum_{i=1}^r n_i\chi_i(x) = 0$.*

Proof.

By the previous theorem this multiplicity is $\langle \mathrm{reg}_G \mid \chi_i \rangle$ and

$$\langle \mathrm{reg}_G \mid \chi_i \rangle = \frac{1}{|G|} \sum_{x \in G} \mathrm{reg}_G(x)\overline{\chi_i(x)} = \chi_i(1) = n_i.$$

□

Class functions

A function $\varphi : G \to \mathbb{C}$ is a *class function* if $\varphi(xy) = \varphi(yx)$ for all $x, y \in G$; equivalently φ is constant on conjugacy classes.

Theorem

Let φ be a class function on G and let $\rho : G \to \mathrm{GL}(V)$ be an irreducible linear representation of degree n with character χ. Then

$$\sum_{x \in G} \varphi(x)\rho(x) = \frac{|G|}{n} \langle \varphi \mid \overline{\chi} \rangle I.$$

Proof.

Let $h = \sum_{x \in G} \varphi(x)\rho(x)$. Then $\rho(y)h\rho(y)^{-1} = \sum_{x \in G} \varphi(x)\rho(y)\rho(x)\rho(y)^{-1} = \sum_{x \in G} \varphi(x)\rho(yxy^{-1}) = h$. It follows from Schur's lemma that $h = \lambda I$ and on taking the trace we find that $n\lambda = \sum_{x \in G} \varphi(x)\chi(x) = |G|\langle \varphi \mid \overline{\chi} \rangle$.

□

The space of class functions

Theorem

The irreducible characters $\chi_1, \chi_2, \ldots, \chi_r$ form an orthonormal basis for the vector space $\mathscr{H}$ of class functions on G.

Proof.

The first orthogonality relations show that the characters $\chi_1, \chi_2, \ldots, \chi_r$ are orthonormal. To see that they span $\mathscr{H}$ we show that if $\varphi \in \mathscr{H}$ is orthogonal to all χ_i, then $\varphi = 0$.
For each representation ρ of G put $\rho_\varphi = \sum_{x\in G} \varphi(x)\rho(x)$. If ρ is irreducible with character χ, then $\rho_\varphi = 0$ because $\langle \varphi \mid \overline{\chi} \rangle = 0$.
The regular representation R is a sum of irreducible representations and therefore $R_\varphi = 0$. Consequently,

$$0 = R_\varphi e_1 = \sum_{x\in G} \varphi(x)R(x)e_1 = \sum_{x\in G} \varphi(x)e_x$$

and so $\varphi(x) = 0$ for all x. □

The number of irreducible characters

Theorem

The number of irreducible characters of G equals the number of conjugacy classes of G.

Proof.

Let $C_1, C_2, \ldots, C_\ell$ be the conjugacy classes of G. Then $\varphi : G \to \mathbb{C}$ is a class function if and only if φ is constant on the C_i. Thus φ is determined by its values λ_i on C_i. These values can be chosen arbitrarily, hence $\ell = \dim \mathscr{H} = r$. □

Application

If G is abelian, then G has $|G|$ conjugacy classes and hence $|G|$ irreducible characters. The sum of the squares of their degrees is $|G|$ and therefore every irreducible character of G has degree 1.

Second orthogonality relations

The *centraliser* of $x \in G$ is the subgroup $C_G(x) = \{y \in G \mid xy = yx\}$. If $\mathrm{ccl}_G(x)$ is the set of conjugates of x in G, then $|\mathrm{ccl}_G(x)| = |G|/|C_G(x)|$.

G acts on ccl(x) by conjugation

Theorem

$$\sum_{i=1}^{r} \chi_i(x)\overline{\chi}_i(y) = \begin{cases} |C_G(x)| & x \text{ is conjugate to } y \\ 0 & x \text{ is not conjugate to } y \end{cases}$$

Proof.

Let φ_x be the class function equal to 1 on the class of x and 0 elsewhere. Then

$$\varphi_x = \sum_{i=1}^{r} \lambda_i \chi_i \quad \text{with} \quad \lambda_i = \langle \varphi_x \mid \chi_i \rangle = \frac{|\mathrm{ccl}_G(x)|}{|G|}\overline{\chi}_i(x).$$

and therefore

$$\varphi_x(y) = |C_G(x)|^{-1} \sum_{i=1}^{r} \overline{\chi}_i(x)\chi_i(y).$$ □

Orthogonality revisited

Let $x_1, x_2, \ldots, x_r$ represent the conjugacy classes of G and define $h_i = |\mathrm{ccl}_G(x_i)|$ for $1 \le i \le r$.

Because characters are constant on conjugacy classes, the first orthogonality relations can be written as

$$|G|^{-1} \sum_{k=1}^{r} h_k \chi_i(x_k)\overline{\chi}_j(x_k) = \delta_{ij}.$$

In matrix form this is $\overline{X}^{\top} D X = |G| I$, where $X = (\chi_i(x_j))$ and $D = \mathrm{diag}(h_1, h_2, \ldots, h_r)$.

Consequently

$$X\overline{X}^{\top} D = X(\overline{X}^{\top} D X)X^{-1} = X(|G|I)X^{-1} = |G|I$$

and therefore $X\overline{X}^{\top} = |G|D^{-1}$, which is the matrix form of the second orthogonality relations.

Summary

In the character table

Class	C_1	$\dots$	C_j	$\dots$	C_r
Size	1	$\dots$	h_j	$\dots$	h_r
χ_1	1	$\dots$	1	$\dots$	1
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
χ_i	n_i	$\dots$	$\chi_i(x_j)$	$\dots$	$\chi_r(x_r)$
	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
χ_r	n_r	$\dots$	$\chi_r(x_j)$	$\dots$	$\chi_r(x_r)$

we have

- $\sum_{k=1}^{r} h_k \chi_i(x_k) \overline{\chi}_j(x_k) = |G| \delta_{ij}$
- $\sum_{i=1}^{r} \chi_i(x_j) \overline{\chi}_i(x_k) = |C_G(x_j)| \delta_{jk}$