Nelson Lectures

Lecture 3

Loose ends from last day:

$$\begin{tikzcd} \rho \arrow[d, mapsto] & R(\Gamma) \arrow[r] \arrow[d, "t"] & R(\Gamma)/Sl_2(\mathbb{C}) \\ \chi_\rho & X(\Gamma) & \end{tikzcd}$$

$\mathrm{Red}(\Gamma)$ (reducible reps) $\quad \mathrm{Irr}(\Gamma)$ (irreps)

$$X(\Gamma) = \{\chi_\rho : \rho \in R(\Gamma)\}$$

$$= t(\mathrm{Red}(\Gamma)) \cup t(\mathrm{irr}(\Gamma))$$

Recall: $\exists\, g_1, \ldots, g_\ell \in \Gamma$ s.t. $\chi_\rho(\gamma)$ is determined by $\chi_\rho(\gamma_i)$ for $i = 1, \ldots, \ell$.

We need to describe $t(\mathrm{Red}(\Gamma))$ and $t(\mathrm{irr}(\Gamma))$ as algebraic sets.

$t(\mathrm{Red}(\Gamma))$ is handled by the following:

<u>Lemma 2.4</u> ρ is reducible $\Leftrightarrow$ $\chi_\rho(c) = 2$ $\forall c \in [\Gamma, \Gamma]$.

<u>Proof</u> ($\Rightarrow$) $\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & 1/a' \end{pmatrix}\begin{pmatrix} 1/a & -b \\ 0 & a \end{pmatrix}\begin{pmatrix} 1/a' & -b' \\ 0 & a' \end{pmatrix}$

$= \begin{pmatrix} aa' & ab' + b/a' \\ 0 & 1/aa' \end{pmatrix}\begin{pmatrix} 1/aa' & -b'/a - ba' \\ 0 & aa' \end{pmatrix}$

$= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $\square$

$\Leftarrow$ (ex) Show $\rho([\Gamma,\Gamma])$ has a 1-dim'l invariant subspace. $\square$

To deal with irreps, a key point to ensure $R(\Gamma)/SL_2(\mathbb{C}) \hookrightarrow X(\Gamma)$ on cpts containing chars of irreps;

Theorem 2.5 Let $\rho, \rho' : \Gamma \to SL_2(\mathbb{C})$ be irreps s.t. $\chi_\rho(\gamma) = \chi_{\rho'}(\gamma)$ $\forall \gamma \in \Gamma$. Then ρ, ρ' are equivalent.

Proof. Key Claim is:

Claim If $\rho \in R(\Gamma)$ is an irrep and $\rho(g) \in \rho(\Gamma)$, $(\neq \pm I)$ then $\exists h \in \Gamma$ s.t. $\rho|_{\langle g, h \rangle}$ is irreducible and $\chi_\rho(h) \neq \pm 2$.

Assume this claim.

Let $G = \langle g, h \rangle$

Have $\chi_\rho(\gamma) = \chi_{\rho'}(\gamma) \quad \forall \gamma \in \Gamma$

and hence $\chi_\rho(\gamma) = \chi_{\rho'}(\gamma) \ \forall \gamma \in G$.

Can conjugate $\rho(\Gamma)$ and $\rho'(\Gamma)$ s.t.

$$\rho(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \rho'(h)$$

In addition

$$\rho(g) = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}, \quad \rho'(g) = \begin{pmatrix} a' & 1 \\ c' & d' \end{pmatrix}$$

and $a + d = a' + d'$.

Now consider

$$\rho(gh) = \begin{pmatrix} \lambda a & \lambda \\ \lambda^{-1} c & \lambda^{-1} d \end{pmatrix}$$

$$\rho'(gh) = \begin{pmatrix} \lambda a' & \lambda \\ \lambda^{-1} c' & \lambda^{-1} d' \end{pmatrix}$$

and so $\lambda a + \lambda^{-1} d = \lambda a' + \lambda^{-1} d'$.

Using $a+d = a'+d'$ gives

$$\lambda(a-a') - \lambda'(d-d') = 0$$

$$\left.\begin{array}{l} \lambda^2 \neq 1 \Rightarrow a = a' \\ \qquad\quad \Rightarrow d = d' \end{array}\right\} \Rightarrow c = c'$$

$$\therefore \rho|_G = \rho'|_G.$$

Now take $\gamma \in \Gamma$ arbitrary

$$\rho(\gamma) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

$$\rho'(\gamma) = \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}$$

and $p+s = p'+s'$.

Argue as above with $h\gamma$

$\Rightarrow p = p'$, $s = s'$

Now consider $g\gamma$ and use the above:

$$\left.\begin{array}{l} ap + r = ap' + r' \\ cq + ds = cq' + ds \end{array}\right] \begin{array}{l} r = r' \\ q = q' \end{array} \quad \square$$

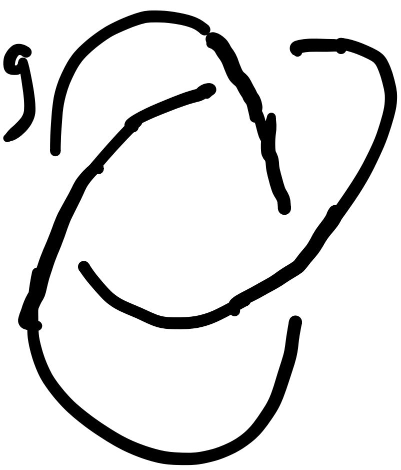
We now turn to connections to the geometry + topology of 3-mflds and connections with number theory.

Theorem 3.1 (Thurston) Let M be a 1 cusped hyperbolic 3-mfld. Then $X(M) := X(\pi_1 M)$ contains a curve cpt X_0 which contains the character of a faithful discrete rep.

X_0 = canonical curve/cpt.

Remarks

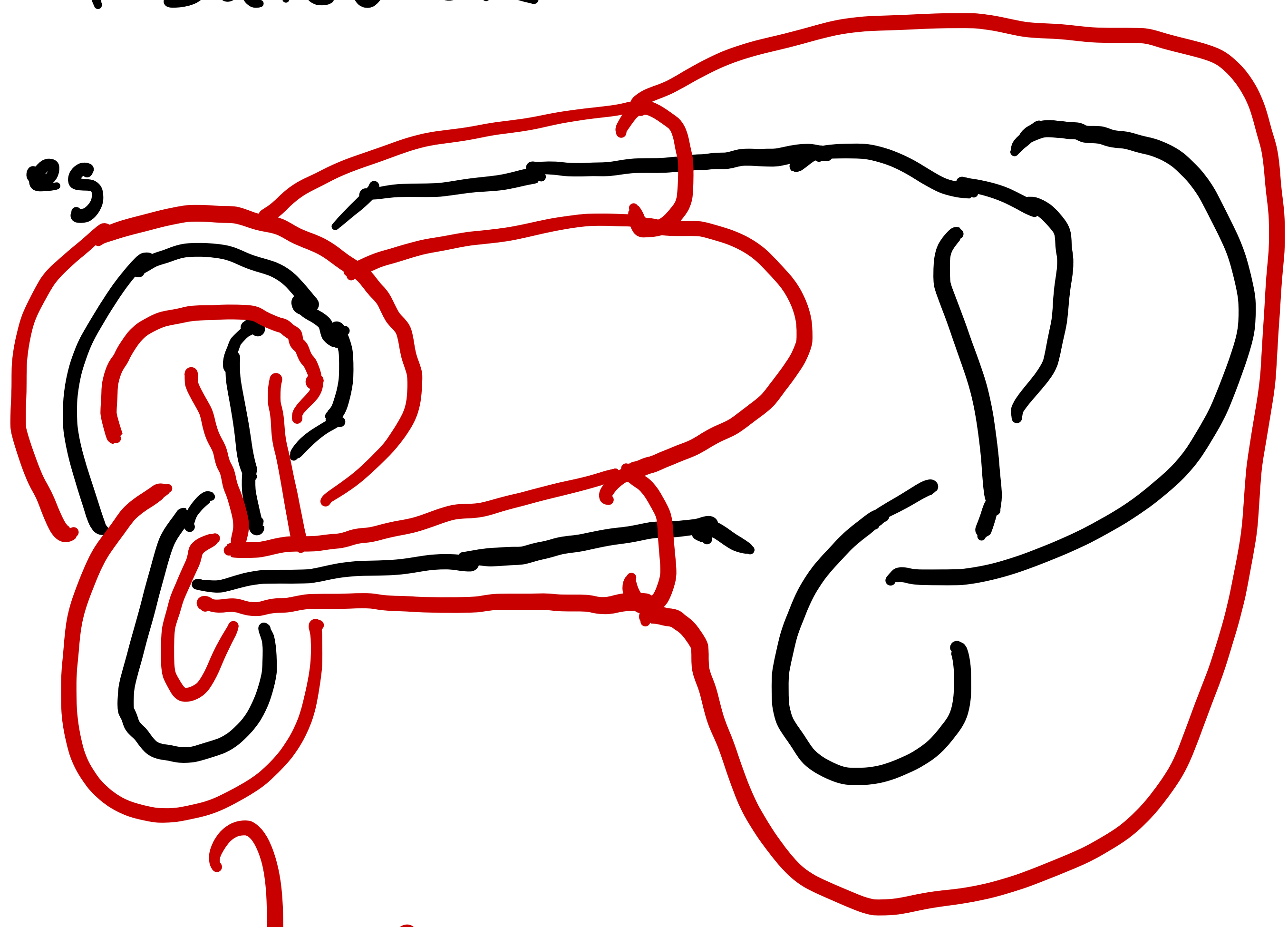
1. $K \subset S^3$ is hyperbolic so long as $K \neq$ torus knot eg



or

$K \neq$ satellite

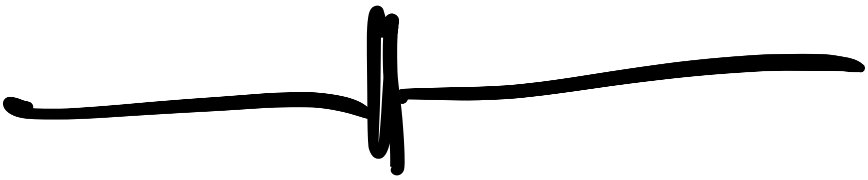
eg



T^2 is essential and not bdy parallel

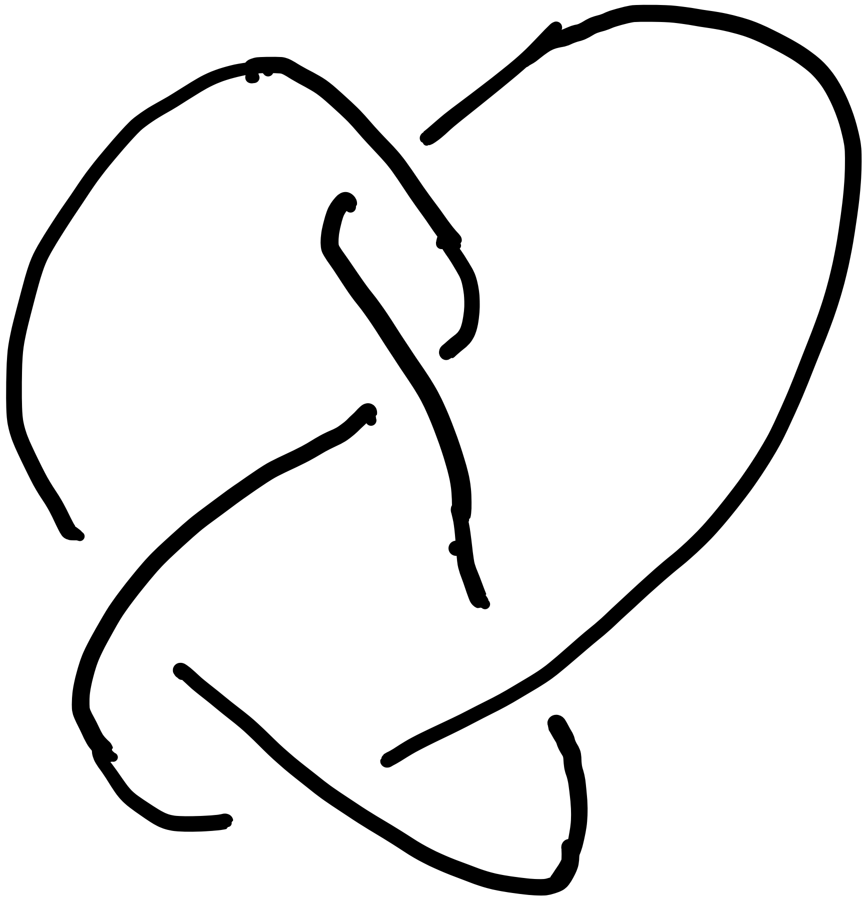
ie $\pi_1 T^2 \to \pi_1(S^3 \setminus K)$ is 1-1,

2. Of 1 701 936 prime knots (ie not connect sums) up through 16 crossings all but 32 are hyperbolic.



Some other examples of canonical curves. (See beamer pdf)

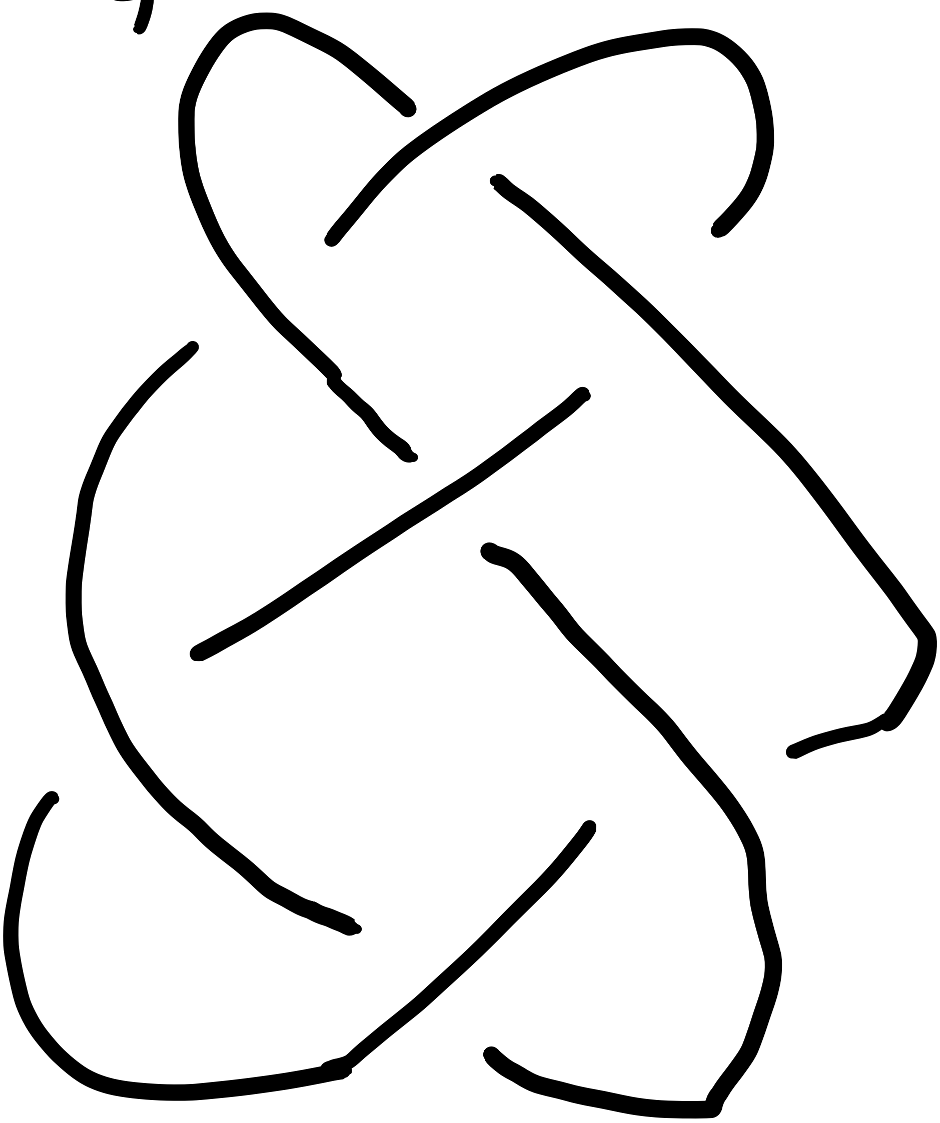
1. 5_2



X_0 defined by vanishing set of $p(z,T)=0$ where

$$p(z,T) = z^2(T-T^2)+T^3+T^2-2T-1$$

2. 7_4



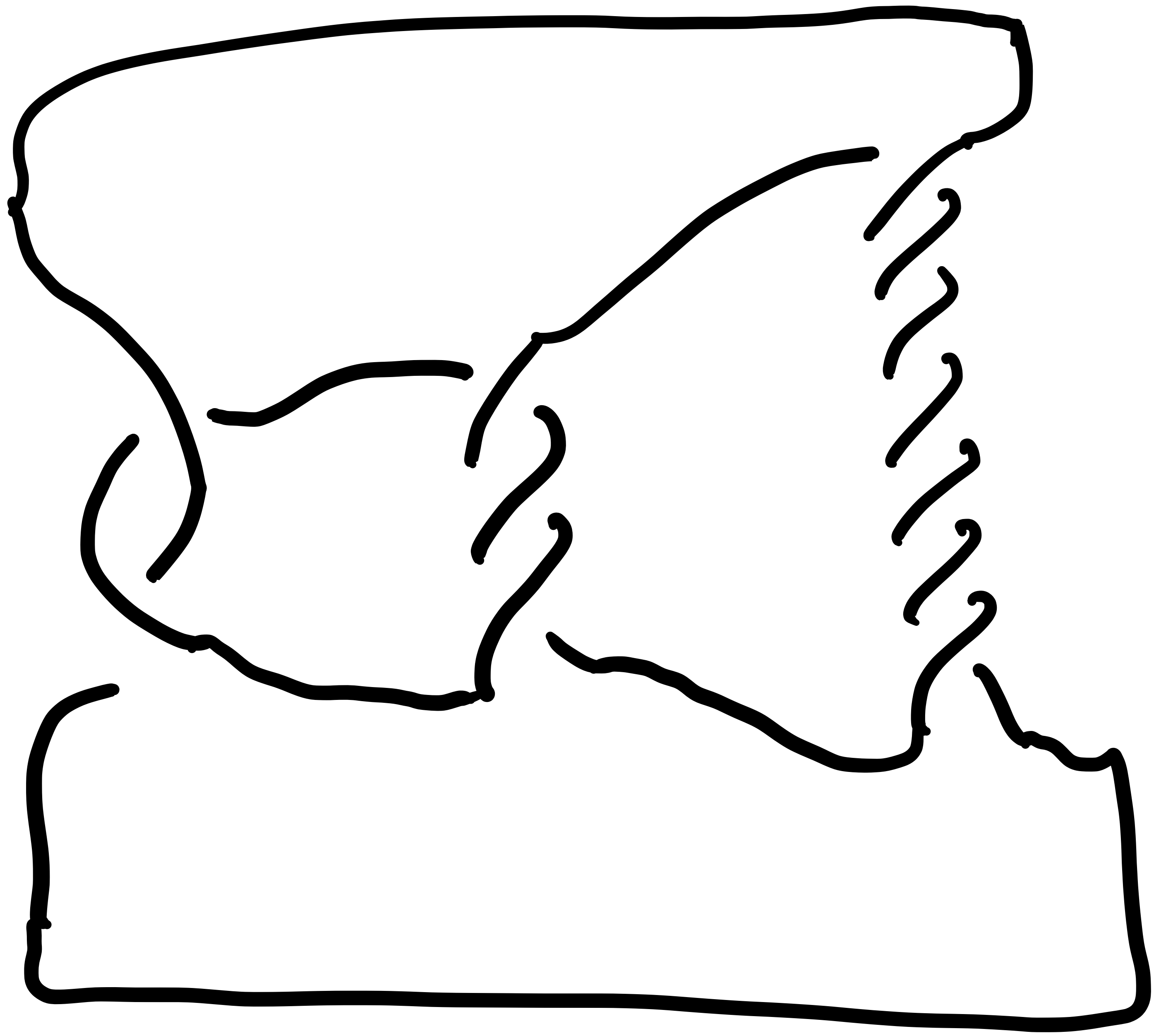
There are 2 cpts containing the character of an irrep.

$p(z,T)=0$ where

X_0

$$p(z,T)=(-1+2T^2+T^3-T^2z^2)(1+4T-4T^2-T^3+T^4-2Tz^2+3T^2z^2-T^3z^2)$$

EX 3 $(-2, 3, 7)$ Pretzel knot



X_0 cut out by rational functions of Q

$$P = \frac{Q}{Q^2-1}, \quad R = \frac{1-2Q^2}{Q^2(Q^2-1)}$$

Question What curves arise as X_0?

Sample theorem:

Theorem 4.2 (Macasieb, Petersen, Van Luijk)

K_n = n-twist knot. (hyp)

$X_0(n)$ = canonical cpt of K_n.

$X_0(n)$ has genus

$$g(n) = c(n) - 3$$

crossing # of K_n

eg. Fig 8, $n=4$, $g(n) = 1$.

Some other remarks

1. $\exists$ M 1 cusped hyperbolic s.t $X(M) \supset$ large dim'l cpts containing the character of an irrep.

2. 1 is true even for knots.

eg $K = K_1 \# K_2$ $K_i =$ fig 8

$X(\pi_1(S^3 \setminus K)) \supset$ a cpt of dim ≥ 2

Construct hyperbolic $J \subset S^3$ with $\pi_1(S^3 \setminus J) \twoheadrightarrow \pi_1(S^3 \setminus K)$

3 <u>Theorem</u> (Kronheimer-Mrowka)
$K \neq$ unknot. Then $X(\pi_1(S^3 \setminus K)) \supset$ curve of chars of irreps.

Topology

Theorem (Culler–Shalen, Bass–Serre)

Let Γ = f.g gp.

If $X(\Gamma) \supset$ curve of chars

$\Rightarrow \Gamma$ splits as graph of gps.

If $\Gamma = \pi_1 M^3$ then M contains a properly embedded essential surface.

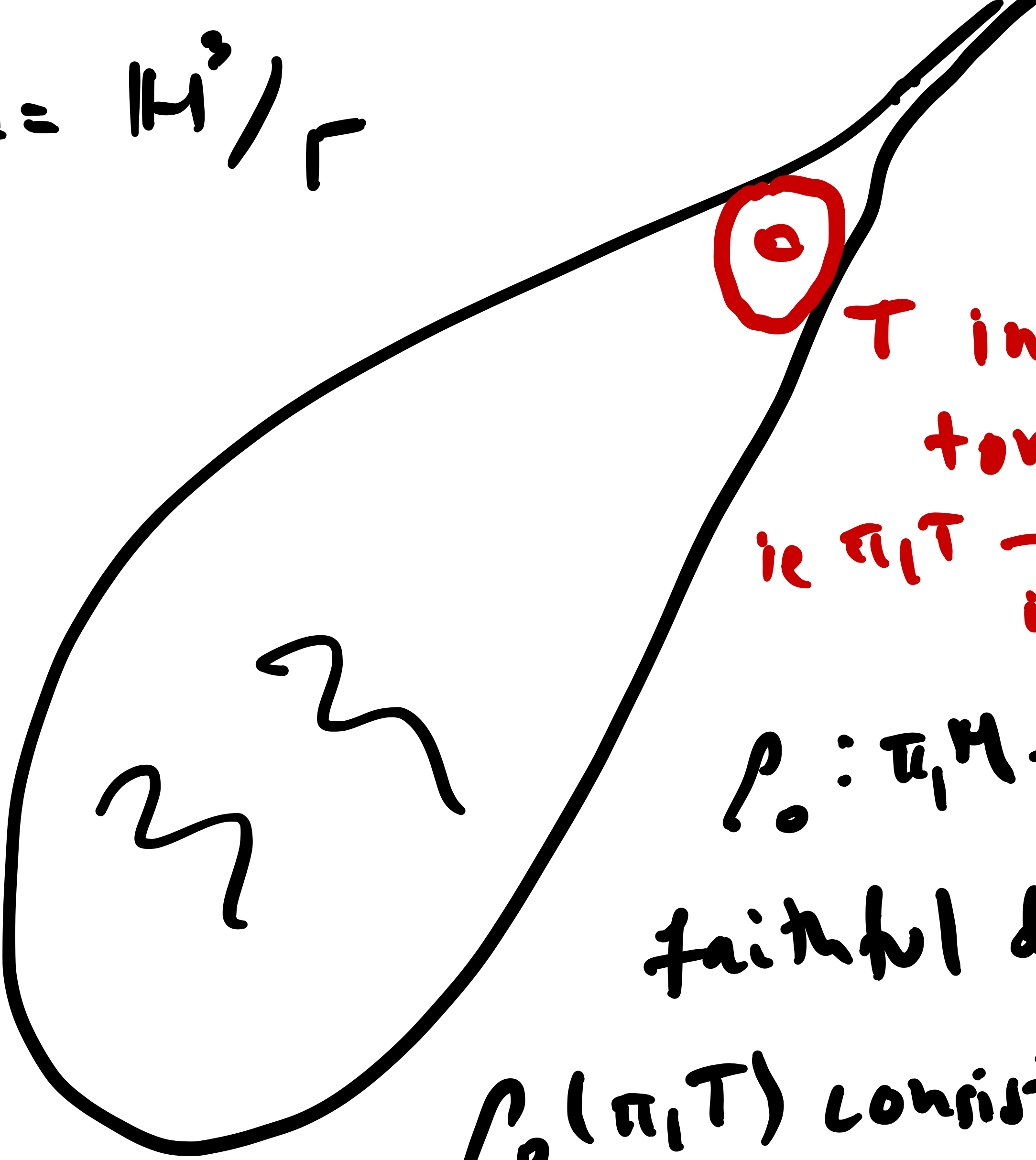
Lecture 4.

Recap

Theorem (Thurston)
Let M be a 1 cusped hyperbolic 3-manifold (see below). Then $X(M) \supset$ an irreducible curve component X_0 (the canonical curve/cpt) which χ_{ρ_0} the character of a faithful discrete repⁿ.

Remark See below for more on other interesting geometric chars on X_0.

$M = \mathbb{H}^3/\Gamma$



Schematic

T incomp. torus

ie $\pi_1 T \to \pi_1 M$ is 1–1.

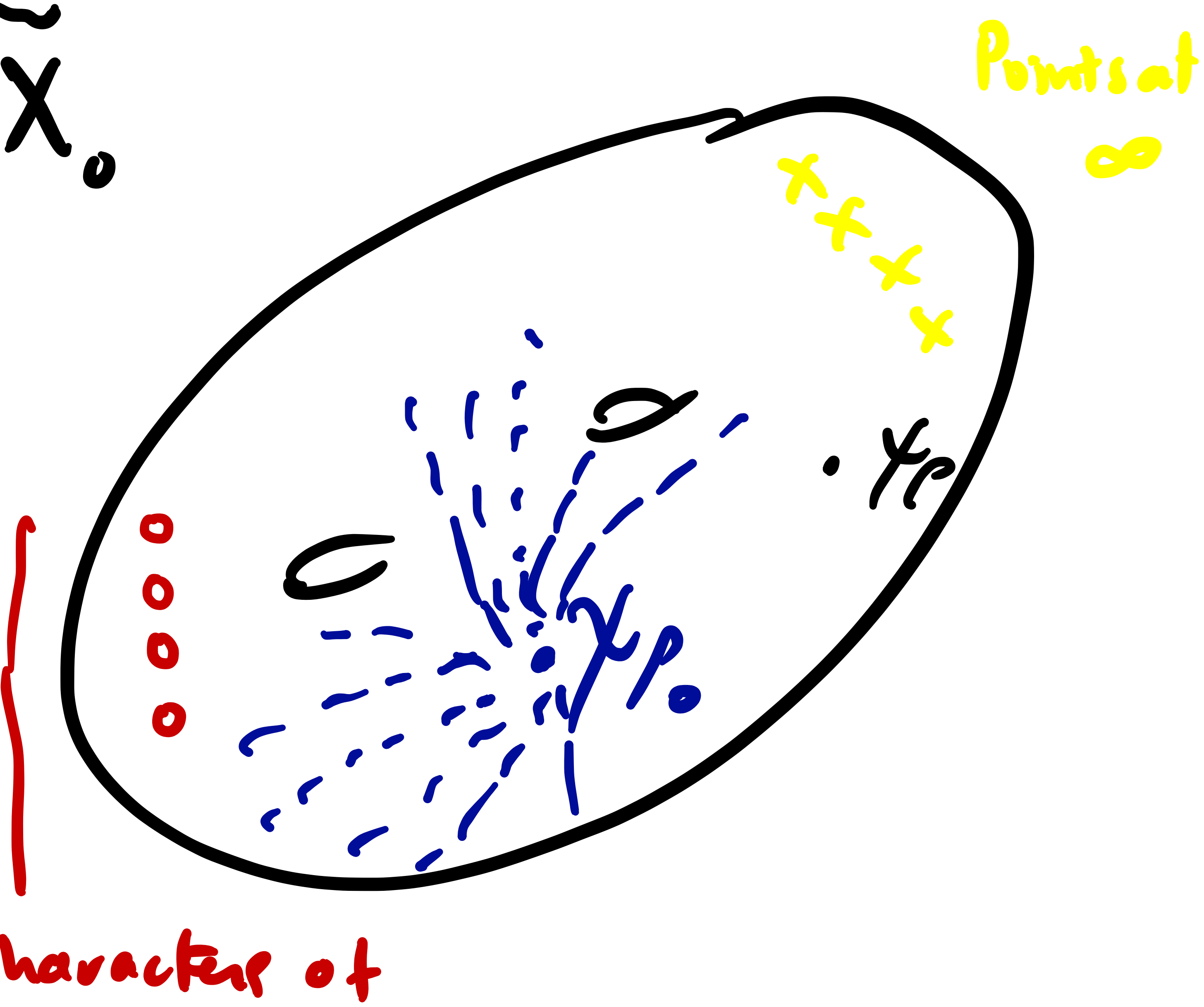
$\rho_0 : \pi_1 M \to SL_2(\mathbb{C})$

faithful discrete

$\rho_0(\pi_1 T)$ consists of parabolic elements (ie $\rho_0(\pi_1 T)$ is conj into $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

Cartoon of X_0 (affine curve)

Smooth projective model $\tilde{X}_0$



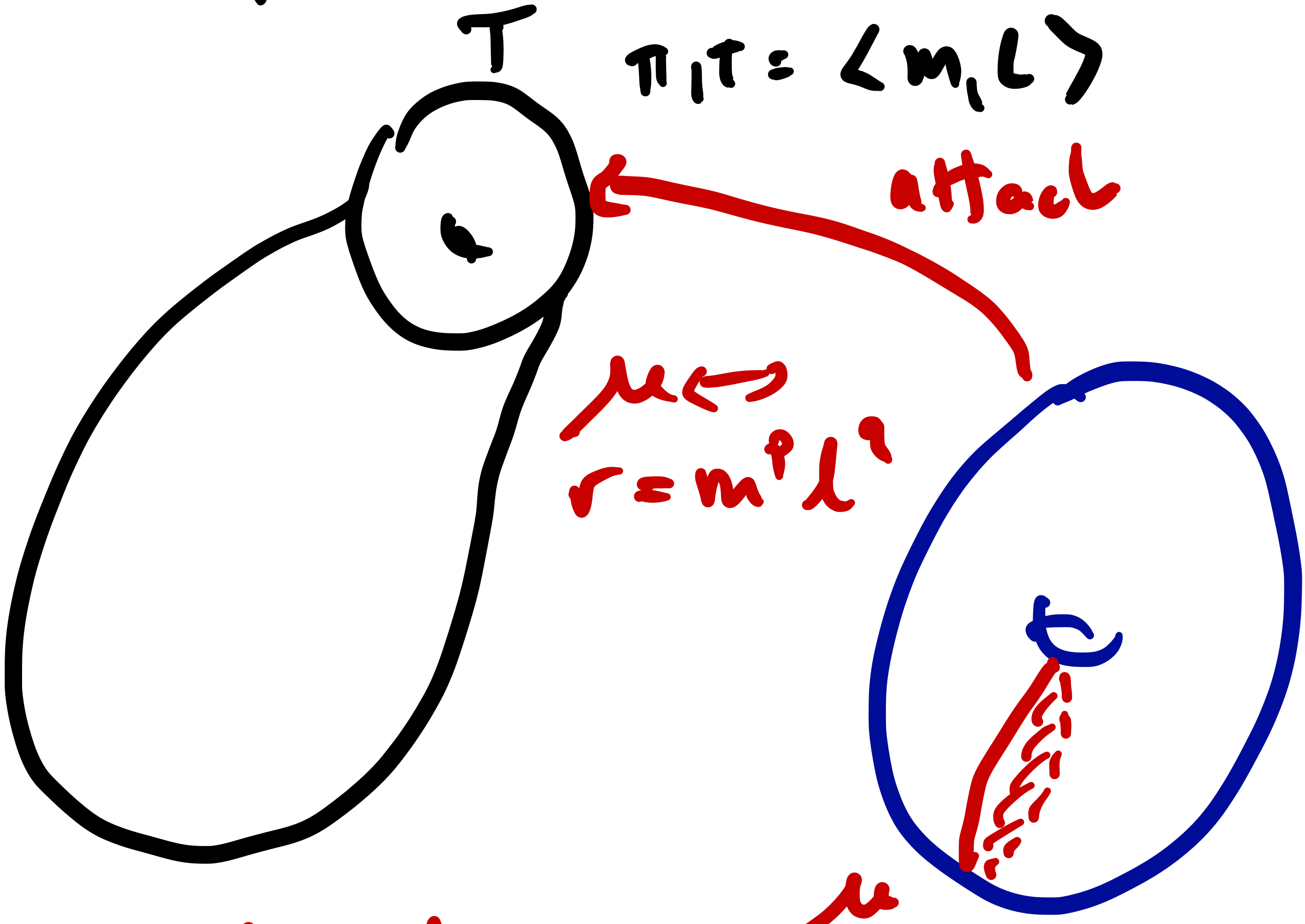
Ex
Since ρ_0 is 1-1 "the generic"
χ_ρ is the character of a faithful irrep of $\pi_1 M$.
Dont have much geometry!

Other interesting characters

Characters of Dehn Surgery Representations (in blue).

Dehn Surgery: Cut off cusp of M get compact 3-mfld with T^2 ∂.

$\pi_1 T = \langle m, l \rangle$



μ bounds a disc in V

The resultant closed 3-mfld is $M(r) = r$ - Dehn Surgery

<u>Thurston's Dehn Surgery Thm</u>

M as above:

All but finitely many r produce $M(r)$ which is hyperbolic.

<u>Note</u>: $\pi_1 M(r) = \pi_1 M \big/ \langle\langle m^p l^q \rangle\rangle$

$$
\begin{tikzcd}
\pi_1 M \arrow[r] \arrow[dr, "\rho_r"'] & \pi_1 M(r) \arrow[d, dashed, "(\text{hyp})"] \\
& SL_2(\mathbb{C})
\end{tikzcd}
$$

Dehn Surgery rep.

Arithmetic of Dehn Surgery Points

$M = \mathbb{H}^3/\Gamma$ fte vol

The trace field of Γ is

$$\mathbb{Q}(\operatorname{tr}\gamma : \gamma \in \Gamma) = k_\Gamma$$

Rigidity $\Rightarrow$ k_Γ is a number field

$[k_\Gamma : \mathbb{Q}] < \infty$

Ex $K =$

$S^3 \setminus K = \mathbb{H}^3/\Gamma$

$$\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \right\rangle$$

$\omega^2 - \omega + 1 = 0$

$$k_\Gamma = \mathbb{Q}(\sqrt{-3}).$$

Ex 5_2 $\quad k_\Gamma = \mathbb{Q}(\theta)$

$$\theta^3 - \theta^2 + 1 = 0$$

Ex $(-2,3,7)$ Pretzel

$k_\Gamma =$ same cubic as 5_2

Ex 7_4 $\quad k_\Gamma = \mathbb{Q}(t)$, $t^3 + 2t - 1 = 0$

[See Snap data for more].

Also attached to Γ is a quaternion algebra

$$A_0\Gamma = \left\{ \sum_{\text{fte}} a_i \gamma_i : \begin{array}{l} a_i \in k_\Gamma \\ \gamma_i \in \Gamma \end{array} \right\}$$

i.e. a 4 dim'l CSA.

<u>EX</u> figure eight.

$\begin{pmatrix}1&1\\0&1\end{pmatrix} \in \Gamma \Rightarrow \begin{pmatrix}1&1\\0&1\end{pmatrix} - Id$

$\in A_0\Gamma$

$\Rightarrow A_0\Gamma = M_2(\mathbb{Q}(\sqrt{-3}))$

<u>Lemma</u> M cusped

$= \mathbb{H}^3/\Gamma$

$A_0\Gamma \cong M_2(k_\Gamma)$.

<u>Proof</u> $\Gamma \ni \begin{pmatrix}1&1\\0&1\end{pmatrix}$ up to

conjugacy.

Quaternion algebras

over $\mathbb{C}$ $\longrightarrow$ $M_2(\mathbb{C})$

$\mathbb{R}$ $\nearrow$ $M_2(\mathbb{R})$

$\mathbb{R}$ $\searrow$ $\mathbb{H}$ (Hamiltonians)

$\mathbb{Q}_p$ $\nearrow$ $M_2(\mathbb{Q}_p)$

$\mathbb{Q}_p$ $\searrow$ unique iso class of div alg.

(and more generally for $[k:\mathbb{Q}_p]<\infty$.

Over number fields more subtle. k a no. field

B/k quat. algebra

B is ramified at $\mathfrak{p}$ prime if $B \otimes_k k_p$ = division.

If for a real embedding of k

$$B \otimes_{\sigma(k)} \mathbb{R} \cong \mathbb{H} \text{ also } B$$

is ramified

<u>Classification Thm</u> Every quat. algebra over a number field k is determined up to iso by a finite # of prime + embeddings that ramify.

[See Swap].

Arithmetic

Theorem (Long–R)

Let M = 1 cusped hyp. 3-mfld.
Given a positive integer D
$\exists$ only finitely many
$\chi_\rho \in X_0$ with ρ discrete
s.t. $[\mathbb{Q}(\operatorname{tr}\rho(\pi_1 M)) : \mathbb{Q}] \leq D$

Compare with Faltings solution
to the Mordell Conjecture.

Cor. Let d be sq. free positive
integer. Let $\mathcal{O}_d$ = ring of integers
in $\mathbb{Q}(\sqrt{-d})$. Then the # $\mathcal{O}_d$ points
on X_0 is finite. ie $\left|\bigcup_d X_0(\mathcal{O}_d)\right| < \infty$

Now see beamer pdf for additional discussion of arithmetic data assoc. to Dehn surgeries.