Nelson Lectures

$SL_2(\mathbb{C})$-character varieties in low-dimensional topology

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The figure-eight knot

Lecture 1 :

Notation : Γ, G groups, Γ will be infinite and f.g.

Def A representation of Γ to G is a homomorphism $\varphi : \Gamma \longrightarrow G$.

EX 1. For any Γ, G $\exists$ trivial representation $\varphi : \Gamma \longrightarrow G$ with $\varphi(\gamma) = 1$ $\forall \gamma \in \Gamma$.

2. If $\varphi : \Gamma \longrightarrow G$ is a representation, can construct a new rep as follows:

Let $g \in G$ $(\neq 1)$ and set $\varphi'(\gamma) = g\varphi(\gamma)g^{-1}$ $\forall \gamma \in \Gamma$.

Say reps. φ, φ' are equivalent if φ, φ' differ by conjugation as above.

Motivating Idea:

Illuminate the structure of Γ by using representations to various G.

Examples 1. Take $|G| < \infty$ $(\neq 1)$

and $\varphi : \Gamma \longrightarrow G$ onto.

Gives finite index subgps of Γ.
if $\Gamma = \pi_1 M$, get finite covers of M.

2. Linear Reps

Let $\mathbb{F}$ = field

$G = GL_n(\mathbb{F})$

$\rho : \Gamma \longrightarrow G$ is a linear representation over $\mathbb{F}$.

If $\mathbb{F} = \mathbb{C}$, say linear rep.

<u>Note</u>: Reps. to finite groups are special examples of linear reps.

Γ is called linear (over $\mathbb{F}$) if $\exists \rho : \Gamma \longrightarrow GL_n(\mathbb{F})$ which is faithful.

(Γ is called linear when $\mathbb{F} = \mathbb{C}$)

Important Property

If $\rho : \Gamma \longrightarrow GL_n(\mathbb{C})$ is a linear rep which is "sufficiently complicated" then get lots of interesting finite quotients.

eg. 1. $\rho : \Gamma \longrightarrow GL_n(\mathbb{Z})$

$$\begin{array}{c} GL_n(\mathbb{Z}) \\ \downarrow \\ GL_n(\mathbb{Z}/p\mathbb{Z}) \end{array}$$

p prime.

2. Γ linear $\Rightarrow$ Γ is Residually finite.

3. An alternative viewpoint of 2.

V = n-dim'l vector space/$\mathbb{F}$.

$GL(V)$ = bijective linear transformations.

Choosing a basis can identify $GL(V) = GL_n(\mathbb{F})$.

Advantage of this viewpoint, a linear rep. $\Gamma \xrightarrow{\rho} GL(V)$ yields <u>an action</u> of $\rho(\Gamma)$ (ie Γ) on V.

<u>Def</u> Say ρ is <u>irreducible</u> if only invariant subspaces for $\rho(\Gamma)$ are $\{0\}$, V. Otherwise <u>reducible</u>.

Example $\Gamma = \pi_1(S^3, K)$

$$\Gamma = \langle a, b \mid waw^{-1} = b, \; w = ab^{-1}a^{-1}b \rangle$$

Note $\Gamma^{ab} = \Gamma/[\Gamma,\Gamma] \cong \mathbb{Z}$

(gen. by image of a or b)

Consider $\Gamma/\langle\langle a^2 \rangle\rangle$ — normal closure.

$$= \langle a, b \mid a^2 = 1, \; b^2 = 1, \; \underbrace{waw^{-1}b^{-1} = 1}_{(ab)^5 = 1} \rangle$$

$$\cong D_5 \quad \Rightarrow K \neq \bigcirc.$$

What about reps to $SL_2(\mathbb{C})$.

Try. $\rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\rho(b) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$

To get a rep. need

$$\rho(wa) = \rho(bw)$$

Exercise $x = \dfrac{1 \pm \sqrt{-3}}{2}$ gives a solution.

Note $\rho : \Gamma \longrightarrow SL_2(\mathbb{Z}[x])$ above is irreducible, discrete.

More: From geometry ρ is 1-1.

Another rep — reducible
not abelian.

Try. $\rho(a) = \begin{pmatrix} t & 1 \\ 0 & 1/t \end{pmatrix}$

$\rho(b) = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$

Evaluate $\rho(wa) = \rho(bw)$

Check if t satisfies

$$t^4 - 3t^2 + 1 = 0$$

get a solution.

Note: The Alex Poly of K is $u^2 - 3u + 1$ (set $u = t^2$)

4. Generalize actions from linear reps. on a vector space X

Take X
- set, perms of X
- top space, homeos of X
- metric space, isoms of X
- graph/tree, graph autos of X

Focus now on $Sl_2(\mathbb{C})$ reps.

Ex $\Gamma = F_2 = \langle a, b \rangle$

$\rho : \Gamma \longrightarrow Sl_2(\mathbb{C})$

is completely determined b

$$\rho(a) = \begin{pmatrix} x_1 & y_1 \\ z_1 & t_1 \end{pmatrix} \quad x_1 t_1 - y_1 z_1 = 1$$

$$\rho(b) \begin{pmatrix} x_2 & y_2 \\ z_2 & t_2 \end{pmatrix} \quad x_2 t_2 - y_2 z_2 = 1$$

$$\xleftrightarrow{\text{identify with}}$$

$$\left\{ (x_1\, y_1\, z_1\, t_1, x_2\, y_2\, z_2\, t_2) : \begin{array}{l} x_i t_i - y_i z_i = 1 \\ i = 1, 2 \end{array} \right\}$$

Algebraic Set cut out by polys.

Lecture 2

Organizing reps of $\Gamma \longrightarrow Sl_2(\mathbb{C})$
↑
(f.g gp).

Def^n $R(\Gamma) = \{ \rho : \Gamma \longrightarrow Sl_2(\mathbb{C})$ a rep. $\}$

Theorem 2.1 $R(\Gamma)$ is an algebraic set defined over $\mathbb{Q}$ (indeed $\mathbb{Z}$).

$R(\Gamma)$ is called the Representation Variety.

Recall: $V \subset \mathbb{C}^n$ is an algebraic set if $\exists$ a finite collection of polynomials $p_1, \ldots, p_m \in \mathbb{C}[x_1 \ldots x_n]$ s.t. V is the vanishing locus of $p_1 \ldots p_m$.

Indeed $\exists$ ideal $I(V) \subseteq \mathbb{C}[x_1, \ldots, x_n]$ s.t $p \in I(V) \iff p$ vanishes on V.

By Hilberts Basis theorem $I(V)$ has a finite set of generators.

V is defined over a subring $R \subset \mathbb{C}$ if a set of generators for $I(V)$ can be taken in $R[x_1, \ldots, x_n]$.

Proof of Theorem 2.1 :

Let $\Gamma = \langle \gamma_1 - \gamma_r \mid w_1, w_2, \cdots \rangle$ be a presentation on the finite generating set $S = \{\gamma_1 - \gamma_r\}$

Let $\rho \in R(\Gamma)$, then

$$\rho(\gamma_i) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

$$a_i d_i - b_i c_i = 1$$

Note $\rho(\gamma_i^{-1}) = \begin{pmatrix} d_i & -b_i \\ -c_i & a_i \end{pmatrix}$

Now $w_k = w_k(\gamma_1^{\pm 1}, -, \gamma_r^{\pm 1})$ and evaluating ρ, we must have

$$\rho(w_k) = w_k(\rho(\gamma_1^{\pm 1}) -, \rho(\gamma_r^{\pm 1}))$$
$$= I$$

This gives a matrix eqn:

$$\begin{pmatrix} p_{j_1} & p_{j_2} \\ p_{j_3} & p_{j_4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for each j, p_{j_ℓ} = polys in a_i, b_i, c_i, d_i

Hence poly eqns:

$$p_{j_1} - 1 = 0, \quad p_{j_2} = 0$$

$$p_{j_3} = 0, \quad p_{j_4} - 1 = 0$$

This defines $R(\Gamma)$ as an alg. set. $\square$.

<u>Remark</u>. $R(\Gamma)$ is not a variety in the sense that $R(\Gamma)$ need be irreducible as an algebraic set.

<u>Drawback</u> : Depends on a generating set.

However if $S' = \{\delta_1 - \delta_m\}$ is also a gen set for Γ we have

alg set constructed above

Same with S'.

$$\begin{tikzcd}
\rho(\gamma_i) & V_S \arrow[r, bend left, "g \circ f^{-1}"] & V_{S'} \arrow[l, bend left, "f \circ g^{-1}"] & \rho(\delta_j) \\
\gamma_i & R(\Gamma) \arrow[u, "f"] & R(\Gamma) \arrow[u, "g"'] & \delta_j
\end{tikzcd}$$

<u>EX</u>: This defines poly iso. over $\mathbb{Z}$ $V_S \longrightarrow V_{S'}$.

$\therefore$ $R(\Gamma)$ is well-defined in this sense.

Def^n. Let $\rho \in R(\Gamma)$, the <u>character</u> of ρ is a function $\chi_\rho : \Gamma \longrightarrow \mathbb{C}$ s.t. $\chi_\rho(\gamma) = \operatorname{tr}\rho(\gamma)\ \forall \gamma \in \Gamma$.

$$
\begin{tikzcd}
\rho & R(\Gamma) \arrow[r] \arrow[d, "t"'] & R(\Gamma)/SL_2(\mathbb{C}) \arrow[dl, dashed] \\
t(\rho) = \chi_\rho & X(\Gamma) = \{\chi_\rho : \rho \in R(\Gamma)\} &
\end{tikzcd}
$$

See discussion below. (*)

<u>Theorem 2.2</u> $X(\Gamma)$ is an algebraic set defined over $\mathbb{Q}$. The Character Variety.

Regarding (*)

Saw last day that the trivial rep. has the same character as the following

$$\Gamma \longrightarrow \mathbb{Z} \longrightarrow SL_2(\mathbb{C})$$
$$a \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

But not equivalent.

Also any reducible rep^{n} has the same character as a diagonal rep.

Before proving this, recall that ρ is reducible if $\rho(\Gamma)$ leaves invt a 1-dim'l subspace of $\mathbb{C}^2$.

Suppose $\rho(\Gamma).L = L$ where $L = \mathbb{C}.v$

Map v to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

This conjugates $\rho(\Gamma)$ to have image in $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$\boxed{\text{Thus reducible} \equiv \text{upper } \Delta' \text{ image.}}$

Thus if $\rho(\Gamma)$ is reducible can assume

$$\rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ 0 & y_{a_\gamma} \end{pmatrix}$$

Define $\rho' : \Gamma \longrightarrow SL_2(\mathbb{C})$

by $\rho'(\gamma) = \begin{pmatrix} a_\gamma & 0 \\ 0 & y_{a_\gamma} \end{pmatrix}$

<u>EX</u>: ρ' is a representation

It visibly has the same character as ρ.

Again not equivalent.

Thus to understand all characters need to understand reducible $\Leftarrow$ diagonal reps

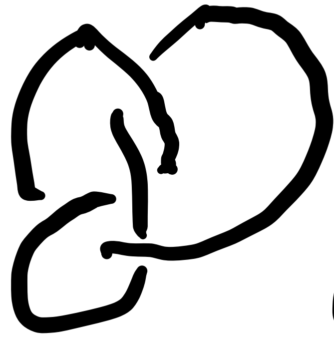
and components $\supset$ characters of irred. reps

See later:

Show $\rho, \rho' \in R(\Gamma)$ irreps with $\chi_\rho(\gamma) = \chi_{\rho'}(\gamma)$ $\forall \gamma \in \Gamma$.

Then ρ, ρ' are equivalent.

Example

$K =$ 

(from last day)

$\Gamma = \pi_1(S^3 \setminus K)$

$= \langle a, b \mid waw^{-1} = b,$
$w = ab^{-1}a^{-1}b \rangle$

What do components of $X(\Gamma)$ containing characters of irreps look like

[Now refer to beamer pdf]

Comment about $X(\Gamma)$:

Using trace identities it can be shown that $\exists\, g_1, \ldots, g_t \in \Gamma$ s.t $\forall g \in \Gamma$, $\chi_\rho(g)$ is completely determined by $\chi_\rho(g_i)$ $i = 1, \ldots, t$

EG: If Γ is 2-generator $\Gamma = \langle S \mid R \rangle$ $S = \gamma_1, \gamma_2$

Then $\chi_\rho(\gamma_1), \chi_\rho(\gamma_2)$
$\chi_\rho(\gamma_1 \gamma_2)$ suffice

Note: Cayley-Hamilton identity

$(\dagger)$ $\operatorname{tr} AB + \operatorname{tr} AB^{-1} = \operatorname{tr} A \operatorname{tr} B$

allows one to choose $\operatorname{tr} AB^{-1}$ in the previous.

Proof of $(\dagger)$

B satisfies its char poly

$$B^2 - (\operatorname{tr} B) B + I = 0$$

$$\Rightarrow B^2 + I = (\operatorname{tr} B) B$$

$$B + B^{-1} = (\operatorname{tr} B) . I$$

$$\Rightarrow A(B + B^{-1}) = (\operatorname{tr} B) A$$

Take trace and result follows $\square$

Using this and other trace ids. prove the general statement.

Summary for Figure Eight

$X(\Gamma)$ has a unique component X_0 containing the character of an irrep. It is the vanishing locus of:

$$z^2(T-1) = T^2 + T - 1$$

$$z = \chi_\rho(a), \quad T = \chi_\rho(ab^{-1})$$

Set $Y = z(T-1)$

$$Y^2 = (T-1)(T^2+T-1)$$

"an elliptic curve $/\mathbb{Q}$".