On the $N=1$ Supergravity Action from Type *II* String Theories with Background

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Niels Bernhardt

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Zusammenfassung in deutscher Sprache

Die vorliegende Diplomarbeit ist im Rahmen der wissenschaftlichen Veröffentlichung "*Scattering of Gauge, Matter and Moduli Fields from Intersecting Branes*", siehe [1], entstanden. Darin geht es darum, die $N = 1$ Supergravitationswirkung für stringtheoretische Modelle mit sich schneidenden D-Branes bzw. D-Branes mit Hintergrundfluss in vier Raum-Zeit-Dimensionen zu bestimmen. Da Stringtheorie in zehn Raum-Zeit-Dimensionen formuliert ist, hängt diese Wirkung von den sechs kompaktifizierten Raum-Dimensionen ab. Genauer gesagt, bestimmen wir die Abhängigkeit der Komponenten der Supergravitationswirkung von den Moduli des Kompaktifizierungsraums im vorliegenden Modell. Die Geometrie des Kompaktfizierungsraums (im betrachteten Fall die Mannigfaltigkeit $T^2 \times T^2 \times T^2$) parametrisieren wir durch sog. Modulifelder, wobei es sich um Skalarfelder handelt, die durch geschlossene Strings dargestellt werden. Die Abhängigkeit des Kählerpotentials, des Superpotentials und der eichkinetischen Funktion, aus welchen sich die Wirkung zusammensetzt, lässt sich in diesem Bild durch Streuprozesse mit ebendiesen Modulifeldern bestimmen. Wir berechnen dazu Streuamplituden zwischen Moduli-Feldern, zwischen Materiefeldern sowie gemischte Amplituden.

Diese Arbeit ist wie folgt aufgebaut. Das erste Kapitel stellt eine Art Vorwort mit anschliessender Kurzzusammenfassung dar.

Im zweiten Kapitel werden die fünf Stringtheorien und die Stringdualitäten kurz vorgestellt. Ferner werden die beiden möglichen vereinheitlichenden Theorien, d.h. M- und F-Theorie , sowie der Zusammenhang zum Standardmodell der Teilchenphysik angesprochen.

Das dritte Kapitel ist als Einführung in die Idee von sog. D-Branes anzusehen. Es wird erklärt, wo und wie diese entstehen. Ferner wird der Komplex der BPS-Zustände angesprochen, zu denen D-Branes gehören.

Das vierte Kapitel beschäftig sich mit Kompaktifizierungen. Es werden allg. Kompaktifizierungsräume angesprochen, worauf insbesondere auf toroidale Kompaktifizierungen eingegangen wird. Für den konkreteren Fall der Kompaktifizierung auf $T^2 \times T^2 \times T^2$ werden die beiden Moduli der jeweiligen Tori dargestellt. Im weiteren leiten wir die Vertexoperatoren der Modulifelder sowie deren duale Vertexoperatorn her, die wir für spätere Rechungen benötigen.

Im fünften Kapitel kehren wir zu D-Branes zurück. Dabei gehen wir auf zwei sehr moderne Modelle ein. Diese sind D-Branes mit einem Hintergrundfluss und D-Branes, die sich unter einem (bel.) Winkel schneiden. Der Kernpunkt beider Modelle ist, dass es zur Vermischung von Dirichlet- und Neumann-Randbedigungen kommt. Sinn dieser Modelle ist es, das Standardmodell besser als bisher zu beschreiben. Ferner führen wir in diesem Kapitel Materiefelder und deren Vertexoperatoren ein.

Das sechste Kaptitel ist der niederenergetischen Feldtheorie gewidmet. Wir wollen mit unserem Modell eine vierdimensionale $N = 1$ Supergravitation beschreiben. Die Wirkung einer solchen Theorie setzt sich aus ganz bestimmten Funktionen zusammen, die in diesem Kapitel vorgestellt werden.
Bevor wir nun die benötigten Amplituden berechnen, führen wir im siebten Kapitel ein, wie man Streuamplituden in der Stingtheorie berechnet. Wir gehen dabei jedoch nicht zu sehr ins Detail. Wichtig ist es uns zu zeigen, wie man vom eher abstrakten Polyakov-Pfadintegral auf die Ausdrücke kommt mit denen wir später rechnen.
Im achten Kaptitel berechnen wir die Amplitude zwischen Materiefeldern und Modulifeldern. Aus den Ergebnissen ziehen wir Rückschlüsse auf die Metriken der Materiefelder im Kählerpotential. Wir berechnen diese Amplitude im Modell mit Hintergrundfluss sowie im Winkelmodell.
Im neunten Kapitel berechnen wir die Amplituden von vier Materiefeldern im Fluss- und Winkelbild. Aus den Ergebnissen schliessen wir über Yukawa-Kopplung auf das Superpotential.
Im zehnten Kapitel bestimmen wir das Kählerpotential der Modulifelder, indem wir dies an den Branes streuen. In diesem Fall koinzidieren Fluss- und Winkelbild.
In elften Kapitel werden wir unsere Ergebnisse soweit möglich auswerten. Hierbei ist anzumerken dass [1] zur Zeit noch nicht fertiggestellt ist, weshalb einige Ergebnisse hier noch nicht präsentiert werden können.
Wir haben ferner zwei Appendizes beigefügt, wobei sich Appendix A mit Twistfeldern beschäftigt. Diese tauchen im Winkelbild auf und ziehen eine äusserst nich-triviale CFT mit sich. In Appendix B stellen wir schliesslich einige der in den Amplituden verwendeten Funktionen dar.

Selbstständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbstständig und ohne fremde Hilfe verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet zu haben.

Berlin, den

(Niels Bernhardt)

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Contents

1. Introduction and Overview

Theoretical physics has undergone many extensions and revisions in its comparatively short history. The original notions introduced by Newton were refined and systemized by Lagrange, Hamilton and Jacobi. Whereas the advent of quantum mechanics and general relativity led to a completely new understanding of physics. Unfortunately these two theories seemed to be incompatible although they formidably describe physics at low scales and large scales respectively.

In the 1960ies a rather exotic theory emerged to originally describe strong interactions. The idea was to use strings rather than point particles which led to the famous Veneziano amplitude. However, the original goal was not achieved. Strong interactions are today described by QCD. Yet string theory managed to survive as it showed a salient feature. In its spectrum there is a spin-2 particle which can be thought of as a quantum particle of gravity. This led to the conjecture that string theory might comprise gravitation.

Unfortunately string theory contained some insufficiencies at that time. So it was not able to reproduce fermions. This problem could be fixed in the 1980ies by making string theory supersymmetric which is often referred to as first string revolution. Henceforth string theory was known as superstring theory.

After the first string revolution there existed five consistent string theories which seemed to be disconnected from each other. Furthermore string theory showed severe difficulties reproducing the standard model. It was in the 1990ies when string dualities were discovered which showed that the five string theories were certain limits of one underlying theory. Moreover these considerations led to extended objects, so-called D-branes. These branes offered a new way to search for the standard model. This era is called second string revolution.

Nowadays these brane world are refined to gain models that reproduce the standard model more realistically. There are two especially promising models, branes with background flux and intersecting brane worlds. These models are the main concern of this work. We will compute certain amplitudes to gain information about the ingredients to a low energy effective action in the regime of intersecting branes or branes with fluxes respectively. Low energy basically refers to massless states in string theory. Note that massive states are extremely heavy in comparison to masses known from the standard model. Thus the standard model particles have to originate from massless string states.

This work is organized as follows. In the second chapter we will give a very short review on the various string theories that have been discovered so far. Furthermore we will address the web of dualities which connect the five string theories and leads

to the conjecture that all are part of one unified theory. Also we will introduce the connection between string theory and the standard model of particle physics.
In the third chapter we will introduce D-branes. We will show how and where they arise and present some basic properties like the fact that D-branes are BPS-states.
The fourth chapter will be dedicated to compactification. Compactification is needed to produce realistic theories in four space time dimensions. The surplus of six dimensions will however not go unseen. They will rather produce so called moduli fields which are in the four dimensional picture a non-vanishing background.
In the fifth chapter we will refine the notion of D-branes. To obtain at least semi-realistic theories, that is theories reproducing the features of the standard model, we have to turn on a flux on the brane or respectively rotate the branes by non-trivial angles.
In the sixth chapter we are approaching the actual purpose of this thesis, i.e. the determination of the moduli dependence of the low energy action. This chapter will present the ingredients to be determined.
The seventh chapter introduces string amplitudes. This is a very basic chapter presenting the major idea behind string perturbation theory, i.e. we work out the link between the abstract Polyakov path integral and how string amplitudes are actually computed.
Chapters eight, nine and ten present the actual calculation of the ingredients to the low energy action as shown in chapter six. Therefore we perform various string amplitude calculations.
In chapter eleven we will finally draw our conclusions and collect the results from the calculations. Furthermore we will address the limitations our findings face and present an outlook of what still has to be done.
Additionally we have attached appendices in which we present some ingredients of the string calculations which are of no major importance but might find interest in some readers.

2. String Theories and Dualities

So far there exist five consistent string theories which will be introduced in a moment. These are not independent from each other, moreover one knows that they reproduce the same physics in certain limits. The key to understand this dependence are string dualities.

2.1. Heterotic String Theories

Considering closed strings the holomorphic and anti-holomorphic modes are completely independent. The heterotic string makes use of this fact in a rather drastic way. One mode is made $N = 1$ supersymmetric while the other one is kept non-supersymmetric. That is we have a superstring in one sector and a bosonic string in the other. Of course the bosonic string lives in 26 space time dimensions and the superstring in 10. Say we made the right-moving sector supersymmetric and the left-moving sector bosonic, then one way to make the heterotic string consistent is to compactify 16 bosonic fields in the left-moving sector on a T^{16} torus [2]. Another way would be to extend the right-moving sector by 32 Lorenz scalars[1] [3]. However, both alternatives come in two shapes giving rise to two possible gauge groups $SO(32)$ and $E_8 \times E_8$, where in the first case the gauge group is determined by modular invariance which translates into the condition for the toroidal lattice to be self dual. The only self dual lattices in 16 dimensions are the root lattices of $SO(32)$ and $E_8 \times E_8$. In the latter case the gauge groups stem from imposing different boundary conditions on the Lorenz scalars.

2.2. Type I String Theory

The type I theory is the only one initially containing open strings. These can come oriented with gauge group $U(N)$ or unoriented with gauge group $O(N)$ or $Sp(N)$, where the latter is obtained by exchanging two endpoints [4]. However, to avoid anomalies one has to couple the open strings with appropriate closed strings[2]. The only anomaly free possibility happens to be the unoriented open string with gauge group $SO(32)$ where the reduction of $O(32)$ to $SO(32)$ can be explained by the orientifold reflection to be introduced in the next chapter. The choice of $N = 32$ is necessary for tadpole cancellation. The open strings in the type I theory possess $N = 1$ world sheet supersymmetry.

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[1] This means they are space time scalars and world sheet spinors. Note that giving them a vector index in space time would lead us rather to a type II superstring.

[2] Indeed one uses type IIB closed string orientifolds, see [5].

2.3. Type II String Theories

Type II string theories consist of closed superstrings. Since in a closed string left- and right-moving modes are independent we face an $N = 1$ supersymmetry from each mode which sum up to a total of $N = 2$ supersymmetries for the closed superstring. This agrees with the fact that one finds two gravitini in the type II spectrum. Thus type II strings are the only strings with extended supersymmetry.

One further distinguishes between two types, namely type IIA and IIB, which differ in the chiralities of the spectrum. That is, type IIA is non-chiral (i.e. the two supercharges have opposite chiralities) while type IIB is chiral (i.e. the two supercharges have same chirality).

A striking property of type II strings is that their (R,R)-spectrum contains massless tensors which are of even order in IIA and odd order in IIB. Consequently we find a 1-form and a 3-form potential in type IIA and accordingly 0-, 2- and 4-form potentials in type IIB. This translates into the existence of p-brane solitons,which are natural objects charged under $(p+1)$-form potentials, where p is even in IIA and odd in IIB. We will come back to type II theories in the next chapter.

2.4. String Dualities

As already mentioned above there is a whole web of dualities connecting the five string theories. There are several such dualities from which the so-called T-duality[3] is longest known.

T-duality expresses the equivalence of two string theories compactified on different background spaces. Equivalence indeed means that these two theories can be considered as one single theory, i.e. both contain the same physics. T-duality is perturbative in a sense that regions of weak coupling on one theory are mapped into regions of weak coupling of the other theory [6]. A prominent example for T-duality is:

$$\begin{array}{ccc} \text{type } II\text{A on } S^1 & \leftrightarrow & \text{type } II\text{B on } S^1 \\ \dfrac{R}{\sqrt{\alpha'}} & \leftrightarrow & \dfrac{\sqrt{\alpha'}}{R}, \end{array}$$

where essentially the radius R of compactification was inverted. This generalizes for theories to be T-dual when one theory compactified on a space with large volume equals the other theory compactified on a space with small volume and vice versa [7]. We will make extensive use of this duality in this work.

[3] = target space duality

Additionally there is a non-perturbative duality, called S-duality[4], where the weak coupling region of one theory is mapped to the strong coupling region of the same theory. Examples of S-dual string theories are the heterotic string on T^6 and type IIB [6]. This generalizes to two different theories, i.e. a theory at weak coupling is equivalent to another theory at strong coupling and vice versa [7].
Yet another duality, the so-called U-duality, comprises T- and S-duality and embeds them into a larger symmetry group. Basically this means a theory at weak (or strong) coupling is equivalent to another theory compactified on a space of small (or large) volume [7].

2.5. M-Theory

The fact that different string theories are connected through various dualities leads to the conjecture that all superstring theories are depending on each other.

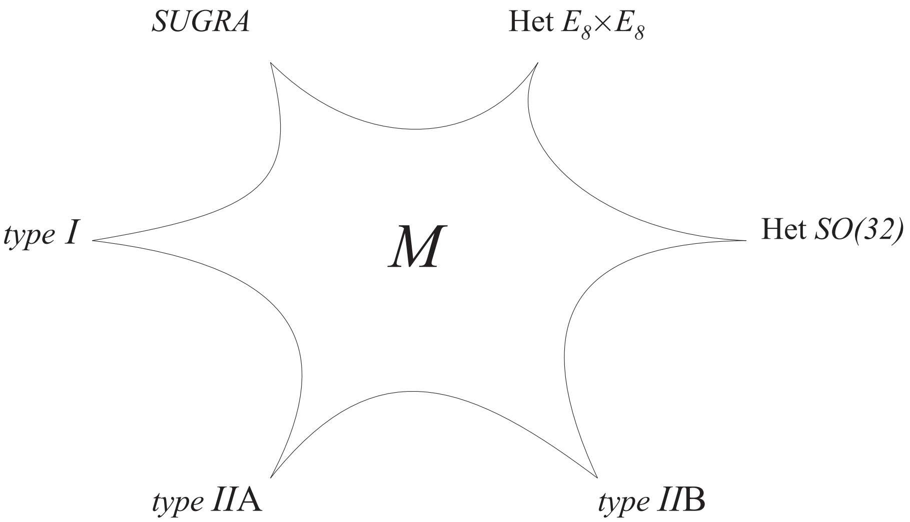

Fig. 1 M-Theory: The known string theories are conjectured to be limits of one underlying theory, the so-called M-theory. The low energy limit is that of 11-dimensional SUGRA.

The theory that comprises all five superstring theories is called M-theory. This theory should be an 11-dimensional theory and its low energy effective action is supposed to coincide with that of 11-dimensional supergravity, see fig. 1.

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[4] = self duality

2.6. F-Theory

Among the five string theories type *II*B superstring theory with gauge group $SO(32)$ takes an especially important role as it contains type I open string theory and the heterotic $SO(32)$ string. Moreover it possesses an $Sl(2, Z)$ symmetry in ten dimensions, which coincides with the geometrical symmetries of a torus. This leads to the idea that type *II*B string theory might be regarded as another theory compactified on a torus. This would lead to a 12 dimensional theory, the so-called F-theory [5]. Although F-theory faces many obstacles in its realization it is considered to be on an equal footing with M-theory [7].

2.7. Looking for the Standard Model

String theory itself is of appealing elegance but without reproducing the standard model of particle physics or its at least its supersymmetric extension (the so-called minimal supersymmetric standard model) in its low energy limit it would be nothing more than a toy theory. The most important features of the standard model to be reproduced are:

- gauge group $U(1) \times SU(2) \times SU(3)$,
- three quark-lepton generations,
- $N = 0$ or $N = 1$ supersymmetry,
- chiral spectrum.

Previous brane model constructions have achieved this goal up to the fact that they always contain unwanted extra particles. More recently intersecting brane models or models with non-vanishing fluxes respectively have proven promising to reproduce the standard model in a more satisfactory way.
One should however not hope to find "the" string theory in a sense that its low energy exactly describes "the" standard model, at least not in the near future. Reason for this is the enormous arbitrariness in the choice of the background configuration. We are currently lacking the understanding of whether any compactification is in any sense preferred over the rest. For these and other reasons we are very unlikely to find the "correct" string theory by trial and error [8]. This is why modern theories reproducing some standard model features are called semi-realistic rather than realistic.

3. Introduction to Dirichlet-Branes

The present chapter is dedicated to basic facts about D-branes which are among the protagonists of this work. This introduction comprises the origins of D-branes as they arise as solitons or in T-duality. Furthermore we will give some basics about supersymmetry breaking and preserving in context of branes. Finally we will discuss the case of a stack of coincident branes.

3.1. Massless States from Type II Superstring Theories

In this section we are interested in the massless states of closed superstrings, from which we expect hints on solitons.
Let us start with the NS-sector. The ground state itself turns out to be a tachyon, while the first excited state is a massless space time vector. One can get rid of the tachyon with the so-called GSO[5]-projection where one sums over the boundary conditions of fermions on the worldsheet and only keeps those with a fixed fermion number $(-1)^F$. This results in the tachyon to be projected out. The lightest state is now the massless space time vector string v.
In the R-sector the ground state itself is massless spinor s. The GSO-projection amounts to projecting to spinors of a given chirality, that is we end up with two spinors s and s' of opposite chirality.
To obtain the actual type II massless spectrum we have to combine left- and right-moving states. From the mixed sectors (NS-R) and (R-NS) we get the fermions including gravitini. Here one has to differ between type IIA and type IIB. That is in the fist case the two gravitini are of opposite chirality while in the latter case they are of the same. The (NS-NS)-sector of both theories is the same:

$$NS \otimes NS: \quad v \otimes v \ \rightarrow \ (g_{\mu\nu}, B_{\mu\nu}, \varphi), \tag{3.1}$$

where g is the graviton, B is the anti-symmetric tensor and φ the dilaton. Accordingly we find for the (R-R)-sectors of IIA and IIB:

$$\begin{aligned} &II\mathrm{A} \quad R \otimes R: \quad s \otimes s' \ \rightarrow \ (A_\mu, C_{\mu\nu\varrho}), \\ &II\mathrm{B} \quad R \otimes R: \quad s \otimes s \ \ \rightarrow \ (\chi, B'_{\mu\nu}, D_{\mu\nu\varrho\lambda}), \end{aligned} \tag{3.2}$$

where all tensors in (3.2) are totally anti-symmetric which allows for them to be form potentials to some field strength. Note that the field strength of $D_{\mu\nu\varrho\lambda}$ is self-dual[6].

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[5] named after Gliozzi, Scherk and Olive.

[6] w.r.t. Hodge duality.

3.2. *String Solitons*

In general solitons may arise in the classical limit of a field theory. A soliton is then a smooth, localized and topologically non-trivial field configuration. In contrast to solitons we have the usual quanta whose ground states represent in the classical limes small fluctuation around the vacuum, see e.g. [9].
For a point soliton this works as follows [5]. Given a one-form potential A the coupling of a particle amounts to weighing the partition function with the background A:

$$Z \to Z \cdot \exp\Big(\mathrm{i} \int\limits_{\gamma} A \Big), \tag{3.3}$$

where γ is the world line of the particle. This is called minimal coupling. The according action reads:

$$S = \int F \wedge *F, \tag{3.4}$$

where $F = \mathrm{d}A$ and $*F$ is the (Hodge-) dual form of F. This concept can now be generalized to extended solitons. Let p denote the dimensions of the soliton which is then called p-brane, then its world volume is a $(p+1)$-dimensional subspace of space time which we shall call γ_{p+1}. In this spirit (3.3) generalizes to:

$$Z \to Z \cdot \exp\Big(\mathrm{i} \int\limits_{\gamma_{p+1}} A_{p+1} \Big). \tag{3.5}$$

The according action is given by (3.4), where F now denotes a totally antisymmetric $(p+2)$-tensor. Given a dual charge the above considerations can be repeated under interchange of $F \leftrightarrow *F$ and $\mathrm{d}\widetilde{A} = *F = *\mathrm{d}A$ [7]. Since $*F$ is a $(d-p-2)$-form the dual soliton will be a $(d-p-4)$-brane. Suppose we have an (electrically charged) p-brane and a (magnetically charged) dual q-brane, then we get the very important relation:

$$p + q = d - 4. \tag{3.6}$$

From (3.6) one can tell that only in four space time dimensions there are both electric and magnetic point charges. Furthermore, if a p-brane is wrapped around an r-dimensional compact object then the $(p+1)$-dimensional potential decomposes into a $\big((p+1-r)+r\big)$-form, where the $(p+1-r)$-part is in the non-compact directions while the r-part lives in the compact directions. The resulting state is charged under the leftover part, that is we end up with a $(p-r)$-brane.

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[7] Note that this is not trivial on the quantum level. Indeed this concept has been pioneered by Nepomechie and Teitelboim via integrating over spheres around the charges.

Let us now try to find solitons in the type *II* string spectra. Clearly, from the (NS-NS)-sector we obtain an anti-symmetric 2-form $B_{\mu\nu}$, see (3.1), giving rise to an electric 1-brane which is identified with the fundamental string (F-string). The dual brane is according to (3.6) a magnetic 5-brane.
From (3.2) we see that in the (R-R)-sector we have form potentials as well. Take type *II*A where we have a 1- and a 3-form potential plus the according dual potentials which are after (3.6) a 7- and a 5-form[8]. Note that we can also include a 9-form potential since this would have trivial dynamics in 10 dimensions. Essentially we have a form potential in each odd dimension. Accordingly we have the 0-, 2- and 4-forms in type *II*B and their duals. These cover the even dimensions. Thus it is natural to define a generalized potential $\mathcal{A}$ which is simply the sum over all odd/even form potentials. This leads to a field strength $\mathcal{F} = \mathrm{d}\mathcal{A}$ which has the property to be a self-dual form, i.e. $\mathcal{F} = *\mathcal{F}$.
From the above considerations one expects branes of even dimension in type *II*A and branes of odd dimension in type *II*B accordingly. These branes are charged under the according (R-R)-fields. Since the elementary string excitations are all neutral under the (R-R)-charges we are led to look for solitonic states. Such solitons are called D-branes and are introduced in the next section.
In all three $N = 1$ theories there exist branes similar to the (NS-NS) branes of type *II* string theory. The originate from a 2-form which is in heterotic string theories the Kalb-Ramond field from the NS sector while in the case of type *I* string theory the 2-form comes from the R-sector, see section five of this chapter. However, in both cases one finds a 1-brane and its dual 5-brane. The 1-brane of heterotic string theories is again identified with the F-string while 1-branes in the R- or (R-R)-sectors are called D-strings.

3.3. Dirichlet-Branes

The question now is how to implement these solitons into string theory. Indeed all we have to take care of is to ensure that conformal invariance on the world sheet remains unbroken. So to say, a string propagating with background must have the same conformal symmetries as in a free theory. The point is now how the $(p+1)$-dimensional world volume of a p-brane would affect the string propagation. The simplest way is to allow the closed string to open up and end on the $(p+1)$-world volume. This means that we introduce an open string sector where the open string ends are fixed on a $(p+1)$-dimensional subspace of space time. Basically this amounts to imposing Dirichlet boundary conditions on $10 - p - 1$ coordinates of the string

[8] Note that these are not independent degrees of freedom.

endpoints [5]. Note that imposing Dirichlet boundary conditions on open string ends is all but a new concept. Dirichlet conditions were discarded in the 1980ies since they do not conserve momentum in the directions they are imposed on. So open string were restricted to have Neumann boundary conditions.
Now, since the (R-R)-charged p-branes are realized by p-dimensional Dirichlet-branes one usually denotes these solitons by Dp-branes or simply D-branes. As mentioned in the preceding section only even branes yield consistent boundary conditions for fermions in type IIA, whereas odd branes do for type IIB. On the other side we have the 1-branes and 5-branes in the (NS-NS)-sector of type II and in the $N = 1$ theories.

3.4. D-Branes as BPS-States

In general BPS[9]-states or "short states" are states where some supersymmetry operators vanish and thus cannot create new states. Note that BPS-states can only occur in massive multiplets of extended supersymmetry. The vanishing of some supercharges depends on the relation between the mass of a multiplet and some central charges from the supersymmetry algebra[10]. On the other side the central charges depend on electric and magnetic charges and on moduli which relates them to branes. The condition for BPS-states is the so-called Bogomol'ny bound:

$$M \geq |Z_i|, \tag{3.7}$$

where M is the mass and Z_i are the eigenvalues of the central charge matrix Z^{IJ}. In the case of equality the BPS-multiplet contains just as many states as the massless multiplet. Such multiplets are called $\frac{1}{2}$-BPS or "ultrashort". BPS-states have two important properties.
First they are stable[11] because their mass depends solely on conserved charges. Thus charge and energy conservation prohibits their decay.
Secondly, the mass formula is exact. That means quantum corrections cannot spoil the relation between mass and charges.
These considerations for point-like BPS-states can now be generalized to BPS p-branes. In the presence of p-branes the supersymmetry algebra contains central charges which are not Lorenz scalars but anti-symmetric tensors of the type $Z_{\mu_1 \ldots \mu_p}$. The according eigenvalues show proportionality to the p-brane charges. Such p-branes

[9] named after Bogomol'ny, Prasad and Sommerfield

[10] Let Q^I and Q^J denote super charges then we have $\{Q^I, Q^J\} \sim Z^{IJ}$ with Z^{IJ} being the anti-symmetric matrix of central charges, where further spinor indices have been neglected.

[11] Except for some exotic cases. However, stability is generically ensured.

can thus be regarded as extended BPS solitons leaving some of the supersymmetries unbroken [4].
This can as well be seen from more fundamental considerations. As mentioned above closed strings are in the regime of D-branes realized as open strings with their endpoints attached to the brane. In an open string left- and right-moving modes get mixed because they are reflected from the endpoints. This applies to the super charges in a way that only one linear combination is a good symmetry for the full state. That is, from the $N = 2$ supersymmetry in the bulk we get an $N = 1$ supersymmetry on the brane. Thus exactly one half of the supersymmetries are broken which makes the brane a $\frac{1}{2}$-BPS state as introduced above [10].

3.5. Open Strings and D-Branes

Let us have a look on type I superstrings. Its spectrum can be obtained from the type IIB superstring. In the case of open strings left- and right-moving components are not independent as they experience refections from the endpoints of the string which results in stationary waves. The concept can be realized by introducing a reflection $P = \frac{1}{2}(1 + \Omega)$, where Ω exchanges left- and right-movers. As this amounts to inverting the orientation of the world sheet, Ω is called orientifold operator [5].
From the (NS-NS)-sector we thus get after applying P the following massless states:

$$NS \otimes NS: \quad (v \otimes v)_{\text{symm}} \;\rightarrow\; (g_{\mu\nu}, \varphi), \tag{3.8}$$

where the anti-symmetric Kalb-Ramond field has been projected out. This leaves us with the graviton and the dilaton.
On the other side we have the (R-R)-sector. Here we receive an extra minus sign because we are dealing with fermionic degrees of freedom in each sector. This translates into keeping the anti-symmetric part:

$$R \otimes R: \quad (s \otimes s)_{\text{anti-symm}} \;\rightarrow\; \widetilde{B}_{\mu\nu}. \tag{3.9}$$

However, this theory still suffers from inconsistencies caused by so-called tadpoles[12]. To get rid of the tadpoles we have to introduce an open string sector. Furthermore it is necessary to assign 32 Chan-Paton factors to each endpoint of the open string. In the NS-sector these open string give vector gauge fields A^μ while in the R-sector they yield the according gaugino. Note that the projector P applied to A^μ results in only keeping its anti-symmetric component [5].

[12] These stem from a 10-form potential in the (R-R)-sector. This has no propagating degree of freedom but causes unwanted tadpoles.

Let us now compactify one dimension on a circle of radius R (see next chapter for more details on compactification). If we now consider the limit of $R \to 0$ we are left with a theory of one dimension less than expected. This happens due to the fact that an open string has no conserved winding number around the compact dimension which results in the states with non-zero momentum to become infinitely massive. Furthermore we have to heed the fact that any open string theory must comprise closed strings as well. At a first look this seems rather awkward as the open string lives in one dimension less than the corresponding closed string but at a second look we see that the only part of an open string which is at all distinguishable from the closed string are its endpoints. The fact that the open string lives in one dimension less now translates into the existence of a hyperplane on which its endpoints are attached. Clearly this amounts to impose Dirichlet boundary conditions $d-1$ space time directions when d is the dimension of the space time under consideration [10]. This leaves us with the notion of D-branes again.

An interesting question is what happens to the gauge fields we introduced for tadpole cancellation. Consider the internal component given by:

$$A^i(X^\mu) = -\frac{\theta}{\pi R} = -\mathrm{i}\Lambda^{-1}\partial_{X^i}\Lambda,$$

where $\Lambda(X^i) = \mathrm{e}^{-\mathrm{i}\theta X^i/\pi R}$ and $\theta = \mathrm{diag}(\theta_1, -\theta_1, ..., \theta_{N/2}, -\theta_{N/2})$ denote the generators of the gauge group. From this we get the following gauge invariant quantity:

$$W_q = \exp\Big(\mathrm{i}\int A^i\Big) = \mathrm{e}^{-\mathrm{i}\theta},$$

which is called Wilson line [10], [11]. These generically break down the Chan-Paton gauge group $SO(N)$ to $U^{N/2}(1)$. Although A can be gauged away all fields charged under A will pick up a phase $\mathrm{e}^{\mathrm{i}\theta}$ when moving around the circle, i.e. $X^i \to X^i + \pi R$. Reason for this is the fact that the periodicity of X^i results in a non-trivial holonomy of A^i. This translates into the following shift in the momentum:

$$P^i = \frac{n}{R} + \frac{\theta}{\pi R},$$

which are now fractional.

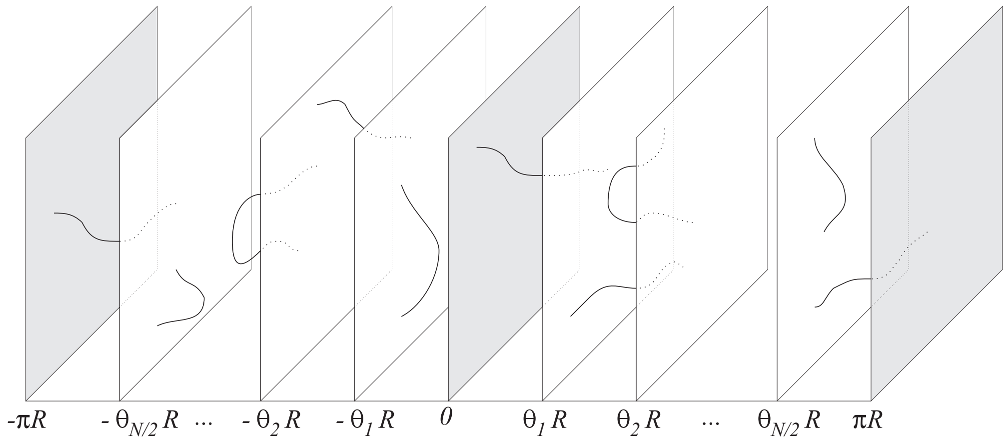

Fig. 2 The dual picture of Wilson lines are $\frac{1}{2}N$ D-branes and another $\frac{1}{2}N$ at their image points under the orientifold identification. The grey planes are the orientifold planes fixed at 0 and πR. Strings can stretch between the various branes as shown in the figure.

In the T-dual picture this becomes fractional winding numbers. This means that the open string endpoints are no longer attached to a single hyperplane but on different ones, see fig. 2. Indeed, say a string with endpoints in the a-th and b-th plane picks up a phase $\mathrm{e}^{\mathrm{i}(\theta_b-\theta_a)}$ and the momentum is:

$$P^i = \frac{n}{R} + \frac{\theta_b - \theta_a}{\pi R}.$$

In other words the T-dual picture of Wilson lines with $SO(N)$ Chan-Paton factors at the endpoints are $\frac{1}{2}N$ D-branes and their orientifold mirrors.
We have seen that we need 32 Chan-Paton factors from which we concretize the gauge group to be $SO(32)$. According to our considerations for the T-dual picture these become 16 $D8$-branes and their orientifold mirrors along with two orientifold $O8$-planes located at 0 and πR, see fig. 2. Note however that breaking down the $SO(32)$ to $U^{16}(1)$ will only work under certain conditions, i.e. the 16 $U(1)$ groups will restore the full $SO(32)$ if all 16 branes coincide at a single orientifold plane [10].

3.6. Multiple Branes

In the previous section we were faced with stacks of multiple parallel or coincident branes. In the present section we will refine this notion.
For a single type *II* p-brane the open string spectrum fills out a vector super multiplet with respect to the unbroken supersymmetries consisting of a $U(1)$ gauge boson and additional scalars/fermions which can be regarded as Goldstones/Goldstinos of the translational symmetries of the vacuum broken by the presence of the D-brane. Since

D-branes are BPS-states their (R-R)-charge is related to their tension which implies that their is no net force between parallel branes. Thus they can be superposed. Let us now consider N coincident branes. There are $N \times N$ possible ways in which brane open strings can start and end. Thus we have N^2 sectors which renders interactions between open string non-abelian. Indeed the open massless spectrum now fills out a $U(N)$ vector multiplet with respect to the unbroken supersymmetries [8].

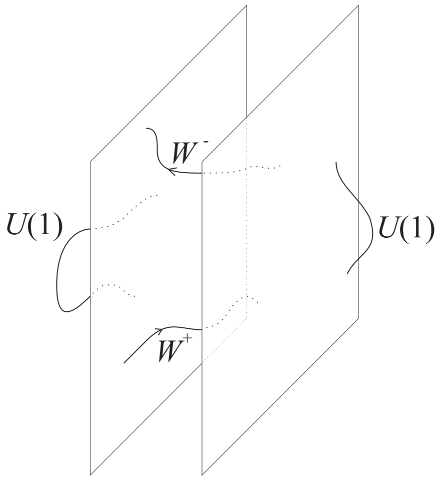

Fig. 3 Two coincident D-branes: Shown are the abelian $U(1)$ gauge bosons which start and end on the same brane and the non-abelian $U(2)$ gauge bosons (named W^+ and W^-) which and start and end on different branes as shown in [8].

In fig. 3 the setup is depicted for two coincident branes. More general setups with non-parallel branes will be discussed in chapter five.

4. Compactification and Moduli Fields

String theory is a theory in ten space-time dimensions. This number arises from considerations in conformal field theory, that is the cancellation of certain conformal anomalies. Clearly, ten space-time dimensions are in a severe contrast to the reality we perceive. We experience the universe to be a four dimensional space time. So we consider the six extra dimensions to be tiny enough not to be detected. This concept can be accomplished by rolling up dimensions on some manifold which is called compactification. However, compactifying on arbitrary manifolds breaks supersymmetry and thus spoils perturbation theory. Fortunately there are certain classes of manifolds which preserve at least some supersymmetries. The number of unbroken supercharges is related to the existence of covariantly constant spinors[13], i.e. spinors parallel with respect to the connection: $\nabla\Psi = 0$. Thus we are led to look for 6-dimensional manifolds which allow for such type of spinors.

- First we have manifolds with trivial holonomy[14]. This case is only fulfilled by a product of flat tori. Toroidal compactifications have the maximum number of covariantly constant spinors. We will later work with the $T^6 = T^2 \times T^2 \times T^2$ compactification.
- Secondly, there is a class of manifolds with $SU(3)$ holonomy. These manifolds are called Calabi-Yau 3-folds (CY3). It is known that any CY admits a Ricci-flat metric which explains the existence of parallel spinors.
- Next we have a manifold with $SU(2) = Sp(1)$ holonomy, the so-called $K3$. Note that $K3$ is a four-dimensional manifold, i.e. it yields a 6-dimensional theory. However, $K3$ compactifications arise in duality considerations or in the combination $K3 \times T^2$. The $K3$ itself can be obtained as an orbifold of T^4, namely $K3 \sim T^4/Z_2$, as a complex surface in the projective space P^3 or an elliptic manifold [5].

Other examples of supersymmetry preserving manifolds are G_2, $Spin(7)$ or manifolds with $Sp(2)$ holonomy. However, those manifolds have more than 6 dimensions and would thus yield a theory below four space time dimensions. They may however be interesting for compactifications of M- or F-theory.
One can now wonder if a compactification space can be deformed in a way leaving its topology invariant, i.e. leaving the property of preserving some supersymmetries

[13] Note that space time spinors are also spinors of internal space. So if we wish to define a constant supersymmetry transformation, we need an additional constant internal spinor.

[14] The holonomy group of a manifold M at a point p is the set of transformations $T_pM \to T_pM$ where elements of T_pM are parallel transported along closed curves in M, c.f. [12].

invariant. The answer is yes. The space of these deformations is called moduli space. It can be shown that the moduli space is a Cartesian product of two moduli:

- The *complex structure modulus* describing the shape of the manifold,
- The *Kähler modulus* describing the size of the manifold,

where the above given properties are only basic ideas. The actual role of moduli can be much more intricate.

4.1. Toroidal Compactifications

The simplest example of a CY is a two torus T^2. Since the torus can be represented by the complex plane C modded out by a lattice Λ, that is $T^2 \sim C/\Lambda$, we are dealing with a flat compactification space here. The lattice:

$$\Lambda : z \sim z + n\lambda_1 + m\lambda_2, \quad n, m \in Z, \quad \lambda_1, \lambda_2 \in C$$

can by conformal invariance be expressed through a single parameter $\tau = \frac{\lambda_1}{\lambda_2}$ which is called Teichmüller parameter and takes values in the so-called Teichmüller space which is in our case the upper half plane $\mathrm{H}_+ = \{z \in \mathrm{C} : \mathrm{im}\ z \geq 0\}$.

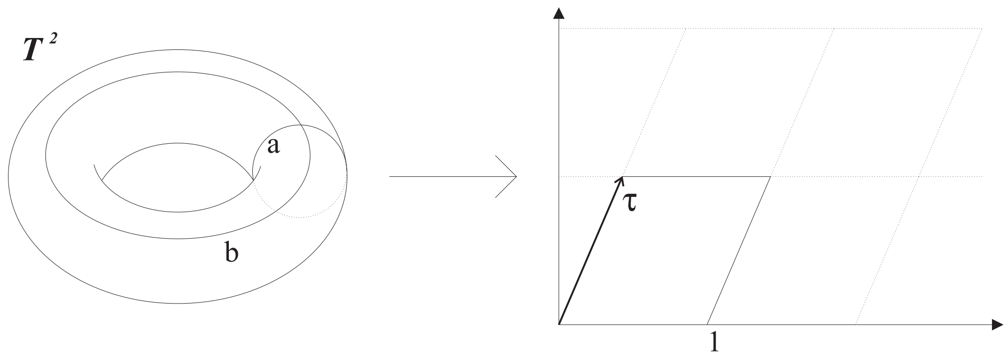

Fig. 4 The torus T^2 with its two cycles (left) is equivalent to a lattice on the complex plane (right). By conformal invariance the lattice can be scaled and rotated such that one parameter τ suffices to describe the torus.

Thus we have for the toroidal lattice:

$$\Lambda : z \sim z + n + m \cdot \tau, \quad n, m \in Z, \quad \tau \in C,$$

where opposite sides are identified, see fig. 4. The imaginary part of τ gives the volume of the torus, while its real part corresponds to a non-vanishing B flux.

Let us now examine the toroidal compactification. To do this we remember that topologically a torus T^2 is equivalent to a product of two circles $T^2 \sim S^1 \times S^1$. This generalizes straightforwardly to the three-torus comprising six circles. Let us thus consider the compactification on a circle of radius R. Note that the string can wrap around the circle which might result in $X(0,\tau) \neq X(2\pi,\tau)$. So it is best not to insist on the condition $P_L = P_R$. In particular we have on the bosonic side:

$$(P_L, P_R) = \Big(\frac{n}{2R} + mR, \frac{n}{2R} - mR\Big),$$

where n labels the center of mass momentum along the circle while m denotes the winding number around the circle. Note that the (P_L, P_R) spectrum is invariant under $R \to \frac{1}{2R}$ with interchange of $n \leftrightarrow m$. This is nothing but the already mentioned T-duality.

Let us now generalize this concept to tori T^d of arbitrary dimension d. In this case (P_L, P_R) belong to a $2d$-dimensional, self-dual[15], integral[16] and even[17] lattice. The concept of T-duality in the S^1 case generalizes to transformations under [5]:

$$\frac{SO(d,d)}{SO(d) \times SO(d) \times O(d,d;Z)},$$

where $SO(d,d)/O(d,d;Z)$ denotes Lorentz boosts on the lattice vectors where boosts of integer coefficient have been divided out as they can be compensated by lattice translations, $SO(d) \times SO(d)$ denotes rotations of the lattice. Here $O(d,d;Z)$ is the generalized T-duality group.

Accordingly we can treat the fermionic side where we assume $N = 2$ supersymmetry, i.e. type *II* theories. Again we start with compactifying on a circle S^1. The T-duality operation $R \to \frac{1}{R}$ acts on the fermions along the circle like:

$$\psi_L \to -\psi_L, \qquad \psi_R \to \psi_R,$$

which amounts to changing the overall sign of the left-moving chirality. Thus under the GSO projection we will keep the opposite chirality to what we had. What we have just done is exchanging *II*A and *II*B under T-duality.

This generalizes to the group $SO(d,d;Z)$ which does not exchange *II*A and *II*B, while parts of $O(d,d;Z)$ which are not in $SO(d,d;Z)$ do exchange *II*A and *II*B and are thus

[15] Self-duality means that any vector having integer products with all lattice vectors is a lattice vector itself.

[16] A lattice is integral if the product between any lattice vectors is integral.

[17] Evenness means that $P_L^2 - P_R^2$ is even for all lattice vectors.

not symmetries. On the other hand we have the coset $SO(d,d)/SO(d) \times SO(d)$ which corresponds to the moduli space of the T^d lattice. Let us summarize [5]:

$$G^{T^d}_{T-\text{duality}} = SO(d,d;Z), \qquad G^{T^d}_{\text{Moduli}} = \frac{SO(d,d)}{SO(d) \times SO(d)}.$$

Let us now assume $N = 1$ supersymmetry which leaves us with the heterotic string theories or the type I superstring. From the previous chapter we know that type I string theory has gauge fields for tadpole cancellation. Compactifying results in Wilson lines which have to be included into our picture here. From the $SO(32)$ Chan-Paton factors we receive 16 $U(1)$ Wilson lines along each compactified dimension. Thus the moduli and T-duality groups are [5]:

$$G^{T^d}_{T-\text{duality}} = SO(d+16,d;Z), \qquad G^{T^d}_{\text{Moduli}} = \frac{SO(d+16,d)}{SO(d+16) \times SO(d)}.$$

Note that in heterotic string theories these 16 Wilson lines come from compactifying the surplus of 16 dimensions on the bosonic side.

4.2. The Moduli Space of Toroidal Compactifications

In the preceding section we have learned that the moduli group of the considered T^6 compactification of a type II theory is:

$$G^{T^6}_{\text{Moduli}} = \frac{SO(6,6)}{SO(6) \times SO(6)}.$$

We will however assume the six torus to be factorizable, i.e. $T^6 = T^2 \times T^2 \times T^2$. This enables us to treat the moduli of each two torus separately. Indeed in this case the moduli of each torus are independent.

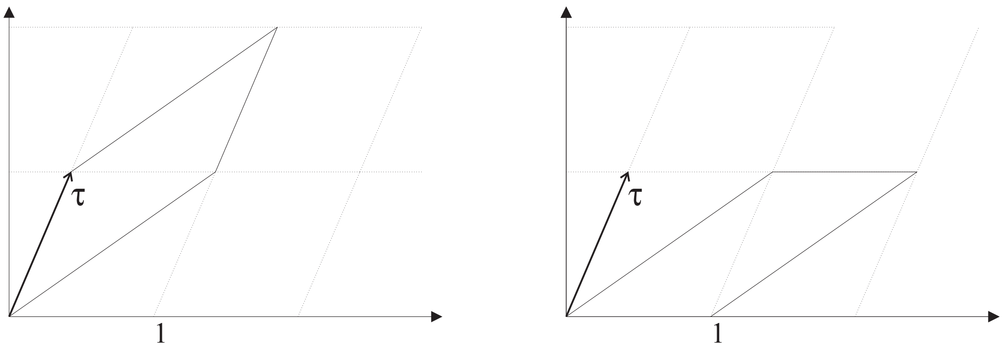

Fig. 5 Disconnected diffeomorphisms of the torus: Dehn twists along the cycles b (left) and a (right). On the lattice this translates into $\tau \to \frac{\tau}{\tau+1}$ (left) and $\tau \to \tau + 1$ (right).

The moduli of the torus are subject to standard considerations. A fine review can be found in [2] and [12]. For arbitrary surfaces the Riemann-Roch theorem states:

$$\dim_C \ker \nabla_z^{(n)} - \dim_C \ker \nabla^z_{(n+1)} = -(2n+1)(g-1), \tag{4.1}$$

where $\nabla_z^{(n)} : \mathcal{T}^{(n)} \to \mathcal{T}^{(n+1)}$ and $\nabla^z_{(n)} : \mathcal{T}^{(n)} \to \mathcal{T}^{(n-1)}$ are covariant derivatives. $\mathcal{T}^{(n)}$ denotes the space of holomorphic tensors of rank n. g denotes the genus of the surface. Basically this theorem relates the number of moduli to the number of conformal Killing transformations. For the torus the conformal Killing group (CKG) is $U(1) \times U(1)$ which has one complex generator. Thus the torus is described by one complex modulus τ which is the Teichmüller parameter we have already introduced in the previous section.

The complex modulus τ is basically given by the ratio of the two radii of the torus. We know however that there are transformations of τ which leave the torus invariant. These are so-called Dehn twists and are obtained from cutting a cycle, twisting it by 2π and gluing it back together. This corresponds to disconnected diffeomorphisms[18]. As shown in fig. 5 Dehn twists translate into lattice translations. These both translations can be combined:

$$\tau \to \frac{a \cdot \tau + b}{c \cdot \tau + d}, \qquad a, b, c, d \in Z. \tag{4.2}$$

With the additional condition of $ad - cb = 1$ which preserves the area this generates the special linear group $Sl(2,Z)/Z_2 = PSl(2,Z)$. This means that the complex structure modulus τ takes values in $\mathrm{H}_+/PSl(2,Z)$ which is called fundamental region.

Let us now consider type IIB strings compactified on factorizable $T^6 = T^2 \times T^2 \times T^2$ torus. The metric and antisymmetric tensor of the j-th torus is then given by:

$$g^j = \begin{pmatrix} g^j_{11} & g^j_{12} \\ g^j_{12} & g^j_{22} \end{pmatrix} = \begin{pmatrix} (R^j_1)^2 & R^j_1 R^j_2 \cos\alpha^j \\ R^j_1 R^j_2 \cos\alpha^j & (R^j_2)^2 \end{pmatrix}, \qquad b^j = \begin{pmatrix} 0 & b^j_{12} \\ -b^j_{12} & 0 \end{pmatrix}, \tag{4.3}$$

where R_1 and R_2 denote the radii of the two cycles while α denotes the angle between the Teichmüller parameter τ and the real axis. The index $j = 1, 2, 3$ labels the three tori. The Kähler moduli T^j describe the area of each torus and if tilted the antisymmetric tensor. That is, we can express it as [1], [13]:

$$T^j = b^j + \mathrm{i}\sqrt{\det g^j} = b^j + \mathrm{i}\, R^j_1 R^j_2 \sin\alpha^j, \tag{4.4}$$

18 These are transformations that cannot be smoothly connected to the identity, i.e. they are global diffeomorphisms.

where we have suppressed matrix indices of b^j_{12} since there is only one independent entry in b^j at all. Furthermore we have the complex structure moduli U^j describing the shape of each torus [1], [13]:

$$U^j = \frac{g^j_{12}}{g^j_{11}} + \mathrm{i}\,\frac{\sqrt{\det g^j}}{g^j_{11}} = \frac{R^j_2}{R^j_1}\,\mathrm{e}^{\mathrm{i}\alpha^j}, \tag{4.5}$$

which essentially consists as already mentioned of the ratio between the two radii. Putting (4.3), (4.4) and (4.5) together the metrics can then be written in terms of the moduli:

$$g^j = \frac{\mathrm{Im}(T^j)}{\mathrm{Im}(U^j)}\begin{pmatrix} 1 & \mathrm{Re}(U^j) \\ \mathrm{Re}(U^j) & |U^j|^2 \end{pmatrix}. \tag{4.6}$$

Accordingly the components of the anti-symmetric tensors are given by the real part of the Kähler moduli. Note that the fact that the moduli can be parameterized by means of the metric and Kalb-Ramond field of the tori is all but a surprising fact.

4.3. Moduli Fields

The coordinates of the moduli space of the compactification manifold appear as massless scalar fields in the low energy limit. Taking vacuum expectation values of such fields can be thought of as deforming the compactification space accordingly. To understand this we start with the uncompactified case where we have a ten dimensional metric. Now after singling out six dimension to be compactified on three two tori, the metric breaks down into several components from the lower dimensional perspective. Of course there is the metric of the usual four dimensional space time. Then there are components that measure angles between tangent vectors of space time and those of the compactification manifold. These contain information about the shape of the compactification space. Finally there are components that measure the length of tangent vectors of the compactification space. That is, these components measure the size of the tori. From the lower dimensional view they are just scalar fields. Since all tori are equally fine for compactification the scalar fields are massless. This is how a modulus of the geometry of the compactification space can be regarded as a massless scalar field from the lower dimensional perspective. As such it can have usual interactions with other fields and the compactification manifold can be deformed dynamically.

Put into simpler words moduli fields are compactified components of the spin-2 particles describing the geometry of space time, that is the graviton, the Kalb-Ramond field and the dilaton.

This enables us to determine the vertex operators of both moduli. Let M^j denote any modulus. We have seen that the toroidal moduli are related to the graviton and the

Kalb-Ramond field. This leads to the conjecture that moduli fields are closed strings. Indeed moduli vertex operators in the $(-1,-1)$ -picture[19] can be calculated using the following expression [1]:

$$V_{M^j}^{(-1,-1)}(\overline{z},z) = \frac{1}{2}\, \mathrm{e}^{-\varphi(z)} \mathrm{e}^{\widetilde{\varphi}(\overline{z})}\, \partial_{M_j}(g^j + b^j)_{kl}\, \psi^k(z)\, \widetilde{\psi}^l(\overline{z})\, \mathrm{e}^{\mathrm{i}k\cdot X(z,\overline{z})}. \tag{4.7}$$

Introducing complex internal fermionic fields:

$$\Psi^j = \sqrt{\frac{\mathrm{Im}(T^j)}{2\,\mathrm{Im}(U^j)}}\, \left(\psi^{2j-1} + \overline{U}^{\,j}\psi^{2j}\right), \qquad \overline{\Psi}^{\,j} = \sqrt{\frac{\mathrm{Im}(T^j)}{2\,\mathrm{Im}(U^j)}}\, \left(\psi^{2j-1} + U^j\psi^{2j}\right), \tag{4.8}$$

the Kähler moduli vertex can be written as:

$$V_{T^j}^{(-1,-1)}(z,\overline{z},k) = \frac{1}{T^j - \overline{T}^{\,j}}\, \mathrm{e}^{-\varphi(z)} \mathrm{e}^{-\widetilde{\varphi}(\overline{z})}\, \overline{\Psi}^{\,j}(z) \widetilde{\Psi}^{\,j}(\overline{z})\, \mathrm{e}^{\mathrm{i}k\cdot X(z,\overline{z})}. \tag{4.9}$$

Accordingly we find for the complex structure moduli:

$$V_{U^j}^{(-1,-1)}(z,\overline{z},k) = \frac{-1}{U^j - \overline{U}^{\,j}}\, \mathrm{e}^{-\varphi(z)} \mathrm{e}^{-\widetilde{\varphi}(\overline{z})}\, \Psi^j(z) \widetilde{\Psi}^{\,j}(\overline{z})\, \mathrm{e}^{\mathrm{i}k\cdot X(z,\overline{z})}. \tag{4.10}$$

This can as well be understood from looking at the uncompactified case, where the spin-2 particles are created by acting with $b^\mu_{-1/2} \otimes \widetilde{b}^\nu_{-1/2}$ on the ground state $|k\rangle$. This creates a product of space time vectors and world sheet spinors $\psi^\mu\, \widetilde{\psi}^\nu\ |k\rangle$. Adding a polarization tensor $\varepsilon_{\mu\nu}(k)$ yields the graviton, Kalb-Ramond field and dilaton for different choices for $\varepsilon_{\mu\nu}$. After compactifying six dimensions the creators $b^\mu_{-1/2}$ and $\widetilde{b}^\nu_{-1/2}$ decompose into external components describing the four dimensional graviton, Kalb-Ramond field and dilaton and internal components $b^j_{-1/2}$ and $\widetilde{b}^j_{-1/2}$ which create the moduli. The internal components of the j-th torus now give rise to $\Psi^j\, \widetilde{\Psi}^j\ |k\rangle$ as well as $\overline{\Psi}^{\,j}\, \widetilde{\Psi}^j\ |k\rangle$, $\Psi^j\, \overline{\widetilde{\Psi}}^{\,j}\ |k\rangle$ and $\overline{\Psi}^{\,j}\, \overline{\widetilde{\Psi}}^{\,j}\ |k\rangle$ since the internal spinors are complex. This is in agreement with (4.9) and (4.10).

To obtain the vertices in the $(0,0)$ -picture[20] we have to act with the picture changing operator $P(w,\overline{w})$ on the $(-1,-1)$ -picture. The picture changing operator is given by:

$$P(w,\overline{w}) = \mathrm{e}^{\varphi(w)}\, \mathrm{e}^{\widetilde{\varphi}(\overline{w})}\, T_F(w)\, \widetilde{T}_F(\overline{w}), \tag{4.11}$$

[19] w.r.t. the charge of the bosonized super charges $\mathrm{e}^{q\cdot\varphi}$.

[20] The charges of the super ghosts have to sum up to -2 in any amplitude to ensure superdiffeomorphism invariance. That is why we will need the (0,0)-picture.

where $T_F(w)$ and $\widetilde{T}_F(\overline{w})$ are components of the super current. Introducing internal bosonic fields:

$$Z^j = \sqrt{\frac{\mathrm{Im}(T^j)}{2\,\mathrm{Im}(U^j)}}\,\Big(X^{2j-1} + \overline{U}^{\,j} X^{2j}\Big), \qquad \overline{Z}^{\,j} = \sqrt{\frac{\mathrm{Im}(T^j)}{2\,\mathrm{Im}(U^j)}}\,\Big(X^{2j-1} + U^j X^{2j}\Big), \tag{4.12}$$

$T_F(w)$ can be written as sum of the external and internal super current:

$$T_F(w) = \partial X_\mu(w)\;\psi^\mu(w) + \sum_{j=1}^{3}\Big[\partial\overline{Z}^{\,j}(w)\;\Psi^j(w) + \partial Z^{\,j}(w)\;\overline{\Psi}^{\,j}(w)\Big],$$

while for $\widetilde{T}_F(\overline{w})$ we find:

$$\widetilde{T}_F(\overline{w}) = \overline{\partial} X_\mu(\overline{w})\;\widetilde{\psi}^\mu(\overline{w}) + \sum_{j=1}^{3}\Big[\overline{\partial Z}^{\,j}(\overline{w})\;\widetilde{\Psi}^j(\overline{w}) + \overline{\partial} Z^{\,j}(\overline{w})\;\overline{\widetilde{\Psi}}^{\,j}(\overline{w})\Big].$$

Now that we have all necessary ingredients for the picture changing operator we can apply it to the $(-1,-1)$ -pictures and find for the Kähler modulus:

$$\begin{aligned} V_{T^j}^{(0,0)}(z,\overline{z},k) = \frac{1}{T^j - \overline{T}^{\,j}}\;&\Big[\partial\overline{Z}^{\,j}(z) + \mathrm{i}\;k\cdot\psi(z)\;\overline{\Psi}^{\,j}(z)\Big] \\ &\times\Big[\overline{\partial} Z^j\,(\overline{z}) + \mathrm{i}\;k\cdot\widetilde{\psi}(\overline{z})\;\widetilde{\Psi}^{\,j}(\overline{z})\Big]\;\mathrm{e}^{\mathrm{i}k\cdot X(z,\overline{z})}, \end{aligned} \tag{4.13}$$

while for the complex structure modulus we have:

$$\begin{aligned} V_{U^j}^{(0,0)}(z,\overline{z},k) = \frac{-1}{U^j - \overline{U}^{\,j}}\;&\Big[\partial Z^j(z) + \mathrm{i}\;k\cdot\psi(z)\;\Psi^j(z)\Big] \\ &\times\Big[\overline{\partial} Z^j(\overline{z}) + \mathrm{i}\;k\cdot\widetilde{\psi}(\overline{z})\;\widetilde{\Psi}^{\,j}(\overline{z})\Big]\;\mathrm{e}^{\mathrm{i}k\cdot X(z,\overline{z})}. \end{aligned} \tag{4.14}$$

4.4. Correlators of Internal Fields

The internal fields defined in (4.12) and (4.8) follow standard correlators known from conformal field theory. The correlators on the sphere are given by (see also section 8.1):

$$\begin{aligned} \langle \partial Z^i(z_k)\;\partial\overline{Z}^{\,j}(z_l)\rangle &= -\frac{\delta^{ij}}{(z_k - z_l)^2}, \\ \langle \Psi^i(z_k)\;\overline{\Psi}^{\,j}(z_l)\rangle &= \frac{\delta^{ij}}{z_k - z_l}, \end{aligned} \tag{4.15}$$

and accordingly for the anti-holomorphic fields. Additionally we have the following vanishing correlators:

$$\begin{aligned} \langle \partial Z^i(z_k)\;\partial Z^j(z_l)\rangle &= 0, \\ \langle \Psi^i(z_k)\;\Psi^j(z_l)\rangle &= 0, \end{aligned} \tag{4.16}$$

where the fermion correlator is obtained from bosonizing their fermionic fields $\Psi(z) = \mathrm{e}^{\mathrm{i}H(z)}$ and $\overline{\Psi}(z) = \mathrm{e}^{-\mathrm{i}H(z)}$ respectively. Furthermore we have correlators on the disc where the internal fields receive a further mixing of left- and right-moving components. The non-vanishing correlators are:

$$\begin{aligned} \langle \partial Z^i(z_k)\, \overline{\partial Z}^{\,j}(\overline{z}_l)\rangle &= -\frac{D^{ij}}{(z_k-\overline{z}_l)^2}, &\qquad \langle \partial \overline{Z}^{\,i}(z_k)\, \overline{\partial} Z^{\,j}(\overline{z}_l)\rangle &= -\frac{\overline{D}^{\,ij}}{(z_k-\overline{z}_l)^2} \\ \langle \Psi^i(z_k)\, \overline{\widetilde{\Psi}}^{\,j}(\overline{z}_l)\rangle &= \frac{D^{ij}}{z_k-\overline{z}_l}, & \langle \overline{\Psi}^{\,i}(z_k)\, \widetilde{\Psi}^{\,j}(\overline{z}_l)\rangle &= \frac{\overline{D}^{\,ij}}{z_k-\overline{z}_l}, \end{aligned} \tag{4.17}$$

while for the vanishing contractions we have:

$$\begin{aligned} \langle \partial Z^i(z_k)\, \overline{\partial} Z^j(\overline{z}_l)\rangle &= 0 = \langle \partial \overline{Z}^{\,j}(z_k)\, \overline{\partial Z}^{\,j}(\overline{z}_l)\rangle, \\ \langle \Psi^i(z_k)\, \widetilde{\Psi}^j(\overline{z}_l)\rangle &= 0 = \langle \overline{\Psi}^{\,j}(z_k)\, \overline{\widetilde{\Psi}}^{\,j}(\overline{z}_l)\rangle. \end{aligned} \tag{4.18}$$

D^{ij} accounts for Dirichlet or Neumann boundary conditions and is given by:

$$D^{ij} = \begin{cases} g^{ij}, & \text{tangent indices,} \\ -g^{ij}, & \text{normal indices,} \end{cases} \tag{4.19}$$

where we assumed no mixing of boundary conditions for now. Later we will consider cases in which the mixing matrix looks much more intricate. The anti-holomorphic fields $\widetilde{\Psi}(\overline{z}) = \mathrm{e}^{\mathrm{i}\widetilde{H}(\overline{z})}$ and $\overline{\widetilde{\Psi}}(\overline{z}) = \mathrm{e}^{-\mathrm{i}\widetilde{H}(\overline{z})}$ are bosonized introducing an according anti-holomorphic field $\widetilde{H}$.

4.5. Moduli Fields in the T-dual Picture

Performing T-duality in one direction of each torus essentially interchanges the Kähler and complex structure moduli:

$$T^j \to -\frac{\alpha'}{U'^j}, \qquad U^j \to -\frac{\alpha'}{T'^j}, \tag{4.20}$$

where the prime denotes reference to the dual torus for which the metric and Kalb-Ramond field read:

$$\begin{aligned} g'^j &= \frac{U'^j-\overline{U}'^j}{T'^j-\overline{T}'^j}\frac{1}{|U'^j|^2}\begin{pmatrix} |T'^j|^2 & -\frac{1}{2}(T'^j+\overline{T}'^j) \\ -\frac{1}{2}(T'^j+\overline{T}'^j) & 1 \end{pmatrix}, \\ b'^j &= \frac{1}{|U'^j|^2}\begin{pmatrix} 0 & -\frac{1}{2}(U'^j+\overline{U}'^j) \\ \frac{1}{2}(U'^j+\overline{U}'^j) & 0 \end{pmatrix}. \end{aligned} \tag{4.21}$$

For the moduli fields in the $(-1,-1)$ -picture this translates into:

$$\begin{aligned} V^{(-1,-1)}_{-\alpha'/U'^j} &= \frac{1}{\alpha'}\frac{|U'^j|^2}{U'^j-\overline{U}'^j}\, \mathrm{e}^{-\varphi(z)}\, \mathrm{e}^{-\widetilde{\varphi}(\overline{z})}\, \overline{\Psi}'^j(z)\, \widetilde{\Psi}'^j(\overline{z})\, \mathrm{e}^{ik\cdot X(z,\overline{z})} \\ V^{(-1,-1)}_{-\alpha'/T'^j} &= \frac{1}{\alpha'}\frac{|T'^j|^2}{T'^j-\overline{T}'^j}\, \mathrm{e}^{-\varphi(z)}\, \mathrm{e}^{-\widetilde{\varphi}(\overline{z})}\, \Psi'^j(z)\, \widetilde{\Psi}'^j(\overline{z})\, \mathrm{e}^{ik\cdot X(z,\overline{z})}, \end{aligned} \tag{4.22}$$

where the primed fermions refer to internal fermions dual to (4.8) which are given by:

$$
\begin{aligned}
\Psi'^j &= \sqrt{\frac{\mathrm{Im}(U'^j)}{2\,\mathrm{Im}(T'^j)}}\;\frac{|U'^j|}{|T'^j|}\;\Big(\psi^{2j-1}-\frac{\alpha'}{\overline{T}\,'^j}\;\psi^{2j}\Big),\\
\overline{\Psi}\,'^j &= \sqrt{\frac{\mathrm{Im}(U'^j)}{2\,\mathrm{Im}(T'^j)}}\;\frac{|U'^j|}{|T'^j|}\;\Big(\psi^{2j-1}-\frac{\alpha'}{T'^j}\;\psi^{2j}\Big).
\end{aligned}
\tag{4.23}
$$

Accordingly the zero pictures read:

$$
\begin{aligned}
V^{(0,0)}_{-\alpha'/U'^j}(z,\overline{z},k) &= \frac{1}{\alpha'}\frac{|U'^j|^2}{U'^j-\overline{U}\,'^j}\;\Big[\partial\overline{Z}\,'^j(z)+\mathrm{i}\,k\cdot\psi(z)\;\overline{\Psi}\,'^j(z)\Big]\\
&\qquad\times\Big[\overline{\partial}Z'^j\,(\overline{z})+\mathrm{i}\,k\cdot\widetilde{\psi}(\overline{z})\;\widetilde{\Psi}'^j\,(\overline{z})\Big]\;\mathrm{e}^{\mathrm{i}k\cdot X(z,\overline{z})},\\
V^{(0,0)}_{-\alpha'/T'^j}(z,\overline{z},k) &= \frac{1}{\alpha'}\frac{|T'^j|^2}{T'^j-\overline{T}\,'^j}\;\Big[\partial Z'^j\,(z)+\mathrm{i}\,k\cdot\psi(z)\;\Psi'^j(z)\Big]\\
&\qquad\times\Big[\overline{\partial}Z'^j\,(\overline{z})+\mathrm{i}\,k\cdot\widetilde{\psi}(\overline{z})\;\widetilde{\Psi}'^j\,(\overline{z})\Big]\;\mathrm{e}^{\mathrm{i}k\cdot X(z,\overline{z})},
\end{aligned}
\tag{4.24}
$$

where the primed bosons refer to internal bosons dual to (4.12). These are given by:

$$
\begin{aligned}
Z'^j &= \sqrt{\frac{\mathrm{Im}(U'^j)}{2\,\mathrm{Im}(T'^j)}}\;\frac{|U'^j|}{|T'^j|}\;\Big(X^{2j-1}-\frac{\alpha'}{\overline{T}\,'^j}\;X^{2j}\Big),\\
\overline{Z}\,'^j &= \sqrt{\frac{\mathrm{Im}(U'^j)}{2\,\mathrm{Im}(T'^j)}}\;\frac{|U'^j|}{|T'^j|}\;\Big(X^{2j-1}-\frac{\alpha'}{T'^j}\;X^{2j}\Big).
\end{aligned}
\tag{4.25}
$$

The internal fields defined in (4.23) and (4.25) obey the same contractions (4.15) and (4.16) as the original ones did. The disc correlators (4.17) and (4.18) have to be slightly modified before they can be applied to (4.23) and (4.25), i.e. we have to dualize the mixing matrix D^{ij} as well. The correlators follow then after replacing $D^{ij} \to \widetilde{D}^{ij}$. We will thus drop the primes but keep in mind to use the proper mixing matrix. In the next chapter we will give expressions for D^{ij} and $\widetilde{D}^{ij}$ where both become non-trivial due to non-vanishing fluxes or intersecting angles respectively.

5. Advanced Brane Models

The present chapter generalizes the notions from the third chapter in which we introduced D-branes. This generalizations happen to be necessary to gain more realistic models which reproduce the minimal supersymmetric standard model at least on a semi-realistic level. The needed modifications are vastly different in both type II theories, that is in type IIA we are dealing with branes intersecting at angles while in type IIB we turn on a background flux. T-duality ensures equivalence of both ideas and allows us to determine the desired quantities in whatever theory suits us better.

5.1. Type IIB: Branes with Magnetic Fluxes

In toroidal compactifications it happens to be impossible to reproduce the features of the standard model unless one turns on a magnetic flux which produces a mixing between boundary conditions [13]. We consider a compactification on three two-tori with a D-brane wrapped around each cycle of the tori. Each torus comes with a (symmetric) metric g^j and an anti-symmetric tensor b^j which are chosen to have constant entries[21]. Now we turn on an internal gauge flux F^j on each torus which is supposed to be block-diagonal:

$$F^j = \begin{pmatrix} 0 & F^j_{12} \\ -F^j_{12} & 0 \end{pmatrix}.$$

This we put together with the Kalb-Ramond field yielding the total anti-symmetric background of each torus:

$$\mathcal{F}^j = b^j + 2\pi\alpha' F^j, \tag{5.1}$$

whose components are given by:

$$\mathcal{F}^j = \begin{pmatrix} 0 & f^j \\ -f^j & 0 \end{pmatrix}, \qquad f^j = b^j_{12} + 2\pi\alpha' F^j_{12} = \frac{1}{2}(T^j + \overline{T}^j) + 2\pi\alpha' F^j_{12}. \tag{5.2}$$

An interesting property of the flux $\mathcal{F}$ is that it produces a mixing of boundary conditions. Consider the j-th torus we get for bosonic fields [14], [13]:

$$\begin{aligned} \partial_\sigma X^{2j-1} + \mathcal{F}^j\, \partial_\tau X^{2j} &= 0, \\ \mathcal{F}^j\, \partial_\tau X^{2j-1} - \partial_\sigma X^{2j} &= 0, \end{aligned} \tag{5.3}$$

which can be seen from varying the bosonic part of the open string action [13], [15], [16]. In the T-dual picture this mixing will be taken into account by an intersecting angle between two branes in a plane.

[21] The flatness of tori allows for this choice.

In the previous chapter we have introduced a matrix (4.19) to account for boundary conditions. The additional mixing produced by $\mathcal{F}$ can now be implemented by the following change in D^j [1], [16]:

$$D = 2\,(g+\mathcal{F})^{-1} - g^{-1}, \tag{5.4}$$

where the label j was suppressed for the sake of clarity. (5.4) contains (4.19) in the limits of $\mathcal{F} \to 0$ and $\mathcal{F} \to \infty$.
Essentially a non-vanishing background field introduces a non-commutative geometry on the brane, where non-commutativity means [15]:

$$[X^i, X^j] = \mathrm{i}\; \Theta^{ij}. \tag{5.5}$$

X^i and X^j are coordinates in the considered non-commutative space. The matrix Θ^{ij} is called non-commutativity parameter. The non-commutativity of coordinates translates into a modified multiplication for fields:

$$f(X) * g(X) = f(X)\cdot g(X) + \frac{1}{2}\,\mathrm{i}\;\Theta^{ij}\;\partial_i f(X)\,\partial_j\, g(X) + ..., \tag{5.6}$$

where $\cdot$ is the usual multiplication. The non-commutativity parameter itself allows us to write (5.4) as:

$$D = 2\;\Theta^{-1} + 2\;G^{-1} - g^{-1}, \tag{5.7}$$

where G is the open string metric, i.e. the pullback of the bulk metric on the D-brane world volume. The mixing matrix can as well be expressed in terms of moduli which results in a rather lengthy expression for the full matrix, see [1]. However, in most cases throughout this work we will not face contributions from different planes but only from one. The matrix reduces in this simple case to $D^{jj} := D^j$ with:

$$D^j = \frac{T^j - \overline{T}^{\,j} + 2\,f^j}{T^j - \overline{T}^{\,j} - 2\,f^j} = -\frac{T^j + 2\pi\alpha' F^j}{\overline{T}^{\,j} + 2\pi\alpha' F^j}. \tag{5.8}$$

From previous considerations we know that type IIB theory allows for D-branes in odd dimensions, i.e. we have initially 1-, 3-, 5-, 7- and 9-branes. We are however interested in models which reproduce features of the standard model or its supersymmetric extension which is best achieved 5-branes and 9-branes as those can preserve an $N = 1$ supersymmetry in four space time dimensions. For the sake of simplicity we will focus on 9-branes and refer to [1] for the cases with 5-branes. Note that supersymmetry preserving is not a priori granted. Indeed the flux entries f^j from (5.2) have to fulfill the following supersymmetry condition [1]:

$$\sum_{j=1}^{3} \frac{f^j}{\mathrm{Im}(T^j)} = \prod_{j=1}^{3} \frac{f^j}{\mathrm{Im}(T^j)}. \tag{5.9}$$

5.2. Type IIA: Branes Intersecting at Angles

We have already learned that type II theories are T-dual when toroidally compactified. This allows us to study the setup in the type IIA picture where the background flux translates into branes intersecting at angles.
Let us start with the simple case of two branes intersecting at an arbitrary angle θ. Examining string state at the intersections leads to the fact that in certain dimension the string state experiences a change of boundary conditions. That means an open string endpoint is allowed to "change" the brane at the intersection. This possibility has to be included into our view. We will do so by introducing a so-called twisted sector. Actually the notion of intersecting branes is quiet similar to the idea of orbifolds[22] and according twist fields.

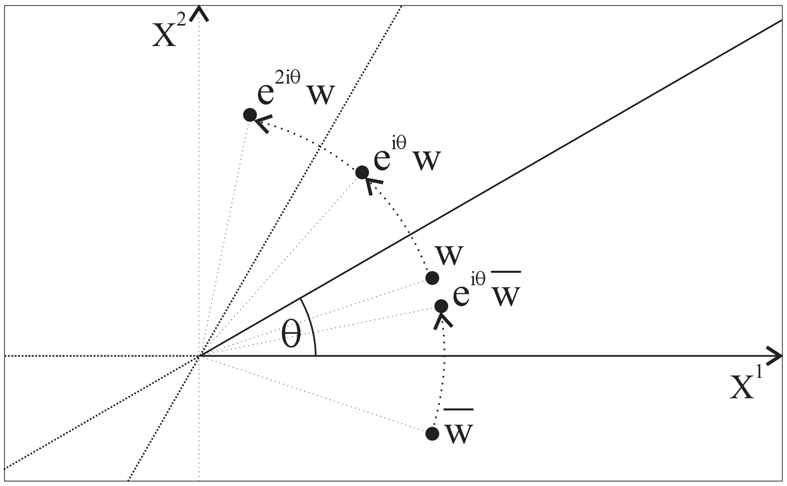

Fig. 6 The image point method: To account for the change of boundary conditions we include to the point w its conjugate $\overline{w}$ and all rotated points. Shown are w and $\overline{w}$, both rotated by θ and additionally w rotated by 2θ. Further image points have been suppressed for the sake of clarity.

Let us now consider string states at the intersection of two branes at angle θ in two dimensions X^1 and X^2. We will work on the complex plane, i.e. $X^1 + \mathrm{i}X^2 \rightarrow \widetilde{z}$. To take into account the different boundary conditions on the planes we have to introduce images as depicted in [17]. Let the angle be $\theta = \frac{\pi}{N}$, then we need N image points for a point $\widetilde{w}$, see fig. 6, and another N for its conjugate point $\overline{\widetilde{w}}$. Note that N is meant

[22] An orbifold O is a manifold M modded out by a finite group (mostly Z_N). The orbifold $O \sim M/Z_N$ shows singularities at the fixed points under the action of Z_N. As such it violates the definition of a manifold and can thus be regarded as a generalization.

in a way such that w rotated by $N \cdot \theta$ is again w. So the notion of is $\theta = \frac{\pi}{N}$ identical to a Z_N symmetry. Putting all images together we find for the Green function [17]:

$$G(\widetilde{z}, \widetilde{w}) = \sum_{k=0}^{N-1} \left(\mathrm{e}^{i\pi \frac{k}{N}} \, G_0\big(\widetilde{z}, \mathrm{e}^{i\pi \frac{k}{N}} \, \widetilde{w}\big) + \mathrm{e}^{i\pi \frac{k}{N}} \, G_0\big(\widetilde{z}, \mathrm{e}^{i\pi \frac{k}{N}} \, \overline{\widetilde{w}}\big) \right).$$

Transforming to the upper half plane $\widetilde{z} \to z^{\frac{1}{N}}$ yields:

$$G(\widetilde{z}, \widetilde{w}) = \sum_{k=0}^{N-1} \left(\mathrm{e}^{i\pi \frac{k}{N}} \, G_0\big(z^{\frac{1}{N}}, \mathrm{e}^{i\pi \frac{k}{N}} \, w^{\frac{1}{N}}\big) + \mathrm{e}^{i\pi \frac{k}{N}} \, G_0\big(z^{\frac{1}{N}}, \mathrm{e}^{i\pi \frac{k}{N}} \, \overline{w}^{\frac{1}{N}}\big) \right). \tag{5.10}$$

This coincides with the Green function under presence of a Z_N twist operator located at the origin of the complex plane. Moreover it corresponds to a fractional mode expansion of string states stretched between two branes intersecting at θ. In this case the boundary conditions read [18]:

$$\begin{aligned} \sin(\pi\,\theta)\;\partial_\sigma X^1 - \cos(\pi\,\theta)\;\partial_\sigma X^2 &= 0, \\ \sin(\pi\,\theta)\;\partial_\tau X^2 - \cos(\pi\,\theta)\;\partial_\tau X^1 &= 0, \end{aligned} \tag{5.11}$$

which is (5.3) expressed in angles instead of fluxes. These boundary conditions determine the holomorphic solutions of the string equation of motion yielding the following mode expansion:

$$\begin{aligned} \partial X(z) &= \sum_k \alpha_{k-\theta} z^{-k+\theta-1}, \\ \partial \overline{X}(z) &= \sum_k \overline{\alpha}_{k+\theta} z^{-k-\theta-1}, \end{aligned} \tag{5.12}$$

which again coincide with what we know from closed string states in the presence of a Z_N twist field. This enables us to deal with the intersection by means of a twist field $\sigma_\theta(w, \overline{w})$ which changes the boundary conditions of the bosonic field X to be those in (5.11). Accordingly we obtain the OPEs between bosons and twist fields [19], [20]:

$$\begin{aligned} \partial X(z)\sigma_\theta(w, \overline{w}) &\sim (z-w)^{\theta-1}\tau_\theta(w, \overline{w}), \\ \partial \overline{X}(z)\sigma_\theta(w, \overline{w}) &\sim (z-w)^{-\theta}\widetilde{\tau}_\theta(w, \overline{w}), \end{aligned} \tag{5.13}$$

where τ and $\widetilde{\tau}$ are excited twist fields. Furthermore we have to heed the local monodromy conditions[23] for transportation around twist fields:

$$\begin{aligned} \partial X(\mathrm{e}^{2\pi \mathrm{i}}(z-w)) &= \mathrm{e}^{2\pi \mathrm{i}\theta} \;\; \partial X(z-w), \\ \partial \overline{X}(\mathrm{e}^{2\pi \mathrm{i}}(z-w)) &= \mathrm{e}^{-2\pi \mathrm{i}\theta} \partial \overline{X}(z-w). \end{aligned} \tag{5.14}$$

This leads to the following mode expansion for X, see [17]:

$$X(z, \overline{z}) = \sqrt{\frac{\alpha'}{2}} \sum_k \left(\frac{\alpha_{k-\theta}}{k-\theta} \; z^{-k+\theta} + \frac{\widetilde{\alpha}_{k+\theta}}{k+\theta} \; \overline{z}^{-k-\theta} \right) \tag{5.15}$$

and a similar one for fermionic states.

[23] Monodromy is in general the multi-valuedness around non-trivial closed paths of a locally single-valued quantity.

5.3. D6-Branes wrapping at Angles on $T^2 \times T^2 \times T^2$

We will now come back to our setup of a type *II* theory compactified on a factorizable T^6 torus. Each of the three two-tori T^2 consists of two cycles (see fig. 4) around which the branes can wrap. Let (n, m) denote the wrapping numbers. Additionally we have to take into consideration several branes. We thus consider a stack of N_a coincident branes, where $N_a = 1, ..., K$. Accordingly we have to relabel the winding numbers to (n_a, m_a).
The only source of chiral fields in the open sector are open strings stretched between different sets of branes. Take a Dp_a-brane and a Dp_b-brane, then the relative angle between them is $\theta_{ab} = \theta_b - \theta_a$. The mass operator now takes the form [13], [18]:

$$\alpha' M_{ab}^2 = \frac{Y^2}{4\pi^2\alpha'} + \nu\Big(\sum_j \theta_{ab}^j - 1\Big) + N_\nu, \tag{5.16}$$

where the sum runs over all tori. In (5.16) Y denotes the length of the stretched string, i.e. the minimal distance between branes for minimum winding numbers. Note that Y is only non-vanishing in dimensions where the branes are parallel. N_ν are the number operators with $\nu = 0$ in the R-sector and $\nu = \frac{1}{2}$ in the NS-sector. N_0 and $N_{\frac{1}{2}}$ are explicitly given in [13] and [18]. The factor θ_{ab}^j in (5.16) is due to normal ordering of twisted ground states. In the R-sector this factor happens to vanish as $\nu = 0$ which renders R-state masses independent of the intersecting angles. We can now label states by introducing a four dimensional twist vector v_θ. Its j-th component is given by θ_{ab}^j. The GSO-projection is then represented by adding constant four vector r to the twist vector components such that it is now $r + v_\theta$. The components of r are integral in the NS-sector and half-integral in the R-sector. Then the GSO-projection amounts to demanding $\sum_j r_j =$ odd. The mass is now given by [18]:

$$\alpha' M_{ab}^2 = \frac{Y^2}{4\pi^2\alpha'} + \frac{1}{2}\Big((r + v_\theta)^2 + \sum_j |\theta_{ab}^j|(1 - |\theta_{ab}^j|) - 1\Big) + N_{bos}(\theta), \tag{5.17}$$

where $N_{bos}(\theta)$ denotes bosonic oscillator states. We will be interested in lowest lying states where $N_{bos}(\theta)$ is vanishing.
As mentioned in chapter two string theories with open strings have to satisfy a crucial consistency condition, i.e. cancellation of (R-R)-tadpoles, which was taken into account by introducing a gauge boson and Wilson lines. How does this look like in intersecting brane models? Let $[a_j]$ and $[b_j]$ denote the homology classes[24] of the j-th

[24] Homology classes are classes of subsets which have no boundary and are not boundaries of any other subset of the torus, c.f. [12].

torus which are its basis cycles. Then $[\Pi_a]$ denotes the homology classes of the cycles wrapped by the $D6$-branes and is given by:

$$[\Pi_a] = \prod_{j=1}^{3} \left(n_a^j\ [a_j] + m_a^j\ [b_j]\right). \tag{5.18}$$

Since in our setup we wrap (R-R)-charged 6-branes around the cycles the homology classes carry (R-R)-charge themselves, consistency requires now the vanishing of the total homology class [8], [18]:

$$\sum_a N_a\ [\Pi_a] = 0, \tag{5.19}$$

which can as well be seen from certain factorization limits of one-loop amplitudes and consistency considerations concerning the 7-form potentials of the $D6$-branes [18].
So far the notions introduced in this chapter apply for any $D(3+n)$-branes wrapping on n-cycles on a T^{2n} compactification. To obtain a four dimensional theory n must not exceed 3. A further interesting fact is that the cases of $n < 3$ do not yield a chiral spectrum [18]. We can thus concentrate on $D6$-branes wrapping on T^6-cycles. According to [18] one can indeed obtain a chiral spectrum from this setup. Moreover the 6-branes follow from acting with (4.20) on one direction of each torus in the 9-brane flux model.
Let us now have a look on the four dimensional spectrum. Strings with both ends in the same stack of N_a 6-branes, i.e. $6_a - 6_a$ or $6_b - 6_b$ strings, have unbroken $N = 4$ supersymmetry. The massless spectrum consists in this case of a $U(N_a)$ (or $U(N_b)$) gauge boson, six real scalars and four Majorana spinors which build an $N = 4$ supermultiplet. Strings with ends in different stacks of branes, i.e. $6_a - 6_b$ or $6_b - 6_a$ strings, occur in certain multiplicities due to multiple intersection numbers I_{ab} of the cycles Π_a and Π_b given by:

$$I_{ab} = [\Pi_a] \cdot [\Pi_b] = \prod_i (n_a^i m_b^i - m_a^i n_b^i). \tag{5.20}$$

In the R-sector one finds I_{ab} chiral fermions while in the NS-sector we find I_{ab} scalars. Note that intersecting numbers I_{ba} describe anti-particles of I_{ab}. In the case of $D6$-branes intersecting on T^6 the twist vector is given by $v_\theta = (\theta^1, \theta^2, \theta^3, 0)$. Thus the above R-sector ground states are [13]:

$$r_{\mathrm{R}} + v_\theta = \left(\theta^1 - \frac{1}{2}, \theta^2 - \frac{1}{2}, \theta^3 - \frac{1}{2}, \frac{1}{2}\right) \qquad \to \alpha' M^2 = 0, \tag{5.21}$$

which are massless (see (5.17) with $Y = 0$ and $N_{\text{bos}} = 0$) independent of the angles. Similarly one obtains the lowest lying masses in the NS-sector [18]:

$$
\begin{aligned}
r_{\text{NS}} - v_\theta &= (\theta^1 - 1, \theta^2, \theta^3, 0) && \to \alpha' M^2 = \frac{1}{2}(-\theta^1 + \theta^2 + \theta^3) \\
&= (\theta^1, \theta^2 - 1, \theta^3, 0) && \to \alpha' M^2 = \frac{1}{2}(\ \ \theta^1 - \theta^2 + \theta^3) \\
&= (\theta^1, \theta^2, \theta^3 - 1, 0) && \to \alpha' M^2 = \frac{1}{2}(\ \ \theta^1 + \theta^2 - \theta^3) \\
&= (\theta^1 - 1, \theta^2 - 1, \theta^3 - 1, 0) && \to \alpha' M^2 = 1 - \frac{1}{2}(\theta^1 + \theta^2 + \theta^3),
\end{aligned}
\tag{5.22}
$$

from which we will work with the last one which implies $\theta^1 + \theta^2 + \theta^3 = 2$ as a condition for masslessness and thus supersymmetry[25]. There are of course solutions with non-vanishing mass. These give rise to an explicitly non-supersymmetric low energy spectrum. Moreover (5.22) allows for tachyonic states which would signal certain instabilities of the system.

Let us come back to the duality between the flux picture and the angle picture. The intersecting angles θ^j can be directly obtained form the flux components (5.2) by [1]:

$$\tan(\pi\,\theta^j) = \frac{f^j}{\mathrm{Im}(T^j)}, \tag{5.23}$$

which can be written in terms of the dual moduli and winding numbers:

$$\tan(\pi\,\theta^j) = \frac{m^j}{n^j}\frac{|U'^j|^2}{\mathrm{Im}(U'^j)} - \frac{\mathrm{Re}(U'^j)}{\mathrm{Im}(U'^j)}, \tag{5.24}$$

where we have expressed the field F^j in f^j from (5.23) by winding numbers n^j and m^j around the two cycles of the j-th torus, that is:

$$F^j = \frac{1}{2\pi}\frac{m^j}{n^j}. \tag{5.25}$$

T-dualizing the supersymmetry condition (5.9) of the flux picture yields:

$$\sum_{j=1}^{3}\left(\frac{m^j}{n^j}\frac{|U'^j|^2}{\mathrm{Im}(U'^j)} - \frac{\mathrm{Re}(U'^j)}{\mathrm{Im}(U'^j)}\right) = \prod_{j=1}^{3}\left(\frac{m^j}{n^j}\frac{|U'^j|^2}{\mathrm{Im}(U'^j)} - \frac{\mathrm{Re}(U'^j)}{\mathrm{Im}(U'^j)}\right), \tag{5.26}$$

which becomes after using (5.24):

$$\tan(\pi\,\theta^1) + \tan(\pi\,\theta^2) + \tan(\pi\,\theta^3) = \tan(\pi\,\theta^1)\cdot\tan(\pi\,\theta^2)\cdot\tan(\pi\,\theta^3). \tag{5.27}$$

---

[25] Note that in the case of a massless scalar boson we have a massless chiral $N = 1$ supermultiplet comprising the massless scalar and the massless Weyl spinor.

This equation translates into the following condition for the angles to preserve $N = 1$ supersymmetry in four space time dimensions:

$$\theta^1 + \theta^2 + \theta^3 = 2 \text{ mod } 2, \tag{5.28}$$

which is equivalent to the supersymmetry condition we extracted from (5.22).
As mentioned in the previous chapter calculations in the angle picture have to be done with the dual mixing matrix $\widetilde{D}^j$ with D^j given by (5.8). After acting with (4.20) on (5.8) we arrive at:

$$\widetilde{D}^j = -\frac{\overline{U}\,'^j}{U'^j}\frac{1-2\pi\alpha' F^j U'^j}{1-2\pi\alpha' F^j \overline{U}\,'^j} = \frac{1-\mathrm{i}\,\tan(\pi\,\theta^j)}{1+\mathrm{i}\,\tan(\pi\,\theta^j)} = \mathrm{e}^{-2\pi\mathrm{i}\theta^j}. \tag{5.29}$$

5.4. Generalization beyond the Torus

Compactifying on tori is the most simple and instructive model but clearly not the most general. Indeed we can compactify on any 6-dimensional manifold which preserves supersymmetry in four dimensions (see chapter four). In such generalized setup one would accordingly pick a set of 3-cycles Π_a on which to wrap N_a $D6$-branes. Tadpole cancellation has to be ensured by imposing (5.19) respectively. In a similar way we introduce N_b rotated 6-branes wrapping the residual 3-cycles Π_b.
Accordingly we find for the two sectors for the open string spectrum where referring to open strings ending on the same 6-brane or stretching between two different ones:

- $6_a - 6_a$ or $6_b - 6_b$ leads to an $N = 1$ supersymmetric $U(N_a)$ or $U(N_b)$ vector multiplet in the NS-sector and a chiral multiplet in the R-sector.
- $6_a - 6_b$ or $6_b - 6_a$ leads to I_{ab} massless scalars in the NS-sector and I_{ab} massless chiral fermions in the R-sector.

This is rather similar to what we found in the toroidal case which holds for other aspects as well [8]. So torus compactifications are in general very good models especially since explicit constructions of intersecting branes from general CY3 are rather difficult to obtain.

5.5. Bosonic Matter Field Vertex Operators

In this section we will clarify how matter is actually realized in the just explained models. (Massless) matter fields themselves in four dimensional space time comprise the fermions we are familiar with from the standard model, that is the three lepton and quark generations. If one extends the standard model by some supersymmetries every matter field is given one superpartner for each supercharge. Our main interest is focused on $N = 1$ supersymmetry in $D = 4$, that is apart from leptons and quarks

(spin $\frac{1}{2}$) our matter fields comprise so-called sleptons and squarks[26] (spin 0). Since the purpose of this work is the determination of the bosonic tree-level supergravity action we will focus on massless bosonic matter.
In the flux picture internal components of compactified matter fields can stretch between the various branes. Chiral matter itself stems from string from 5- and 9-branes. We will for the sake simplicity concentrate on matter fields on 9-branes C_j^9, with $j = 1, 2, 3$ referring to the tori. Most time we will drop the superscript. Matter fields between 5-branes or stretching between 5- and 9-branes are discussed in [1].
To obtain the matter field vertex operator we have a look at the NS-sector of the open string. In ten space-time dimensions the NS ground state is tachyonic. We get rid of it using the GSO-projection. The lowest lying state is then created by acting with $b^{\varrho}_{-1/2}$ ($\varrho = 0, ..., 9$) on the ground state $|k\rangle$[27] yielding $\psi^{\varrho}(z)|k\rangle$. This generates a massless ten dimensional space time vector. Compactifying 6 dimensions results in the creator to decompose into an external part $b^{\mu}_{-1/2}$ ($\mu = 0, ..., 3$) creating a four dimensional space time vector and six internal parts $b^{i}_{-1/2}$ ($i = 4, ..., 9$)[28] creating one real scalars each. The latter may be combined to three complex scalars referring to the three tori we compactified on. The vertex operator is then given by:

$$V^{(-1)}_{\text{Boson}}(z,k) = \lambda\, \mathrm{e}^{-\varphi(z)} \left[\xi_\mu(k)\, \psi^\mu(z) + \sum_{j=1}^{3} \Psi^j(z)\right] \mathrm{e}^{\mathrm{i}k\cdot X(z)}, \tag{5.30}$$

with $\Psi^j(z)$ defined in (4.8). Accordingly the vertex operator of a complex scalar referring to the j-th torus can be read off from (5.30):

$$V^{(-1)}_{C_j}(z,k) = \lambda\, \mathrm{e}^{-\varphi(z)} \Psi^j(z)\, \mathrm{e}^{\mathrm{i}k\cdot X(z)}. \tag{5.31}$$

This is a massless bosonic matter field. Acting with the picture changing operator $P(w, \overline{w})$ which is on the boundary defined as follows:

$$P(w,\overline{w}) = \mathrm{e}^{\varphi(w)}\, T_F(w) + \mathrm{e}^{\widetilde{\varphi}(\overline{w})}\, \widetilde{T}_F(\overline{w}) \tag{5.32}$$

on (5.31) yields the zero ghost picture:

$$V^{(0)}_{C_j}(z,k) = \lambda\Big[\partial Z^j(z) + \mathrm{i}\; k\cdot\psi(z)\Psi^j(z)\Big]\mathrm{e}^{\mathrm{i}k\cdot X(z)}. \tag{5.33}$$

[26] or s-leptons and s-quarks
[27] The ground state $|k\rangle$ with momentum k is generated by $\mathrm{e}^{\mathrm{i}k\cdot X}|0\rangle$.
[28] Note that i is unlike μ not a vector index but a counting index.

Note that the actual setup was shifted into the correlators (4.15) and (4.17) of the internal fields in (5.31) and (5.33). The actual vertex operators themselves do not "feel" the geometry.
In the angle picture chiral matter stems form open strings stretched between two 6-branes intersecting at an angle θ. The found vertex operators (5.31) and (5.33) directly take over to the untwisted sector of the angle picture. The twisted sector deserves some special attention though.
To create states in the twisted sector we have to heed the fact that the mode expansions received a shift from the angles, see (5.15). The internal creators are shifted accordingly, that is the internal scalars are created by $b^i_{\theta-1/2}$, see [21]. As already mentioned the mixing of boundary conditions at the intersection is taken into account by inserting bosonic twist fields σ_{θ^j}. To ensure supersymmetry we have to introduce fermionic twist fields s_{θ^j} as well. The bosonic matter field vertex is then given by [1]:

$$V_{C_\theta}^{(-1)}(z,k) = \lambda \, \mathrm{e}^{-\varphi(z)} \prod_{j=1}^{3} s_{\theta^j}(z) \, \sigma_{-\theta^j}(z) \, \mathrm{e}^{\mathrm{i}k\cdot X(z)}. \tag{5.34}$$

Note the different angles on the bosonic and fermionic twist fields. This is needed to avoid divergencies. Acting with the picture changing operator (5.32) on the (-1) picture yields as usual the zero picture:

$$\begin{aligned} V_{C_\theta}^{(0)}(z,k) =& \lambda \Big[\sum_{l=1}^{3} t_{\theta^l}(z) \, \tau_{-\theta^l}(z) \prod_{j\neq l}^{3} s_{\theta^j}(z) \, \sigma_{-\theta^j}(z) + \\ & + \mathrm{i} \, k \cdot \psi(z) \prod_{i=1}^{3} s_{\theta^i}(z) \, \sigma_{-\theta^i}(z) \Big] \mathrm{e}^{\mathrm{i}k\cdot X(z)}. \end{aligned} \tag{5.35}$$

The twist fields (5.34) and (5.35) have a rather intricate CFT. We will give some more information in appendix A.

6. Low Energy Effective Theories

Now that we have compactified the ten dimensional string theory we are interested the correct description in four space time dimensions. The geometry of the internal six dimensions is taken into account with scalar moduli fields as explained in chapter four. These are expected to considerably influence the four dimensional field theory. Remember that we are interested in models that reproduce the (minimal supersymmetric) standard model. That is, we focus on an $N = 1$ low energy field theory. Low energy means that we only include massless string states into our considerations. Massive string states have masses near the Planck scale, i.e. some 10^{19} GeV which is far beyond the masses of the standard model. Moreover the particles of the standard model gain their masses from symmetry breaking and thus are initially all massless.

6.1. The $N = 1$ Supergravity Action in $D = 4$

It is known that any $N = 1$ supergravity action in four space time dimensions can be written in three functions. These are the Kähler potential K, the superpotential W and the gauge kinetic function f. The according lagrangian reads [22]:

$$\begin{aligned}\mathcal{L} = &\int \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\; K(\Phi^i, \Phi^{j\dagger} \mathrm{e}^{2V}) - \int \left[\mathrm{d}^2\theta W(\Phi^i) + \text{ h.c. }\right] \\ &+ \frac{1}{8\pi}\mathrm{Im}\left[\tau \int \mathrm{d}^2\theta \;\mathrm{Tr} f(\Phi^i) W^\alpha W_\alpha\right],\end{aligned} \tag{6.1}$$

where θ denote super space coordinates[29] , Φ are chiral super fields[30], τ is a complex gauge coupling and V is a complex scalar potential. Since string theory is initially formulated in ten space time dimensions these three functions will inherit a non-trivial dependence on the moduli of the compactification space. It is the main purpose of this work to determine these dependencies.
The occurring of a Kähler potential in (6.1) is a very interesting feature. Any Kähler potential $K(A^i, \overline{A}^j)$ with A^i denoting the scalar field in Φ^i is invariant under Kähler transformations:

$$K(A^i, \overline{A}^j) \to K(A^i, \overline{A}^j) + \Lambda(A^i) + \Lambda^\dagger(\overline{A}^j), \tag{6.2}$$

[29] θ must not be confused with the intersecting angle.

[30] Chiral super fields are defined as $\overline{D}\Phi = 0$ and anti-chiral ones as $D\overline{\Phi} = 0$ where D is the super space covariant derivative. Chiral super fields contain a scalar, a Weyl spinor and an auxiliary scalar field.

where $\Lambda(A^i)$ is again a chiral super field. Moreover one can gain a Kähler metric from the following derivative:

$$G_{A^i\overline{A}^j} = \frac{\partial^2 K(A^i, \overline{A}^j)}{\partial A^i\, \partial \overline{A}^j}. \tag{6.3}$$

Essentially this means that the scalar components A^i of the chiral super field Φ^i define local complex coordinates of the target space which is a Kähler manifold. Note that (6.2) is valid for the spinor part of Φ^i as well while (6.3) is only applicable to the complex scalar of the chiral super field. In (6.1) the Kähler potential plays the role of a kinetic term. Actually it is the non-linear sigma model generalization of $\Phi^\dagger\Phi$ to $G_{ij}\Phi^{\dagger,i}\Phi^j$.

The Kähler potential K in our case consists of a function $\widehat{K}$ describing the potentials of the moduli fields themselves and further entries describing the bosonic matter fields:

$$K = \widehat{K}(M, \overline{M}) + K_{\text{matter}}(C, \overline{C}). \tag{6.4}$$

Both types of fields are complex scalars for which (6.3) can be used. Up to second order in the (bosonic) matter fields the potential is given by:

$$K_{\text{matter}} = \sum_{\text{twisted}} G_{C_\theta \overline{C}_\theta}(M, \overline{M})\, C_\theta \overline{C}_\theta + \sum_{\text{untwisted}} G_{C_i \overline{C}_j}(M, \overline{M})\, C_i \overline{C}_j + ... \tag{6.5}$$

We will determine the matter (Kähler-) metrics $G_{C_\theta \overline{C}_\theta}$ and $G_{C_i \overline{C}_j}$ by calculating tree level scattering amplitudes between matter fields and moduli in both twisted and untwisted sectors. Equation (6.3) then defines differential equations from which we will extract the Kähler potential.

While the Kähler potential K describes the kinetics in (6.1) the superpotential W is as its name suggests a potential. As such it may depend on Φ or $\Phi^\dagger$ but unlike the Kähler potential it must not contain any products $\sim \Phi^\dagger\Phi$. W will be determined in scattering of four matter fields. We will probe certain factorization limits to gain the Yukawa couplings in which the trilinear superpotential is expressed like:

$$W = \sum_{\alpha,\beta,\gamma} W_{\alpha\beta\gamma}\, C_\alpha C_\beta C_\gamma + \sum_{a,b,c} W_{abc}\, C_a C_b C_c, \tag{6.6}$$

where the indices α, β and γ are labels in twisted sector while a, b and c label untwisted matter fields.

Finally the gauge kinetic function $f(\Phi^i)$ emerges if one includes gauge fields which also explains the factor e^{2V} in the kinetic term of (6.1). Indeed replacing usual derivatives by gauge covariant derivatives amounts to replacing $\Phi^\dagger\Phi$ by $\Phi^\dagger \mathrm{e}^{2V}\Phi$. The gauge kinetic function itself accounts for the kinetics of the gauge fields. However, f will not be determined in this treatise. Instead we will refer to [1], where f was calculated in explicit scattering of gauge fields and moduli. Another way obtain f would be to dimensionally reduce the Born-Infeld action of massless NS-states on the brane.

6.2. String Basis and Field Theory Basis

The toroidal moduli (4.4) and (4.5) of the compactification space have a rather geometrical meaning because they stem from ten dimensional string theory. Moreover the dependence of above introduced parts of the supergravity action will be obtained from explicit string calculations. Thus they are referred to as moduli in the string basis. To make contact with a low energy field theory in four space time dimensions these moduli will be regarded as couplings to branes without fluxes [1]. This enables us to compare our results with the Kähler potential found in [23] which is supposed to appear in a certain limit of our findings.
In the flux picture we focused on 5- and 9-branes. The coupling to the 9-brane is now described by the real part of the dilaton field s [1], [23]:

$$\begin{aligned} \mathrm{Re}(s) =& g_{D9}^{-2} = \frac{1}{2\pi}\frac{\mathrm{e}^{-\varphi_{10}}}{\alpha'^3}\,\mathrm{Im}(T^1)\,\mathrm{Im}(T^2)\,\mathrm{Im}(T^3) \\ =& \frac{1}{2\pi}\frac{\mathrm{e}^{-\varphi_4}}{\alpha'^{3/2}}\sqrt{\mathrm{Im}(T^1)\,\mathrm{Im}(T^2)\,\mathrm{Im}(T^3)}, \end{aligned} \tag{6.7}$$

where φ_{10} and $\varphi_4 = \varphi_{10} - \frac{1}{2}\ln[\mathrm{Im}(T^1)\mathrm{Im}(T^2)\mathrm{Im}(T^2)/\alpha'^3]$ are the string dilaton fields which arise in the Einstein term of the string action. The coupling to the 5-branes is respectively expressed as:

$$\mathrm{Re}(t^j) = g_{D5,j}^{-2} = \frac{1}{2\pi}\frac{\mathrm{e}^{-\varphi_{10}}}{\alpha'}\,\mathrm{Im}(T^j) = \frac{\alpha'^{1/2}}{2\pi}\,\mathrm{e}^{-\varphi_4}\sqrt{\frac{\mathrm{Im}(T^j)}{\mathrm{Im}(T^k)\,\mathrm{Im}(T^l)}}. \tag{6.8}$$

The field theory moduli in the angle picture can again be obtained form acting with (4.20) on the flux picture. The resulting quantities are:

$$\begin{aligned} \mathrm{Re}(s') =& \frac{\mathrm{e}^{-\varphi_4}}{2\pi}\,\frac{\sqrt{\mathrm{Im}(U'^1)\mathrm{Im}(U'^2)\mathrm{Im}(U'^3)}}{|U'^1U'^2U'^3|}, \\ \mathrm{Re}(u'^j) =& \frac{\mathrm{e}^{-\varphi_4}}{2\pi}\sqrt{\frac{\mathrm{Im}(U'^j)}{\mathrm{Im}(U'^k)\mathrm{Im}(U'^l)}}\left|\frac{U'^kU'^l}{U'^j}\right|. \end{aligned} \tag{6.9}$$

In the following we will mostly work with the following coupling:

$$\kappa_4^{-2} = \frac{\mathrm{e}^{-2\varphi_4}}{\alpha'\,\pi}. \tag{6.10}$$

7. Calculating Scattering Amplitudes in String Theory

Before starting to calculate the various string amplitudes to obtain the moduli dependencies of the ingredients to the low energy action we deem it appropriate to give a short introduction to string amplitudes. This chapter is meant as a basic introduction and as such it will not comprise advanced notions needed in scattering strings from branes. Instead we will present these ideas in the beginning of each of the following chapters.

Strings of all types are extended objects, that is interaction diagrams are surfaces. If one considers arbitrary scattering processes the smoothness of the surface is not ensured which makes calculations rather intricate. This changes considerably if we restrict ourselves to asymptotic amplitudes (S-matrix), that is we fix incoming strings at $t = -\infty$ and outgoing strings at $t = +\infty$, where t refers to time. Recalling that string theory is a conformal field theory these states can now be mapped to punctures on an ordinary surface which can be integrated over. Furthermore we can map the punctured manifold on the complex plane.

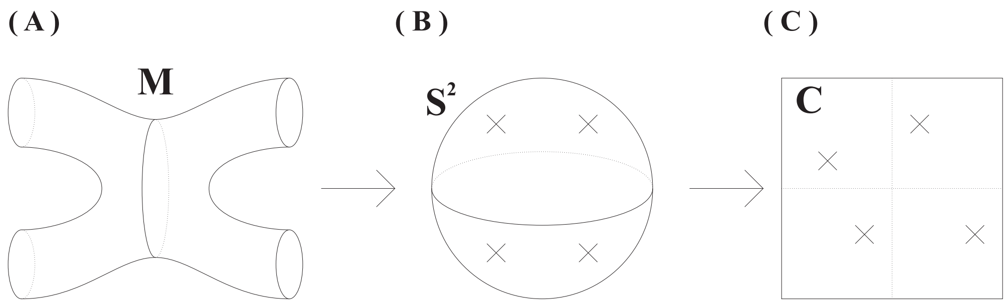

Fig. 7 Two closed string scattering at tree level. In (A) we see the actual process on the world sheet, that is the two closed strings are merging and afterwards splitting again. In (B) we see the process conformally mapped on the sphere, where the asymptotic states are vertex insertions into the surface. (C) describes the setup projected onto the complex plane.

We will now discuss the raised points in more detail for both closed and open string separately. Let us start with the closed string, see fig. 7. As mentioned above we consider the asymptotic case where conformal invariance allows us to map the in- and out-coming stated to points on a Riemannian surface. This surface has a number g of handles corresponding to the order of the interaction. Tree level interactions yield the sphere S^2, while one loop interactions are given by the torus T^2. Integration on these surfaces is well defined but bothersome. So one strives to map the surface on a flat space. Taking e.g. the sphere this can be achieved through the stereographic projection mapping the sphere on the complex plane. Analogously this can be done for higher order interactions.

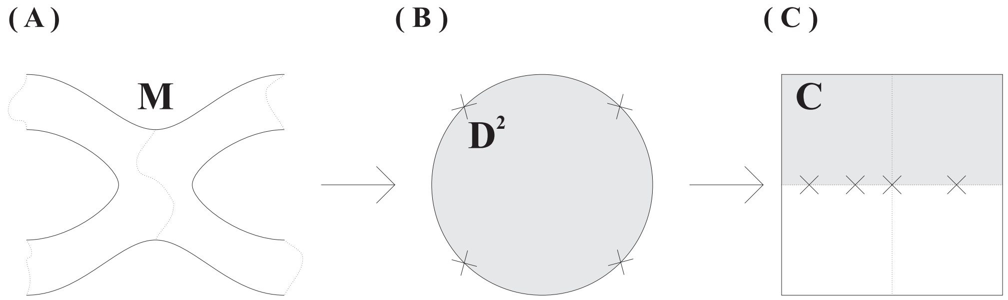

Fig. 8 Two open strings scattering at tree level. The fat line denotes the boundary of the surface. (A) shows the process on the world sheet level. (B) stems from conformally mapping the world sheet on the disc which itself is projected onto the upper half plane (C). The open strings become vertex insertion into the boundary of the disc (B) and are projected on the real line in (C).

For the open string this quiet similar, see fig. 8. Tree level interactions are conformally equivalent to a disc with the asymptotic states inserted into the boundary. One loop interactions are equivalent to the annulus accordingly. We can again map the surfaces on the complex plane. The disc is mapped to the upper half plane $\mathrm{H}_+ = \{z \in \mathrm{C} : \mathrm{im}\, z \geq 0\}$ and its boundary happens to be the real line.

7.1. The Polyakov Path Integral

The scattering amplitudes themselves are calculated by means of path integration. A general n-point amplitude of arbitrary high order is then given by:

$$\mathcal{A} \sim \sum_{\chi(M)} \mathrm{e}^{-\Phi\,\chi(M)} \int_M \frac{\mathcal{D}X\,\mathcal{D}g}{V_{\mathrm{Diff}\times\mathrm{Weyl}}}\, \mathrm{e}^{-S[X,g]} \prod_{i=1}^{n} \int \mathrm{d}^2 z_i\, g^{1/2}(z_i)\, V_i(k_i, z_i), \tag{7.1}$$

where $\chi = 2 - 2g - b - c$ is the Euler number of the surface M with g being the number of handles, b is the number of boundaries and c counts the cross-caps. Essentially χ accounts for the order of interaction. Thus $\mathrm{e}^{-\Phi\,\chi}$ is a weight factor accounting for different topologies of the world sheets with Φ being the dilaton. $\mathcal{D}X$ is a path integral measure referring to the embedding X of the world sheet into space time while $\mathcal{D}g$ is a measure referring to the world sheet[31]. To avoid integrating over equivalent paths we had to divide out the symmetry group of the string action S, that is S is invariant under connected reparameterizations ($\sim V_{\mathrm{Diff}}$) and Weyl rescaling ($\sim V_{\mathrm{Weyl}}$). How this is actually done depends strongly on the topology of the surface.

[31] Do not confuse the world sheet metric in $\mathcal{D}g$ and $S[X,g]$ with the number of handles in χ.

Later in this section we will derive the result for tree level scattering. $S[X,g]$ is the classical string action, that is the Polyakov action for the bosonic string or its supersymmetric extension for the superstring. The second integral accounts for the inserted asymptotic states, where the punctures into the surface are introduced by vertex operators $V_i(k_i, z_i)$. The factor $g^{1/2}(z_i)$ has to be inserted to account for the correct normalization of the vertices in the field theory basis. The path integration can be replaced by vacuum expectation values of the vertex operators, which are correlation function predicted by conformal field theory:

$$\mathcal{A} \sim \sum_{\chi(M)} \mathrm{e}^{-\Phi\,\chi(M)} \int_M \frac{\mathrm{d}^2 z_1 \,...\, \mathrm{d}^2 z_n}{V_{\mathrm{Diff}\times\mathrm{Weyl}}} \,\langle V_1(k_1, z_1) \,...\, V_n(k_n, z_n)\rangle, \tag{7.2}$$

where we have neglected the normalizations.

7.2. Tree-level Amplitudes

As already stated the symmetry volume $V_{\mathrm{Diff}\times\mathrm{Weyl}}$ depends on the topology of the surface. Actually we have to include disconnected diffeomorphisms (Dehn twists) into our considerations as well. These give rise to moduli which have to be added to the measure. Fortunately moduli no not emerge at tree-level [2] where we want to restrict ourselves to in this work. At tree level we are talking about the sphere or disc respectively. From these rather simple cases we know that the symmetry group $V_{\mathrm{Diff}\times\mathrm{Weyl}}$ is equal to the conformal Killing group V_{CKG}. The conformal transformations are easy to tell for the sphere and disc. We have translations, rotations and special conformal transformations. Altogether this makes three conformal Killing vector fields living on the sphere or disc. These three vector fields allow for three vertex positions to be fixed where we have to introduce one so-called c-ghost field $c(z)$ for each fixed location. Doing this V_{CKG} is cancelled out and we arrive at:

$$\begin{aligned}\mathcal{A}^{\mathrm{tree}} \sim \int \mathrm{d}^2 z_{n-3} \,...\, \mathrm{d}^2 z_n \,\langle c(\zeta_1)\, V_1(k_1,\zeta_1)\;\; c(\zeta_2)\, V_2(k_2,\zeta_2)\;\; c(\zeta_3)\, V_3(k_3,\zeta_3) \\ \times V_{n-3}(k_{n-3}, z_{n-3}) \,...\, V_n(k_n, z_n)\rangle,\end{aligned} \tag{7.3}$$

where ζ_i denote fixed locations while z_i are to be integrated over. Usually we are faced with amplitudes involving four strings. In this case only one position is to be integrated which makes the integration doable in most cases. Three point amplitudes are especially simple as they can be completely fixed. Amplitudes involving less than three strings are at tree-level generally vanishing.

8. Amplitudes of Matter Fields and Moduli Fields

It is our goal in this chapter to determine the matter field metrics in both twisted and untwisted sector. These are the second derivatives of the Kähler potential which can thus be extracted from the metrics. In doing so we will compute tree level scattering between two matter fields and one modulus. From the resulting amplitudes one can extract differential equations to determine the metric entries.

8.1. The General Setup for Open-Closed String Scattering from D-branes

Before starting to compute the desired amplitudes it is in order to present some basic properties of scattering processes involving both open and closed strings. Open strings are represented as punctures into the boundary of a disc while closed strings are punctures into a sphere.

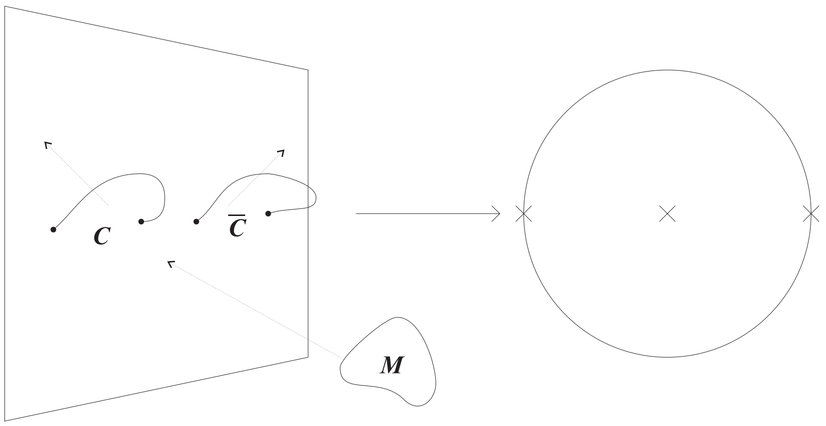

Fig. 9 A closed string is scattered with two open strings on a D-brane. The closed string vertex is inserted into the bulk while the open vertices are located on the boundary of the disk.

To combine this to one picture one has to regard the closed string as two independent open strings, namely the left- and right moving components of the closed string. These are inserted into the bulk, that is the inside of the disc, while the open string are still inserted in the boundary, see fig. 9.

The disc is represented by the upper half plane $\mathrm{H}_+ = \{z \in \mathrm{C} : \mathrm{im}\ z \geq 0\}$. The boundary of the disc then coincides with the real line. The sphere on the other hand is represented by the full complex plane. The upper half plane can now be obtained from applying the Z_2-projection $z \to \overline{z}$. This projection entails an interaction between the left- and right-moving sectors of the closed strings. This means that apart from the usual correlators on the sphere:

$$\begin{aligned}
\langle X^\mu(z_i)\, X^\nu(z_j)\rangle &= -g^{\mu\nu}\ln(z_i - z_j),\\
\langle \psi^\mu(z_i)\, \psi^\nu(z_j)\rangle &= \frac{g^{\mu\nu}}{z_i - z_j},\\
\langle \varphi(z_i)\, \varphi(z_j)\rangle &= -\ln(z_i - z_j)
\end{aligned} \tag{8.1}$$

we have the additional correlators between left- and right-moving fields:

$$\begin{aligned}
\langle X^\mu(z_i)\, \widetilde{X}^\nu(\overline{z}_j)\rangle &= -D^{\mu\nu}\ln(z_i - \overline{z}_j),\\
\langle \psi^\mu(z_i)\, \widetilde{\psi}^\nu(\overline{z}_j)\rangle &= \frac{D^{\mu\nu}}{z_i - \overline{z}_j},\\
\langle \varphi(z_i)\, \widetilde{\varphi}(\overline{z}_j)\rangle &= -\ln(z_i - \overline{z}_j),
\end{aligned} \tag{8.2}$$

where $D^{\mu\nu}$ is a matrix describing whether Dirichlet or Neumann boundary conditions apply to open strings attached to the brane. Here X^μ are bosonic, ψ^μ fermionic and φ super ghost fields. The tildes in (8.2) denote left-moving fields.
The mixing of left- and right-moving fields can be taken into account by yet another method, the so-called "doubling trick". Essentially this means that all left-moving fields are contracted with the matrix $D^{\mu\nu}$:

$$\begin{aligned}
\widetilde{X}^\mu(\overline{z}) &\to D^\mu_{\ \nu}\, X^\nu(\overline{z}),\\
\widetilde{\psi}^\mu(\overline{z}) &\to D^\mu_{\ \nu}\, \psi^\nu(\overline{z}),\\
\widetilde{\varphi}(\overline{z}) &\to \varphi(\overline{z}).
\end{aligned} \tag{8.3}$$

The correlators (8.2) then follow from the correlators (8.1). Later in this chapter we will be faced with twisted matter fields in which there are twist fields. The twist field correlators deserve special attention which will be done in appendix A. Here we just give the needed correlators for the amplitude in the third section. The two point correlators of bosonic twist fields are:

$$\begin{aligned}
\langle \sigma_{\theta^j}(z_1)\sigma_{-\theta^j}(z_2)\rangle &= z_{12}^{-\theta^j(1-\theta^j)},\\
\langle \tau_{\theta^j}(z_1)\tau_{-\theta^j}(z_2)\rangle &= \theta^j \cdot z_{12}^{-\theta^j(3-\theta^j)},
\end{aligned} \tag{8.4}$$

where σ_θ are the bosonic twist fields, while τ_θ are their excited fields. Accordingly the fermionic two twist contractions are:

$$\begin{aligned}
\langle s_{\theta_j}(z_1)\, s_{-\theta_j}(z_2)\rangle &= z_{12}^{-(1-\theta_j)^2},\\
\langle t_{\theta^j}(z_1)\, t_{-\theta^j}(z_2)\rangle &= z_{12}^{-(\theta^j)^2},
\end{aligned} \tag{8.5}$$

where s_θ denotes the fermionic twist. t_θ denote the excited twist fields. Furthermore there are correlators between two internal fields and two twist fields in the fermionic and bosonic sector. The according contractions are:

$$\begin{aligned}
\langle \sigma_{\theta^j}(z_1)\,\sigma_{-\theta^j}(z_2)\,\overline{\partial} Z^j(z_3)\,\partial \overline{Z}^{\,j}(z_4)\rangle &= -\,\frac{\widetilde{\overline{D}}^{\,j}}{z_{34}^2}\, z_{12}^{-\theta^j(1-\theta^j)} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{\theta^j} \left[1-\theta^j\,\frac{z_{12}z_{34}}{z_{13}z_{24}}\right],\\
\langle s_{\theta_j}(z_1)\,s_{-\theta^j}(z_2)\,\overline{\Psi}^{\,j}(z_3)\,\widetilde{\Psi}^j(z_4)\rangle &= \frac{\widetilde{\overline{D}}^{\,j}}{z_{34}}\, z_{12}^{-(1-\theta^j)^2} \left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{(1-\theta^j)}\\
\langle t_{\theta^j}(z_1)\,t_{-\theta^j}(z_2)\,\overline{\Psi}^{\,j}(z_3)\,\widetilde{\Psi}^j(z_4)\rangle &= \frac{\widetilde{\overline{D}}^{\,j}}{z_{34}}\, z_{12}^{-(\theta^j)^2} \left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{\theta^j},
\end{aligned} \tag{8.6}$$

where the other combinations of internal fields give rise to similar correlators, see appendix A for more details. The factor $\widetilde{D}^{\,j}$ accounts for mixing of boundary conditions. Since we are in the angle picture which is dual to the flux picture we have to work with the dual of (5.8) which is given in (5.29).
In the following calculations we have two open strings with momenta p_1 and p_2 and one closed string with momentum q. To translate this into our picture of four open strings we have to think about momentum conservation. Only the momentum parallel to the D-brane is conserved:

$$p_1 + p_2 + q^{\parallel} = 0, \tag{8.7}$$

where $q^{\parallel}$ is given by $q^{\parallel} = \frac{1}{2}(q + Dq)$. This means that one of the right-moving component of the closed string carries the momentum $\frac{1}{2}q$ and is inserted at z, while the left-moving component carries the momentum $\frac{1}{2}Dq$ and is inserted at $\overline{z}$. To circumvent the fractional factors in front of the momenta coming from the closed string one usually doubles the momenta leaving us with:

$$k_1 = 2\,p_1, \qquad k_2 = 2\,p_2, \qquad k_3 = q, \qquad k_4 = Dq. \tag{8.8}$$

Using (8.8) the conservation of momenta (8.7) reads:

$$k_1 + k_2 + k_3 + k_4 = 0. \tag{8.9}$$

Furthermore we remember only to consider massless particles, i.e. $k_i^2 = 0$. Traditionally amplitudes are written in terms of the so-called Mandelstam momenta which we define as:

$$s = k_1 \cdot k_2 = k_3 \cdot k_4, \qquad t = k_1 \cdot k_3 = k_2 \cdot k_4, \qquad u = k_1 \cdot k_4 = k_2 \cdot k_3. \tag{8.10}$$

From calculation one can easily see that those three Mandelstam momenta are not independent from each other. There is only one kinematic invariant in the present scattering process [24]. Commonly one picks t. In terms of t the two residual momenta are expressed by:

$$s = -2\,t, \qquad u = t.$$

The last subject of this section shall be dedicated to the fixing of vertex positions. We are concerned with tree level scattering which allows for three positions to be fixed, see chapter seven. Since we have split the closed string into its left- and right-moving sector and inserted the resulting two open string at different points of the complex plane, we have essentially four vertex positions. It is evident that the vertices of the two original open strings have to be fixed on the real line as they are to be inserted into the boundary of the disc. The two open strings which stem from the closed string are inserted into the bulk with the restriction that the right-mover is inserted at z and the left-mover at the according complex conjugate $\overline{z}$. A very convenient choice is to fix the open strings at x and $-x$ respectively and the strings in the bulk at i and $-$i [24]. This leaves exactly one position to be integrated over, namely x.

8.2. Matter-Moduli-Amplitudes in the Untwisted Sector

We are now ready to determine the metric of untwisted charged matter fields $G_{C_i\overline{C}_j}$. Remember this setup is located in a *II*B model where the mixing of boundary conditions was shifted into the brane geometry, i.e. the mixing matrix (5.4). This means that the calculation itself is completely independent of the presence of a flux until one has to specify the mixing matrix.
In the first place we are able to rule out metric entries with $i \neq j$, i.e. referring to two different internal complex planes. This follows from internal charge conservation and can as well be seen from the vertex operators (5.33). Thus we have:

$$G_{C_i\overline{C}_j} = 0, \qquad i \neq j.$$

The residual matter fields C_i and $\overline{C}_i$ have to be scattered with each modulus M^j, where M^j comprises T^j, $\overline{T}^{\,j}$, U^j and $\overline{U}^{\,j}$. From the low energy expansion of the scattering amplitude we can than read off:

$$\mathcal{A}_{C_i\overline{C}_iM^j} \sim t \cdot \partial_{M^j} G_{C_i\overline{C}_i},$$

from which we may determine $G_{C_i\overline{C}_i}$ up to a certain degree.
As stated before the matter fields are open strings inserted into the boundary of the disk while the moduli field is a closed string inserted into the bulk. In order for the super charges to add up to –2 we have to chose either the two open strings or the closed string in the zero picture where the vertex operators are in both cases

rather unwieldy. We will however chose the closed string in the –1 picture. Hence the amplitude reads:

$$\mathcal{A}_{C_i\overline{C}_i M^j} = \int \frac{\mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}^2 z_3}{V_{\mathrm{CKG}}} \langle V^{(0)}_{C_i}(z_1,k_1) V^{(0)}_{\overline{C}_i}(z_2,k_2) V^{(-1,-1)}_{M^j}(z_3,\overline{z}_3,k_3,k_4)\rangle,$$

with the vertex operators given in (5.33) and (4.9), (4.10).
For the U-modulus as well as for the $\overline{U}$-modulus the amplitude turns out to be vanishing:

$$\mathcal{A}_{C_i\overline{C}_i U^j} = 0 \qquad \text{and} \qquad \mathcal{A}_{C_i\overline{C}_i \overline{U}^j} = 0, \tag{8.11}$$

which can easily be seen from calculation, i.e. the internal spinors only occur in vanishing combinations[32]. Furthermore this property can be understood from internal charge conservation. For the matter field metric this means:

$$\partial_{U^j} G_{C_i\overline{C}_i} = 0 \qquad \text{and} \qquad \partial_{\overline{U}^j} G_{C_i\overline{C}_i} = 0. \tag{8.12}$$

Thus the metric of untwisted charged matter is not at all dependent on the U-moduli. On the other hand we have the amplitude involving the T-modulus which is unlikely to vanish. It will be computed in the following way:

$$\begin{aligned}
\mathcal{A}_{C_i\overline{C}_i T^j} =& \int \frac{\mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}^2 z_3}{V_{\mathrm{CKG}}} \langle V^{(0)}_{A_i}(z_1,k_1) V^{(0)}_{\overline{A}_i}(z_2,k_2) V^{(-1,-1)}_{T^j}(z_3,\overline{z}_3,k_3,k_4)\rangle \\
=& \frac{\mathrm{Tr}\left(\lambda_1\overline{\lambda}_2\right)}{T^j - \overline{T}^j} \int \frac{\mathrm{d}z_1 \mathrm{d}_2 \mathrm{d}^2 z_3}{V_{\mathrm{CKG}}} \langle \mathrm{e}^{-\varphi(z_3)} \mathrm{e}^{-\widetilde{\varphi}(\overline{z}_3)}\rangle \langle \mathrm{e}^{\mathrm{i}k_1 X(z_1)} \mathrm{e}^{\mathrm{i}k_2 X(z_2)} \mathrm{e}^{\mathrm{i}k_3 X(z_3)} \mathrm{e}^{\mathrm{i}k_4 X(\overline{z}_3)}\rangle \\
& \times \left\langle \left[\partial Z^i(z_1) \partial \overline{Z}^i(z_2) + k_1\cdot\psi(z_1) k_2\cdot\psi(z_2) \Psi_i(z_1) \overline{\Psi}_i(z_2)\right] \overline{\Psi}^j(z_3) \widetilde{\Psi}^j(\overline{z}_3)\right\rangle,
\end{aligned}$$

where terms containing only one external fermionic field have been discarded[33]. The bosonized super charges yield the following correlator:

$$\langle \mathrm{e}^{-\varphi(z_3)} \mathrm{e}^{-\widetilde{\varphi}(z_4)}\rangle = \frac{1}{z_{34}},$$

where $\overline{z}_3$ has been renamed to z_4 for reasons of convenience. The correlator of the sources of momenta will occur in all amplitudes under consideration in this treatise. Henceforth we will refer to it by $\mathcal{E}$. For $\mathcal{E}$ we easily find:

$$\mathcal{E} = \langle \mathrm{e}^{\mathrm{i}k_1\cdot x(z_1)} \mathrm{e}^{\mathrm{i}k_2\cdot x(z_2)} \mathrm{e}^{\mathrm{i}k_3\cdot X(z_3)} \mathrm{e}^{\mathrm{i}k_4\cdot X(z_4)}\rangle = \left[\frac{z_{13} z_{14} z_{23} z_{24}}{z_{12}^2 z_{34}^2}\right]^t.$$

[32] Remember: $\langle \Psi(z)\Psi(\overline{z})\rangle = 0$.

[33] Note that the external fermionic fields do not meet any partner to be contracted with.

The first contribution of the internal part is especially simple as it only contains two point correlators:

$$\langle \partial Z^i(z_1)\partial \overline{Z}^{\,i}(z_2)\overline{\Psi}^{\,j}(z_3)\widetilde{\Psi}^j(z_4)\rangle = -\frac{\overline{D}^{\,j}}{z_{12}^2 z_{34}}.$$

The second contribution contains a four point correlator which can be evaluated using Wick's theorem or bosonization:

$$\Big\langle k_1\psi(z_1)k_2\psi(z_2)\Psi^i(z_1)\overline{\Psi}^{\,i}(z_2)\overline{\Psi}^{\,j}(z_3)\widetilde{\Psi}^j(z_4)\Big\rangle = -\frac{2t}{z_{12}}\left[\frac{\overline{D}^{\,j}}{z_{12}z_{34}} - \frac{\delta^{ij}}{z_{13}z_{24}}\right].$$

Putting all this together we find for the amplitude:

$$\mathcal{A}_{C_i\overline{C}_iT^j} = -\frac{\mathrm{Tr}\left(\lambda_1\overline{\lambda}_2\right)\overline{D}^{\,j}}{T^j - \overline{T}^{\,j}}\int \frac{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}^2z_3}{V_{\mathrm{CKG}}}\;\frac{\mathcal{E}}{z_{34}}\left[\frac{1+2t}{z_{12}^2 z_{34}} - \frac{2t\cdot\delta^{ij}}{z_{12}z_{13}z_{24}}\right]$$

Due to modular invariance we can fix three vertex positions. A convenient choice is:

$$z_1 = x,\quad z_2 = -x,\quad z_3 = \mathrm{i},\quad \text{and}\quad \overline{z}_3 = z_4 = -\mathrm{i}.$$

We have to insert ghost fields for each vertex we fix. The ghost field have the following correlator:

$$\langle c(z_2)\,c(z_3)\,c(z_4)\rangle = z_{23}\,z_{24}\,z_{34} = 2\mathrm{i}(x^2+1).$$

The amplitude now takes the form:

$$\begin{aligned}\mathcal{A}_{C_i\overline{C}_iT^j} &= -\frac{\mathrm{Tr}\left(\lambda_1\overline{\lambda}_2\right)\overline{D}^{\,j}}{T^j - \overline{T}^{\,j}}2^{-4t}\int \mathrm{d}x\, x^{-2t}(x-\mathrm{i})^{2t+1}(x+\mathrm{i})^{2t+1}\left[\frac{1+2t}{8\mathrm{i}x^2} + \frac{t\cdot\delta^{ij}}{x(x-\mathrm{i})^2}\right]\\ &= -\frac{\mathrm{Tr}\left(\lambda_1\overline{\lambda}_2\right)\overline{D}^{\,j}}{T^j - \overline{T}^{\,j}}2^{-4t}\left[\frac{1+2t}{8\mathrm{i}}I(-2t-1,0) + t\cdot\delta^{ij}I(-2t,-1)\right],\end{aligned}$$

where the integral $I(\delta,\alpha)$ is defined as:

$$I(\delta,\alpha) = \int \mathrm{d}x\; x^{\delta-1}(x-\mathrm{i})^{\alpha-\delta}(x+\mathrm{i})^{-\alpha-\delta}. \tag{8.13}$$

It has the following representation in terms of gamma functions:

$$I(\delta,\alpha) = \frac{2^{-\delta}}{\sqrt{\pi}}\,\mathrm{e}^{-\frac{1}{2}\pi\mathrm{i}\alpha}\,\frac{\Gamma\left(\frac{\delta}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\delta}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\delta}{2}-\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\delta}{2}+\frac{\alpha}{2}\right)}. \tag{8.14}$$

Using this formula we finally have for the amplitude of two untwisted matter fields and a T-modulus:

$$\mathcal{A}_{C_i\overline{C}_iT^j} = -\operatorname{Tr}\left(\lambda_1\overline{\lambda}_2\right)\frac{\overline{D}^{\,j}}{T^j-\overline{T}^{\,j}}\,2^{-1-2t}\,\frac{\mathrm{i}}{\sqrt{\pi}}\left[1-2\cdot\delta^{ij}\right]\frac{\Gamma\left(\frac{1}{2}-t\right)}{\Gamma(-t)}. \tag{8.15}$$

This calculation can be repeated with the $\overline{T}$-modulus. The result is the same amplitude with opposite sign[34] and the conjugate of $\overline{D}^{\,j}$ [35], that is:

$$\mathcal{A}_{C_i\overline{C}_i\overline{T}^{\,j}} = +\operatorname{Tr}\left(\lambda_1\overline{\lambda}_2\right)\frac{D^j}{T^j-\overline{T}^{\,j}}\,2^{-1-2t}\,\frac{\mathrm{i}}{\sqrt{\pi}}\left[1-2\cdot\delta^{ij}\right]\frac{\Gamma\left(\frac{1}{2}-t\right)}{\Gamma(-t)}. \tag{8.16}$$

Thus we have seen that the matter metric is dependent on both T-moduli. To further extract this dependence we expand the amplitudes for low momenta t [36] and equate them to the according metric derivatives:

$$\begin{aligned}
\mathcal{A}_{C_i\overline{C}_iT^j} &= +\frac{1}{2}\,\mathrm{i}\,t\frac{\overline{D}^{\,j}}{T^j-\overline{T}^{\,j}}(1-2\,\delta^{ij})+\mathcal{O}(t^2) = +t\,\partial_{T^j}G_{C_i\overline{C}_i}+\mathcal{O}(t^2),\\
\mathcal{A}_{C_i\overline{C}_i\overline{T}^{\,j}} &= -\frac{1}{2}\,\mathrm{i}\,t\frac{D^j}{T^j-\overline{T}^{\,j}}(1-2\,\delta^{ij})+\mathcal{O}(t^2) = -t\,\partial_{\overline{T}^{\,j}}G_{C_i\overline{C}_i}+\mathcal{O}(t^2).
\end{aligned} \tag{8.17}$$

Note that equating the amplitudes to metric derivatives makes sense since this expansion agrees with the according effective action of massless NS-states on D-branes, c.f. [1]. Moreover general relations between string amplitudes and geometry have been discussed in [25] for heterotic strings. From (8.17) one can read off the following differential equations for the T and $\overline{T}$ dependence of the metric:

$$\begin{aligned}
\partial_{T^j}G_{C_i\overline{C}_i} &= +\frac{1}{2}\frac{\overline{D}^{\,j}}{T^j-\overline{T}^{\,j}}(1-2\,\delta^{ij})\,G_{C_i\overline{C}_i},\\
\partial_{\overline{T}^{\,j}}G_{C_i\overline{C}_i} &= -\frac{1}{2}\frac{D^j}{T^j-\overline{T}^{\,j}}(1-2\,\delta^{ij})\,G_{C_i\overline{C}_i},
\end{aligned} \tag{8.18}$$

where we had to renormalize the matter field vertex operators $V_{C_i}\to G^{1/2}_{C_i\overline{C}_i}V_{C_i}$ in going from the string basis to the field theory basis. This renormalization explains the $G_{C_i\overline{C}_i}$ on the r.h.s. of (8.18).
As we stated at the beginning of the section the calculation is up to now independent of the flux which is wholly encapsulated in the mixing parameter D^j. However, to solve (8.18) we have to specify D^j.

---

[34] This is because of the normalization of the vertex operator, i.e. $\frac{1}{\overline{T}^{\,j}-T^j} = -\frac{1}{T^j-\overline{T}^{\,j}}$.

[35] Note that $\langle\overline{\Psi}\ \widetilde{\Psi}\rangle \sim \overline{D}^{\,j}$, while $\langle\Psi\ \widetilde{\overline{\Psi}}\rangle \sim D^j$ and accordingly for the internal bosonic fields.

[36] This is best to be done using *Mathematica*.

We will start with turned off flux, i.e. $f^j \to 0$. In this limit the mixing parameter becomes $+1$ for Neumann boundary conditions or -1 for Dirichlet conditions. Considering C_i^9 fields the situation is very simple as all open string components obey Neumann conditions and we have $D^j = +1,\ \forall j$. The equations (8.18) integrate in this case to:

$$G_{C_i^9 \overline{C}_i^9} = \alpha'^{-3/2} \mathrm{e}^{-\varphi_4} \sqrt{\frac{(T^k - \overline{T}^k)\,(T^l - \overline{T}^l)}{T^i - \overline{T}^i}}, \qquad (i,k,l) = \overline{(1,2,3)}, \tag{8.19}$$

where the factor $\alpha'^{-3/2} \mathrm{e}^{-\varphi_4}$ stems from the factor $\mathrm{e}^{\varphi\chi}$ in (7.1) which we have suppressed so far. Note that the Euler number of the disc on which the scattering takes place equals one ($g = c = 0$, $b = 1$). (8.19) can be recast in the field theory basis, we find:

$$G_{C_i^9 \overline{C}_i^9} = \frac{\kappa_4^{-2}}{t^i + \overline{t}^i}, \tag{8.20}$$

with κ_4 defined in (6.10). Let us now turn to the cases in which the flux is non-vanishing. The flux dependence of D^j is given in (5.8). Putting this into (8.18) we have:

$$\begin{aligned} \partial_{T^j} G_{C_i \overline{C}_i} &= + \frac{1}{2} \frac{1}{T^j - \overline{T}^j} \frac{\overline{T}^j + 2\pi\alpha' F^j}{T^j + 2\pi\alpha' F^j} (1 - 2\,\delta^{ij})\, G_{C_i \overline{C}_i}, \\ \partial_{\overline{T}^j} G_{C_i \overline{C}_i} &= - \frac{1}{2} \frac{1}{T^j - \overline{T}^j} \frac{T^j + 2\pi\alpha' F^j}{\overline{T}^j + 2\pi\alpha' F^j} (1 - 2\,\delta^{ij})\, G_{C_i \overline{C}_i}, \end{aligned} \tag{8.21}$$

which integrates to:

$$G_{C_i^9 \overline{C}_i^9} = \alpha'^{-3/2} \mathrm{e}^{-\varphi_4} \sqrt{\frac{(T^k - \overline{T}^k)\,(T^l - \overline{T}^l)}{T^i - \overline{T}^i} \frac{|T^k + 2\pi\alpha' F^k||T^l + 2\pi\alpha' F^l|}{|T^i + 2\pi\alpha' F^i|}}. \tag{8.22}$$

This result can be simplified using the supersymmetry condition (5.9):

$$G_{C_i^9 \overline{C}_i^9} = \alpha'^{-3/2} \mathrm{e}^{-\varphi_4} \sqrt{\frac{(T^k - \overline{T}^k)\,(T^l - \overline{T}^l)}{T^i - \overline{T}^i}} \left| 1 - \frac{f^k f^l}{\mathrm{Im}(T^k)\, \mathrm{Im}(T^l)} \right|. \tag{8.23}$$

From (8.23) one can formidably see that it reproduces (8.19) in the limit of vanishing fluxes f. Expressed in terms of the field theory moduli we finally have:

$$G_{C_i^9 \overline{C}_i^9} = \frac{\kappa_4^{-2}}{t^i + \overline{t}^i} \left| 1 - \frac{f^k f^l}{\mathrm{Im}(T^k)\, \mathrm{Im}(T^l)} \right|. \tag{8.24}$$

This result directly takes over to the untwisted sector of the angle picture. Using (5.23) we get:

$$G_{C_i^9 \overline{C}_i^9} = \frac{\kappa_4^{-2}}{u^i + \overline{u}^i} \left| 1 - \tan(\pi\,\theta^k)\ \tan(\pi\,\theta^l) \right|. \tag{8.25}$$

Next we turn to the twisted sector in which we want to repeat above calculations. The twisted sector amounts to take the calculations into the T-dual picture which is type *II*A. Unlike in the flux picture in the angle picture calculations are well depending on the intersection angle which are introduced through twist fields. These have their very own correlators (see appendix A). In the fourth chapter we have learned that T-dualizing a type *II*B string theory essentially interchanges the moduli. So we expect the trivial findings for the U-modulus in the preceding section to take over to the T-modulus in the present one. Accordingly the non-trivial calculations will have to be done with the U-modulus.
From similar considerations as in the untwisted sector we can conclude that the twisted amplitude with the T-modulus as well as with the anti-modulus $\overline{T}$ is again vanishing:

$$\mathcal{A}_{C_\theta \overline{C}_\theta T^j} = 0 \qquad \text{and} \qquad \mathcal{A}_{C_\theta \overline{C}_\theta \overline{T}^j} = 0, \tag{8.26}$$

where T^j should be seen regarded as $-\alpha'/T'^j$ and similarly for $\overline{T}^j$ and in the following for U^j and $\overline{U}^j$. Before computing the according amplitude involving the U-modulus we have to think about whether to put the twisted matter fields in the zero picture or the moduli field. Putting the matter fields in the zero picture would involve excited twist fields making the calculation even more intricate. On the other hand in the zero picture the moduli fields contain internal bosonic fields which have rather lengthy contractions with the twist fields. We will however present both ways here starting with the latter choice, thus we have to calculate:

$$\mathcal{A}_{C_\theta \overline{C}_\theta U^j} = \int \frac{\mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}^2 z_3}{V_{\mathrm{CKG}}} \langle V^{(-1)}_{C_\theta}(z_1, k_1) V^{(-1)}_{\overline{C}_\theta}(z_2, k_2) V^{(0,0)}_{U^j}(z_3, \overline{z}_3, k_3, k_4) \rangle,$$

with the vertex operators (4.24) and (5.34). The external contribution to the expectation value is rather simple. From the exponential factor we receive the same factor $\mathcal{E}$ as before. The super charges contribute a factor $1/z_{12}$. For the internal part we have to evaluate the following correlator:

$$\Big\langle \prod_{i=1}^{3} s_{\theta^i}(z_1)\, s_{-\theta^i}(z_2)\, \sigma_{-\theta^i}(z_1)\, \sigma_{\theta^i}(z_2) \times \\ \Big[\partial \overline{Z}^j(z_3)\, \overline{\partial} Z^j(z_4) + k_3 \cdot \psi(z_3)\; k_4 \cdot \widetilde{\psi}(z_4)\; \overline{\Psi}^j(z_3) \widetilde{\Psi}^j(z_4) \Big] \Big\rangle,$$

where we have again discarded the terms containing only one external fermionic field. $\overline{z}_3$ has again been renamed to z_4. We will evaluate this correlator in two steps, beginning with the bosonic part of the modulus. We can write:

$$X_1 = \langle \prod_{i=1}^{3} s_{\theta^i}(z_1)\, s_{-\theta^i}(z_2) \rangle \langle \prod_{i \neq j}^{3} \sigma_{-\theta^i}(z_1)\, \sigma_{\theta^i}(z_2) \rangle \langle \sigma_{-\theta^j}(z_1)\, \sigma_{\theta^j}(z_2)\; \partial \overline{Z}^j(z_3) \overline{\partial} Z^j(z_4) \rangle.$$

The first factor is simply a product of three two point functions of fermionic twist fields which are given in (8.5). We find:

$$\langle\prod_{i=1}^{3} s_{\theta^i}(z_1)\, s_{-\theta^i}(z_2)\rangle = \prod_{i=1}^{3} z_{12}^{-(1-\theta^i)^2} = z_{12}^{-\sum_{i=1}^{3}(1-\theta^i)^2} .$$

For the second factor we have to heed the fact that in the j-th dimension the twist fields contract with the internal bosonic fields (8.6) while in the other dimensions we have the usual two point correlators of twist fields (8.4). Note however that the angles in (5.34) are in opposite order than in (8.6) and (8.4). In order to use these we have to either interchange z_1 and z_2 or replace θ by $1-\theta$. This leaves us with:

$$\begin{aligned}&\langle\prod_{i\neq j}^{3} \sigma_{-\theta_i}(z_1)\, \sigma_{\theta_i}(z_2)\rangle\langle\sigma_{-\theta^j}(z_1)\, \sigma_{\theta^j}(z_2)\, \partial\overline{Z}^{\,j}(z_3)\, \overline{\partial} Z^j(z_4)\rangle \\ &= -\prod_{i\neq j}^{3} z_{12}^{-(1-\theta^i)\,\theta^i}\, z_{12}^{-(1-\theta^j)\,\theta^j}\, \frac{\widetilde{\overline{D}}^{\,j}}{z_{34}^2}\left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{1-\theta^j}\left[1-(1-\theta^j)\,\frac{z_{12}z_{34}}{z_{14}z_{23}}\right].\end{aligned}$$

Taking into account the supersymmetry condition (5.28) the bosonic part eventually is:

$$X_1 = -\frac{\widetilde{\overline{D}}^{\,j}}{z_{12}z_{34}^2}\left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{1-\theta^j}\left[1-(1-\theta^j)\,\frac{z_{12}z_{34}}{z_{14}z_{23}}\right].$$

The fermionic part can be done in a similar way. We can write:

$$\begin{aligned}X_2 = {} & \langle k_3\cdot\psi(z_3)\ k_4\cdot\widetilde{\psi}(z_4)\rangle \times \\ & \prod_{i\neq j}^{3}\langle s_{\theta_i}(z_1) s_{-\theta^i}(z_2)\rangle\langle s_{\theta^j}(z_1) s_{-\theta^j}(z_2)\overline{\Psi}^{\,j}(z_3)\widetilde{\Psi}^{\,j}(z_4)\rangle\prod_{i=1}^{3}\langle\sigma_{-\theta^i}(z_1)\sigma_{\theta^i}(z_2)\rangle,\end{aligned}$$

where the first factor is simply $-2t/z_{34}$. Note that since we only move pairs of spinors the permutation is always even, that is we do not encounter any signs. The fermionic contribution is finally:

$$X_2 = -\frac{2t}{z_{12}}\,\frac{\widetilde{\overline{D}}^{\,j}}{z_{34}^2}\left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{1-\theta^j} .$$

Altogether we have for the amplitude:

$$\mathcal{A}_{C_\theta\overline{C}_\theta U^j} \sim -\int\frac{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}^2z_3}{V_{\mathrm{CKG}}}\,\frac{\mathcal{E}}{z_{12}^2}\,\frac{\widetilde{\overline{D}}^{\,j}}{z_{34}^2}\times\left[(1+2t)-(1-\theta^j)\,\frac{z_{12}z_{34}}{z_{14}z_{23}}\right]\cdot\left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{1-\theta^j} .$$

Again we fix three vertex positions inserting according ghost fields which leaves us with:

$$\mathcal{A}_{C_\theta\overline{C}_\theta U^j} \sim \overline{\widetilde{D}}{}^j 2^{-3-4t}\mathrm{i}\int \mathrm{d}x\; x^{-2t-2}(x-\mathrm{i})^{2t+1}(x+\mathrm{i})^{2t+1}$$
$$\times\left[(1+2t)\,\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{2(1-\theta^j)} - 4\mathrm{i}(1-\theta^j)\frac{x}{(x+\mathrm{i})^2}\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^{2(1-\theta^j)}\right].$$

This can be written in terms of the formerly defined integral (8.13) and solved by (8.14):

$$\begin{aligned}\mathcal{A}_{C_\theta\overline{C}_\theta U^j} &= 2^{-3-4t}\,\mathrm{i}\;\mathrm{Tr}\left(\lambda_1\widetilde{\lambda}_2\right)\frac{\overline{\widetilde{D}}{}^j}{U'^j-\overline{U}'^j}\frac{|U'|^2}{\alpha'}\\
&\quad\times\Big[(1+2t)\cdot I(-2t-1,2\theta^j-2)-4\mathrm{i}(1-\theta^j)\cdot I(-2t,2\theta^j-1)\Big]\\
&= 2^{-1-2t}\frac{\mathrm{i}}{\sqrt{\pi}}\mathrm{Tr}\left(\lambda_1\widetilde{\lambda}_2\right)\frac{\overline{\widetilde{D}}{}^j}{U'^j-\overline{U}'^j}\frac{|U'|^2}{\alpha'}\\
&\quad\times\left[\frac{\theta^j+t+1}{\theta^j-t}+\frac{\theta^j-1}{\theta^j+t}\right]\mathrm{e}^{-\mathrm{i}\pi\theta^j}\frac{\Gamma(\frac{1}{2}-t)\Gamma(-t)}{\Gamma(\theta^j-t)\Gamma(-\theta^j-t)}.\end{aligned}$$

We will now repeat this calculation with the matter fields in the zero picture and the modulus in the -1 picture, that is we have:

$$\mathcal{A}_{C_\theta\overline{C}_\theta U_j} = \int\frac{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}^2z_3}{V_{\mathrm{CKG}}}\langle V^{(0)}_{C_\theta}(z_1,k_1)V^{(0)}_{\overline{C}_\theta}(z_2,k_2)V^{(-1,-1)}_{U_j}(z_3,\overline{z}_3,k_3,k_4)\rangle,$$

with the vertex operators given in (4.22) and (5.35). The external contribution is again very simple. We have the usual $\mathcal{E}$ from the exponentials and a factor $1/z_{34}$ from the super charges of the U-modulus. The internal contribution is much more intricate. Fortunately the twist fields do not produce two point correlators with their excited twist fields which safes much work. Thus we have to compute the following expectation value:

$$\Bigg\langle\Bigg[\sum_{l=1}^{3}\tau_{-\theta^l}(z_1)\,\tau_{\theta^l}(z_2)\;t_{\theta^l}(z_1)\,t_{-\theta^l}(z_2)\prod_{\substack{i=1\\i\neq l}}^{3}\sigma_{-\theta^i}(z_1)\,\sigma_{\theta^i}(z_2)\;s_{\theta^i}(z_1)\,s_{-\theta^i}(z_2)+$$
$$k_1\cdot\psi(z_1)\;k_2\cdot\psi(z_2)\prod_{i=1}^{3}\sigma_{-\theta^i}(z_1)\,\sigma_{\theta^i}(z_2)\;s_{\theta^i}(z_1)\,s_{-\theta^i}(z_2)\Bigg]\times\overline{\Psi}{}^j(z_3)\widetilde{\Psi}^j(z_4)\Bigg\rangle$$

We will again evaluate this correlator in two steps beginning with the first term. As there are no further internal bosonic fields the bosonic twist correlator is rather simple:

$$\langle\tau_{-\theta^l}(z_1)\,\tau_{\theta^l}(z_2)\rangle\prod_{\substack{i=1\\i\neq l}}^{3}\langle\sigma_{-\theta^i}(z_1)\,\sigma_{\theta^i}(z_2)\rangle = -(1-\theta^l)\cdot z_{12}^{(\theta^l+1)(\theta^l+2)}\prod_{\substack{i=1\\i\neq l}}^{3}z_{12}^{-(1-\theta^i)\,\theta^i}.$$

On the fermionic side we have contractions between spin twist fields and internal spinors. We have to compute:

$$
\begin{aligned}
&\langle t_{\theta^l}(z_1)\, t_{-\theta^l}(z_2) \prod_{\substack{i=1\\ i\neq l}}^{3} s_{\theta^k}(z_1)\, s_{-\theta^k}(z_2) \overline{\Psi}^{\,j}(z_3) \widetilde{\Psi}^j(z_4)\rangle \\
&= \delta_{lj}\cdot z_{12}^{-(\theta^l)^2} \frac{\overline{\widetilde{D}}^{\,j}}{z_{34}} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{\theta^l} \prod_{\substack{i=1\\ i\neq l}}^{3} z_{12}^{-(1-\theta^i)^2} + \\
&\quad \delta_{ij}\cdot z_{12}^{-(\theta^l)^2} \prod_{\substack{i=1\\ i\neq j,\, i\neq l}}^{3} z_{12}^{-(1-\theta^i)^2} \prod_{\substack{i=1\\ i=j,\, i\neq l}}^{3} z_{12}^{-(1-\theta^i)^2} \frac{\overline{\widetilde{D}}^{\,j}}{z_{34}} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{\theta^i-1} .
\end{aligned}
$$

Putting this together with the bosonic part of the first term yields:

$$
X_1 = -\frac{\overline{\widetilde{D}}^{\,j}}{z_{12}^2 z_{34}} \sum_{l=1}^{3}(1-\theta^l) \left[\delta_{lj}\cdot \left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{-1} + (1-\delta_{lj})\right] \left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{1-\theta^j} .
$$

Performing the sum over the angles get us to:

$$
\begin{aligned}
X_1 &= -\frac{\overline{\widetilde{D}}^{\,j}}{z_{12}^2 z_{34}} \left[(1-\theta^j)\, \frac{z_{13}z_{24}}{z_{14}z_{23}} + \theta^j\right] \left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{1-\theta^j} \\
&= -\frac{\overline{\widetilde{D}}^{\,j}}{z_{12}^2 z_{34}} \left[1-(1-\theta^j)\, \frac{z_{12}z_{34}}{z_{14}z_{23}}\right] \left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{1-\theta^j} ,
\end{aligned}
$$

where we again made use of the supersymmetry condition (5.28). The second part is somewhat easier as it does not comprise excited twist fields. For its bosonic part we readily find:

$$
\prod_{i=1}^{3}\langle\sigma_{-\theta^i}(z_1)\sigma_{\theta^i}(z_2)\rangle = \prod_{i=1}^{3} z_{12}^{-(1-\theta^i)\,\theta^i} ,
$$

while its fermionic part yields:

$$
\langle\prod_{i=1}^{3} s_{\theta^i}(z_1)\, s_{-\theta^i}(z_2)\; \overline{\Psi}^{\,j}(z_3)\; \widetilde{\Psi}^j(z_4)\rangle = \prod_{\substack{i=1\\ i\neq j}}^{3} z_{12}^{-(1-\theta^i)^2}\, z_{12}^{-(1-\theta^j)^2} \frac{\overline{\widetilde{D}}^{\,j}}{z_{34}} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{\theta^j-1} .
$$

Together we find for the non excited part:

$$
X_2 = -\frac{\overline{\widetilde{D}}^{\,j}}{z_{12}^2 z_{34}} 2t \left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{1-\theta^j} .
$$

Finally the amplitude as a whole reads:

$$\mathcal{A}_{C_\theta \overline{C}_\theta U^j} \sim -\int \frac{\mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}^2 z_3}{V_{\mathrm{CKG}}} \frac{\mathcal{E}}{z_{12}^2} \frac{\overline{\widetilde{D}}^j}{z_{34}^2} \times \left[(1+2t) - (1-\theta^j)\, \frac{z_{12} z_{34}}{z_{14} z_{23}}\right] \cdot \left(\frac{z_{14} z_{23}}{z_{13} z_{24}}\right)^{1-\theta^j},$$

which coincides as expected with the according calculation without excited twist fields, where we had found:

$$\begin{aligned}\mathcal{A}_{C_\theta \overline{C}_\theta U^j} &= 2^{-1-2t} \frac{\mathrm{i}}{\sqrt{\pi}} \mathrm{Tr}\left(\lambda_1 \widetilde{\lambda}_2\right) \frac{\overline{\widetilde{D}}^j}{U'^j - \overline{U}'^j} \frac{|U'|^2}{\alpha'} \\ &\quad \times \left[\frac{\theta^j + t + 1}{\theta^j - t} + \frac{\theta^j - 1}{\theta^j + t}\right] \mathrm{e}^{-\mathrm{i}\pi\theta^j} \frac{\Gamma(\frac{1}{2} - t)\Gamma(-t)}{\Gamma(\theta^j - t)\Gamma(-\theta^j - t)}.\end{aligned} \tag{8.27}$$

Repeating the calculation for the according conjugate modulus yields:

$$\begin{aligned}\mathcal{A}_{C_\theta \overline{C}_\theta \overline{U}^j} &= -\, 2^{-1-2t} \frac{\mathrm{i}}{\sqrt{\pi}} \mathrm{Tr}\left(\lambda_1 \widetilde{\lambda}_2\right) \frac{\widetilde{D}^j}{U'^j - \overline{U}'^j} \frac{|U'|^2}{\alpha'} \\ &\quad \times \left[\frac{\theta^j + t + 1}{\theta^j - t} + \frac{\theta^j - 1}{\theta^j + t}\right] \mathrm{e}^{-\mathrm{i}\pi\theta^j} \frac{\Gamma(\frac{1}{2} - t)\Gamma(-t)}{\Gamma(\theta^j - t)\Gamma(-\theta^j - t)}.\end{aligned} \tag{8.28}$$

As in the untwisted case we will now expand the amplitudes (8.27) and (8.28) for low momenta:

$$\begin{aligned}\mathcal{A}_{C_\theta \overline{C}_\theta U^j} &= \frac{\overline{\widetilde{D}}^j}{U'^j - \overline{U}'^j} \frac{|U'|^2}{\alpha'} \frac{\mathrm{i}}{2\pi}\, \mathrm{e}^{-\mathrm{i}\pi\theta^j} \sin(\pi\theta^j) \Big[1 + t\big(2\gamma_E + \psi(\theta^j) + \psi(1-\theta^j)\big) + \mathcal{O}(t^2)\Big], \\ \mathcal{A}_{C_\theta \overline{C}_\theta \overline{U}^j} &= -\frac{\widetilde{D}^j}{U'^j - \overline{U}'^j} \frac{|U'|^2}{\alpha'} \frac{\mathrm{i}}{2\pi}\, \mathrm{e}^{\mathrm{i}\pi\theta^j} \sin(\pi\theta^j) \Big[1 + t\big(2\gamma_E + \psi(\theta^j) + \psi(1-\theta^j)\big) + \mathcal{O}(t^2)\Big].\end{aligned} \tag{8.29}$$

Here γ_E denotes Euler's constant. This constant arises in the so called Euler's infinite product describing the reciprocal of the gamma function and is given by:

$$\gamma_E = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln n\right] \sim 0.577.$$

Further ψ is the digamma function $\psi(z) = \partial_z \ln\Gamma(z)$. From this expansion we can again obtain the matter metrics by picking the terms linear in t, casting the amplitudes into the field theory basis by redefining the matter field vertices $V_{C_\theta} \to G^{1/2}_{C_\theta \overline{C}_\theta} V_{C_\theta}$ which produces an additional factor of $G_{C_\theta \overline{C}_\theta}$ and equating to the U derivatives of $G_{C_\theta \overline{C}_\theta}$. Doing this we arrive with the following differential equations:

$$\begin{aligned}\partial_{U'^j} G_{C_\theta \overline{C}_\theta} &= \frac{\overline{U}'^j}{U'^j} \frac{\overline{\widetilde{D}}^j}{U'^j - \overline{U}'^j} \frac{\mathrm{i}}{2} \mathrm{e}^{-\mathrm{i}\pi\theta^j} \sin(\pi\,\theta^j)\big(2\gamma_E + \psi(\theta^j) + \psi(1-\theta^j)\big) G_{C_\theta \overline{C}_\theta}, \\ \partial_{\overline{U}'^j} G_{C_\theta \overline{C}_\theta} &= -\frac{U'^j}{\overline{U}'^j} \frac{\widetilde{D}^j}{U'^j - \overline{U}'^j} \frac{\mathrm{i}}{2} \mathrm{e}^{\mathrm{i}\pi\theta^j} \sin(\pi\,\theta^j)\big(2\gamma_E + \psi(\theta^j) + \psi(1-\theta^j)\big) G_{C_\theta \overline{C}_\theta},\end{aligned} \tag{8.30}$$

where we used $\frac{\partial}{\partial U^j} = \frac{(U^j)^2}{\alpha'}\frac{\partial}{\partial U'^j}$ with $U^j = -\frac{\alpha'}{U'^j}$. The moduli are related to the intersecting angle (5.24). Thus we can express the moduli derivatives in terms of angles:

$$\begin{aligned}\frac{\partial \theta^j}{\partial U^j} &= \frac{1}{\pi}\frac{\overline{U}^j}{U^j}\frac{1}{U^j - \overline{U}^j}\ \mathrm{e}^{\mathrm{i}\pi\theta^j}\sin(\pi\,\theta^j),\\ \frac{\partial \theta^j}{\partial U^j} &= -\frac{1}{\pi}\frac{\overline{U}^j}{U^j}\frac{1}{U^j - \overline{U}^j}\ \mathrm{e}^{-\mathrm{i}\pi\theta^j}\sin(\pi\,\theta^j).\end{aligned} \tag{8.31}$$

The dual mixing matrix entry $\widetilde{D}^j$ in type IIA was explicitly given in (5.29). With (8.31) and (5.29) the differential equations (8.30) can be put into the following form:

$$\partial_{\theta^j}\ln G_{C_\theta\overline{C}_\theta} = \frac{1}{2}\big(2\gamma_E + \psi(\theta^j) + \psi(1-\theta^j)\big), \tag{8.32}$$

which integrates to:

$$G_{C_\theta\overline{C}_\theta} = \exp\Big(\sum_{j=1}^{3}\theta^j\gamma_E\Big)\prod_{j=1}^{3}\sqrt{\frac{\Gamma(\theta^j)}{\Gamma(1-\theta^j)}}. \tag{8.33}$$

The first factor looks somewhat disturbing but recalling the supersymmetry condition (5.28) the exponent is vanishing which leaves us with:

$$G_{C_\theta\overline{C}_\theta} \sim \prod_{j=1}^{3}\sqrt{\frac{\Gamma(\theta^j)}{\Gamma(1-\theta^j)}}\times\frac{1}{(T^j - \overline{T}^j)^{1-\theta^j}}, \tag{8.34}$$

where the T dependent terms had to be added[37] but cannot be understood from the preceding calculations and are actually in contrast to (8.27). This indicates that the situation is more complex than expected and has to be refined. This is under current investigation in [1] and will not be included here.

[37] One reason is the Kähler potential given in [23]. There the terms describing strings stretched between 5- and 9-branes $\frac{|C^{95}|^2}{(t^1+\bar{t}^1)^{1/2}(t^2+\bar{t}^2)^{1/2}}$ were derived using supersymmetry arguments. Of course our findings should be able to reproduce this result for a certain choice of angles.

9. Four Matter Field Amplitudes

In this chapter we want to determine the Yukawa couplings of matter fields. Knowing these we can construct the superpotential W. Therefore we will calculate the four matter field amplitudes in the twisted and untwisted sector. The Yukawa couplings will appear in a suitable factorization limit.

9.1. The General Setup for Open String Scattering from D-branes

Scattering processes involving only open strings are much easier that mixed scattering. All participating strings are inserted into the boundary of the disk. Again the disk is mapped to the upper half plane H_+ which entails the boundary to be the real line. Thus all vertex positions are real valued. Contractions of any fields are evaluated using (8.1).

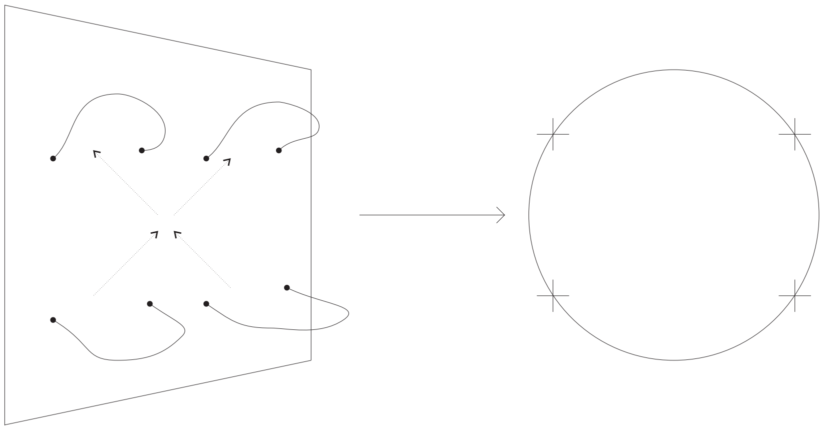

Fig. 10 Scattering of open strings from a D-brane: The open disk endpoints are attached to the brane. The calculation is carried out with the open string vertices all inserted into the boundary of the disk which is the D-brane world volume.

Since we are only dealing with original open strings the setup of the momenta carried by the strings is much simpler. All four strings enter with a momentum k_i each. Conservation of momentum reads:

$$k_1 + k_2 + k_3 + k_4 = 0. \tag{9.1}$$

The Mandelstam momenta are defined in the same manner as in (8.10). Note however that the momenta k_i entering the formula are not the same. Indeed this time the three Mandelstam momenta depend only through momentum conservation, that is:

$$s + t + u = 0. \tag{9.2}$$

Basically this means there are two kinematic invariants in this process. But commonly one keeps all three.
Again we are faced with twisted matter. The correlators in (8.6) are not needed, instead we are confronted with correlators involving four twist fields. The fact that we consider matter fields with different intersecting angles makes the calculation rather intricate. We will thus drop it and instead refer to [26] or similar sources, see appendix A. The correlator of four bosonic twist fields in the considered setup is given by:

$$\langle \sigma_{\theta^j}(z_1)\,\sigma_{-\theta^j}(z_2)\,\sigma_{\nu^j}(z_3)\,\sigma_{-\nu^j}(z_4)\rangle = z_{12}^{-\theta^j(1-\theta^j)} z_{34}^{-\nu^j(1-\nu^j)} \times \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{\frac{1}{2}\theta^j(1-\nu^j)+\frac{1}{2}\nu^j(1-\theta^j)} \times I_j^{-\frac{1}{2}}(x), \tag{9.3}$$

where:

$$I_j(x) = \frac{\sin(p\,\pi\,\theta_j)}{\pi}\Big[B_{1;j}G_{2;j}(x)H_{1;j}(1-x) + B_{2;j}G_{1;j}(x)H_{2;j}(1-x)\Big]. \tag{9.4}$$

The ingredients of formula (9.4) are given in appendix A. On the other side we have the correlator involving four spin twists, which can in contrast to the bosonic case be easily obtained from bosonizing the fermionic fields, see appendix A. The result is:

$$\langle s_{\theta^j}(z_1)\,s_{-\theta^j}(z_2)\,s_{\nu^j}(z_3)\,s_{-\nu^j}(z_4)\rangle = z_{12}^{-(1-\theta^j)^2} z_{34}^{-(1-\nu^j)^2} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{(1-\theta^j)(1-\nu^j)}. \tag{9.5}$$

We will later see that the excited twist fields in (5.35) drop out due to charge conservation. Thus we are not faced with correlators of four excited twist fields.
Again this introductory section will be closed with the fixing of vertices. We have four completely independent vertex positions which are all located on the real line. As usual three positions can be fixed. Traditionally one chooses the points 0, 1 and ∞ leaving the fourth point x to be integrated over. Note however that fixing a point at infinity, one has to carefully check that all divergencies cancel out.

9.2. The Untwisted Four Matter Field Amplitude

In the untwisted sector matter field vertices are given by (5.31) and (5.33). The desired amplitude is then:

$$\mathcal{A}_{C_i\overline{C}_iC_j\overline{C}_j} = \int \frac{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4}{V_{\mathrm{CKG}}}\langle V^{(0)}_{C_i}(z_1,k_1)V^{(-1)}_{\overline{C}_i}(z_2,k_2)V^{(0)}_{C_j}(z_3,k_3)V^{(-1)}_{\overline{C}_j}(z_4,k_4)\rangle,$$

where all vertices are inserted into the boundary of the disk. Later we will add an exchange $k_1 \leftrightarrow k_3$ to account for all possible combinations.

From the external fields we get the usual contribution of $\mathcal{E}/z_{24}$, where $\mathcal{E}$ is similarly defined as in the former case, only we now use three Mandelstam momenta, i.e.

$$\mathcal{E} = z_{12}^s \, z_{13}^t \, z_{14}^u \, z_{23}^u \, z_{24}^t \, z_{34}^s .$$

Regarding the internal contribution we can again discard terms with only one external spinor leaving two terms. For the first one we find:

$$\langle \partial Z^i(z_1)\, \partial Z^j(z_3)\rangle \langle \overline{\Psi}^{\,i}(z_2)\, \overline{\Psi}^{\,j}(z_4)\rangle = 0,$$

which leaves us with only the second term. This one can be computed using Wick's theorem:

$$\langle k_1 \cdot \psi(z_1)\; k_3 \cdot \psi(z_3)\rangle \langle \Psi^i(z_1)\overline{\Psi}^{\,i}(z_2)\Psi^j(z_3)\overline{\Psi}^{\,j}(z_4)\rangle = \frac{t}{z_{13}}\left[\frac{1}{z_{12}z_{34}} + \frac{\delta^{ij}}{z_{14}z_{23}}\right].$$

Putting this together we find:

$$\mathcal{A} \sim \int_{-\infty}^{+\infty} \frac{\mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}z_4}{V_{\mathrm{CKG}}} \frac{\mathcal{E}}{z_{24}} \frac{t}{z_{13}} \left[\frac{1}{z_{12}z_{34}} + \frac{\delta^{ij}}{z_{14}z_{23}}\right].$$

We will again fix three vertex positions. But this time all points are on the boundary, i.e. the real line, so we will fix the positions at 0, 1 and ∞, introducing a ghost contribution of

$$\langle c(z_1)\, c(z_3)\, c(z_4)\rangle = z_{13}\; z_{14}\; z_{34} = -z_\infty^2 .$$

To account for all possible permutations of the four matter fields we split the integration into three ranges and add an exchange of $k_1 \leftrightarrow k_3$:

$$\begin{aligned}\mathcal{A}_{C_i\overline{C}_iC_j\overline{C}_j} &= \left\{ \int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{+\infty} \right\} \mathrm{d}x \; \langle c(0)\; c(1)\; c(z_\infty)\rangle \; \times \\ &\left[\langle V_{C_i}^{(0)}(0)\, V_{\overline{C}_i}^{(-1)}(x)\, V_{C_j}^{(0)}(1)\, V_{\overline{C}_j}^{(-1)}(z_\infty)\rangle + \langle V_{C_i}^{(0)}(1)\, V_{\overline{C}_i}^{(-1)}(x)\, V_{C_j}^{(0)}(0)\, V_{\overline{C}_j}^{(-1)}(z_\infty)\rangle \right].\end{aligned}$$

Then the whole amplitude reads:

$$\begin{aligned}\mathcal{A}_{C_i\overline{C}_iC_j\overline{C}_j} = C\, t \cdot &\left\{ \mathrm{Tr}\,(2134) \int_{-\infty}^{0} + \mathrm{Tr}\,(1234) \int_{0}^{1} + \mathrm{Tr}\,(1324) \int_{1}^{\infty} \right\} \mathrm{d}x \\ &\times \left[x^{s-1}(1-x)^u + x^s(1-x)^{u-1}\,\delta^{ij} \right] + \\ C\, t \cdot &\left\{ \mathrm{Tr}\,(2314) \int_{-\infty}^{0} + \mathrm{Tr}\,(3214) \int_{0}^{1} + \mathrm{Tr}\,(3124) \int_{1}^{\infty} \right\} \mathrm{d}x \\ &\times \left[x^s(1-x)^{u-1} + x^{s-1}(1-x)^u\,\delta^{ij} \right],\end{aligned}$$

where we have added a yet to be determined overall normalization C to the amplitude. Performing a change of variables of $x \to 1-x$ in the second term we can write this as:

$$= C\, t \cdot \left\{ \mathrm{Tr}\,(2134 + 3124) \int\limits_{-\infty}^{0} \mathrm{d}x + \mathrm{Tr}\,(1234 + 3214) \int\limits_{0}^{1} \mathrm{d}x + \mathrm{Tr}\,(1324 + 2314) \int\limits_{1}^{\infty} \mathrm{d}x \right\}$$
$$\times \left[x^{s-1}(1-x)^{u} + x^{s}(1-x)^{u-1}\, \delta^{ij} \right].$$

The ranges of integration can be taken into each other by Möbius invariance. We will pick the range between 0 and 1. The needed transformation for $(-\infty, 0)$ is $x \to \frac{x}{x-1}$:

$$\mathrm{Tr}\,(2134 + 3124) \int\limits_{-\infty}^{0} \mathrm{d}x \left[x^{s-1}(1-x)^{u} + x^{s}(1-x)^{u-1}\, \delta^{ij} \right]$$
$$= \mathrm{Tr}\,(2134 + 3124) \int\limits_{0}^{1} \mathrm{d}x \left[-\, x^{s-1}(1-x)^{t-1} + x^{s}(1-x)^{t-1}\, \delta^{ij} \right],$$

while the interval $(1, \infty)$ can be mapped to $(0, 1)$ via $x \to \frac{1}{x}$:

$$\mathrm{Tr}\,(1324 + 2314) \int\limits_{1}^{\infty} \mathrm{d}x \left[x^{s-1}(1-x)^{u} + x^{s}(1-x)^{u-1}\, \delta^{ij} \right]$$
$$= \mathrm{Tr}\,(1324 + 2314) \int\limits_{0}^{1} \mathrm{d}x \left[x^{t-1}(1-x)^{u} - x^{t-1}(1-x)^{u-1}\, \delta^{ij} \right].$$

Introducing the Euler beta function (c.f. appendix B):

$$B(a, b) = \int\limits_{0}^{1} \mathrm{d}x\; x^{a-1}(1-x)^{b-1}$$

we can summarize the amplitude:

$$\begin{aligned} = \; C\, t \cdot \Big\{ \; & \mathrm{Tr}\,(2134 + 3124) \left[-\, B(s, t) + \delta^{ij} \cdot B(s+1, t) \right] \\ & + \mathrm{Tr}\,(1234 + 3214) \left[B(s, u+1) + \delta^{ij} \cdot B(s+1, u) \right] \\ & + \mathrm{Tr}\,(1324 + 2314) \left[B(u+1, t) - \delta^{ij} \cdot B(u, t) \right] \Big\}. \end{aligned}$$

The Beta functions have a representation in terms of gamma functions which is depicted in appendix B. The amplitude is finally:

$$
\begin{aligned}
\mathcal{A}_{C_i\overline{C}_iC_j\overline{C}_j} \;=\; C\,t\cdot\Bigg\{ & \;\mathrm{Tr}\,(2134+3124)\left[s\cdot\delta^{ij}+u\right]\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t+1)} \\
& +\mathrm{Tr}\,(1234+3214)\left[s\cdot\delta^{ij}+u\right]\frac{\Gamma(s)\Gamma(u)}{\Gamma(s+u+1)} \\
& +\mathrm{Tr}\,(1324+2314)\left[s\cdot\delta^{ij}+u\right]\frac{\Gamma(t)\Gamma(u)}{\Gamma(t+u+1)}\Bigg\}.
\end{aligned} \tag{9.6}
$$

Expanding this for low momenta yields:

$$
\begin{aligned}
\mathcal{A}_{C_i\overline{C}_iC_j\overline{C}_j} = C\Bigg\{ & \;\mathrm{Tr}\,(2134+3124)\left[\left(\frac{\pi^2}{6}\,t\,u-\frac{u}{s}\right)+\delta^{ij}\left(\frac{\pi^2}{6}\,s\,t-1\right)\right] \\
& +\mathrm{Tr}\,(1234+3214)\left[\left(\frac{\pi^2}{6}\,t\,u-\frac{t}{s}\right)+\delta^{ij}\left(\frac{\pi^2}{6}\,s\,t-\frac{t}{u}\right)\right] \\
& +\mathrm{Tr}\,(1324+2314)\left[\left(\frac{\pi^2}{6}\,t\,u-1\right)+\delta^{ij}\left(\frac{\pi^2}{6}\,s\,t-\frac{s}{u}\right)\right]\;+\;\mathcal{O}(k^6)\Bigg\}.
\end{aligned} \tag{9.7}
$$

To interpret these results we refer to [25] where the low energy effective field theory was extracted from heterotic strings. The polar terms stem from contact interactions due to D-terms[38] in the scalar potential and gauge boson exchange in the s- and u-channels. Comparison with the according terms from the field theory predictions in [25] allows us to normalize our amplitude:

$$
C \sim G_{C_i\overline{C}_i}\,G_{C_j\overline{C}_j}, \tag{9.8}
$$

which is similar to the redefinitions of matter field vertex operators we performed in the previous chapter. Additionally we have constant terms in (9.7) which come from the F-terms[39] of the scalar potential. These allow us get information about the Yukawa couplings W_{ijk}. From comparison with [25] we extract the following expression:

$$
G_{C_i\overline{C}_i}\,G_{C_j\overline{C}_j} = \mathrm{e}^{\kappa_4^2\widehat{K}}\sum_{k=1}^{3} W_{ijk}\,G^{C_k\overline{C}_k}\,\overline{W}_{\overline{kji}}, \tag{9.9}
$$

[38] D-terms are terms in $\theta^2\overline{\theta}^{\,2}$ where θ are super space coordinates. This is the highest possible product of expansion of $\Phi^\dagger\Phi$. These determine the kinematic terms of a supersymmetry invariant action.

[39] F-terms are the highest possible components of a product of chiral superfields. Taking into account that super space components are Grassman variables these turn out to be of order in θ^2. F-terms define the superpotential of a supersymmetry action.

where $\widehat{K}$ denotes the Kähler potential of the moduli fields to be determined in the next chapter. To evaluate this further it is convenient to have a look at single components, e.g. for $i = 1,\ j = 2,\ k = 3$ we find:

$$|W_{123}|^2 = G_{C_1\overline{C}_1}\, G_{C_2\overline{C}_2}\, G_{C_3\overline{C}_3}\, \mathrm{e}^{-\kappa_4^2 \widehat{K}}, \tag{9.10}$$

where the metric entries are given in (8.23). Since these contain terms in $\frac{f^j}{\mathrm{Im}(T^j)}$ we can use the supersymmetry condition (5.9) to simplify this expression:

$$W_{123} = \prod_{j=1}^{3} \sqrt{1 + \frac{f^j}{\mathrm{Im}(T^j)}}, \tag{9.11}$$

which can now be generalized to arbitrary values for i, j, k:

$$W_{ijk} = \prod_{j=1}^{3} \sqrt{1 + \frac{f^j}{\mathrm{Im}(T^j)}}\ \varepsilon_{ijk}. \tag{9.12}$$

Accordingly this applies to the untwisted sector of the angle picture. Using (5.23) we have:

$$W_{ijk} = \prod_{j=1}^{3} \sqrt{1 + \tan(\pi\, \theta^j)}\ \varepsilon_{ijk}. \tag{9.13}$$

9.3. The Twisted Four Matter Field Amplitude

Let us now turn to the twisted sector. The according vertex operators for the matter fields are (5.34) and (5.35). Here we consider two different angles θ^i and ν^i. The amplitude reads:

$$\mathcal{A}_{C_\theta \overline{C}_\theta C_\nu \overline{C}_\nu} = \int \frac{\mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}z_4}{V_{\mathrm{CKG}}} \langle V^{(0)}_{C_\theta}(z_1, k_1) V^{(-1)}_{\overline{C}_\theta}(z_2, k_2) V^{(0)}_{C_\nu}(z_3, k_3) V^{(-1)}_{\overline{C}_\nu}(z_4, k_4) \rangle.$$

Again we receive a factor of $\mathcal{E}/z_{24}$ from the external contractions. The internal contribution reads:

$$\begin{aligned}
\Bigg\langle \sum_{l=1}^{3} \Big[t_{\theta^l}(z_1)\, \tau_{-\theta^l}(z_1) + \mathrm{i}\, k_1 \cdot \psi(z_1) s_{\theta^l}(z_1)\, \sigma_{-\theta^l}(z_1) \Big] \times \\
\prod_{k \neq l}^{3} s_{\theta^k}(z_1)\, \sigma_{-\theta^k}(z_1) \prod_{j=1}^{3} s_{-\theta^j}(z_2)\, \sigma_{-\theta^j}(z_2) \times \\
\sum_{l=1}^{3} \Big[t_{\nu^l}(z_3)\, \tau_{-\nu^l}(z_3) + \mathrm{i}\, k_3 \cdot \psi(z_3) s_{\nu^l}(z_3)\, \sigma_{-\nu^l}(z_3) \Big] \times \\
\prod_{k \neq l}^{3} s_{\nu^k}(z_3)\, \sigma_{-\nu^k}(z_3) \prod_{j=1}^{3} s_{-\nu^j}(z_4)\, \sigma_{-\nu^j}(z_4) \Bigg\rangle.
\end{aligned}$$

Fortunately the terms containing excited twist fields do not contribute since their internal $U(1)$ charge, i.e. the intersecting angle, does not add up to zero in the occurring combinations. Take e.g. the correlator $\langle t_{\theta^j}(z_1)s_{-\theta^j}(z_2)t_{\nu^j}(z_3)s_{-\nu^j}(z_4)\rangle$. From bosonizing the twist fields one can easily see that the internal charges θ^j and ν^j add both up to one[40]. This leaves us with:

$$-\langle k_1\cdot\psi(z_1)\ k_3\cdot\psi(z_3)\rangle\times$$
$$\left\langle\prod_{j=1}^{3}\sigma_{-\theta^j}(z_1)\,\sigma_{\theta^j}(z_2)\,\sigma_{-\nu^j}(z_3)\,\sigma_{\nu^j}(z_4)\ s_{\theta^j}(z_1)\,s_{-\theta^j}(z_2)\,s_{\nu^j}(z_3)\,s_{-\nu^j}(z_4)\right\rangle.$$

The first factor yields simply t/z_{13}. The four twist correlators are evaluated using (9.3) and (9.5). The total contribution of the twist fields is:

$$\frac{z_{13}z_{24}}{z_{12}z_{14}z_{23}z_{34}}\prod_{j=1}^{3}I_j^{-\frac{1}{2}}(x),$$

with $I_j(x)$ defined in (9.4). Putting all together the amplitude reads:

$$\mathcal{A}_{C_\theta\overline{C}_\theta C_\nu\overline{C}_\nu}\sim-\int\frac{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4}{V_{\mathrm{CKG}}}\frac{\mathcal{E}\cdot t}{z_{12}z_{14}z_{23}z_{34}}\prod_{j=1}^{3}I_j^{-\frac{1}{2}}(x).$$

After fixing the three vertex positions we arrive at:

$$\begin{aligned}\mathcal{A}_{C_\theta\overline{C}_\theta C_\nu\overline{C}_\nu}\sim\ & t\cdot\left\{\mathrm{Tr}\,(2134)\int_{-\infty}^{0}+\mathrm{Tr}\,(1234)\int_{0}^{1}+\mathrm{Tr}\,(1324)\int_{1}^{\infty}\right\}\mathrm{d}x\\ &\times x^{s-1}(1-x)^{u-1}\prod_{j=1}^{3}I_j^{-\frac{1}{2}}(x)+\\ & t\cdot\left\{\mathrm{Tr}\,(2314)\int_{-\infty}^{0}+\mathrm{Tr}\,(3214)\int_{0}^{1}+\mathrm{Tr}\,(3124)\int_{1}^{\infty}\right\}\mathrm{d}x\\ &\times(1-x)^{s-1}x^{u-1}\prod_{j=1}^{3}I_j^{-\frac{1}{2}}(1-x).\end{aligned}$$

[40] Note that this is only true for the generic case of $\theta^j\neq-\nu^j$ which we will assume for now. After all we are mainly interested in certain limits of this amplitude to normalize the matter metric.

Performing a change of variables $x \to 1-x$ in the second term we can again summarize the amplitude:

$$\mathcal{A}_{C_\theta \overline{C}_\theta C_\nu \overline{C}_\nu} = t \cdot \left\{ \mathrm{Tr}\,(2134{+}3124) \int_{-\infty}^{0} + \mathrm{Tr}\,(1234{+}3214) \int_{0}^{1} + \mathrm{Tr}\,(1324{+}2314) \int_{1}^{\infty} \right\} \mathrm{d}x$$
$$\times\, x^{s-1}(1-x)^{u-1} \prod_{j=1}^{3} I_j^{-\frac{1}{2}}(x) \times g_o^4 C_{D_2} (2\pi)^4 \delta(k_1+k_2+k_3+k_4), \tag{9.14}$$

where we have added standard constants, i.e. the disc coupling C_{D_2}, one open string coupling g_o for each vertex and a delta expression to ensure conservation of momenta.

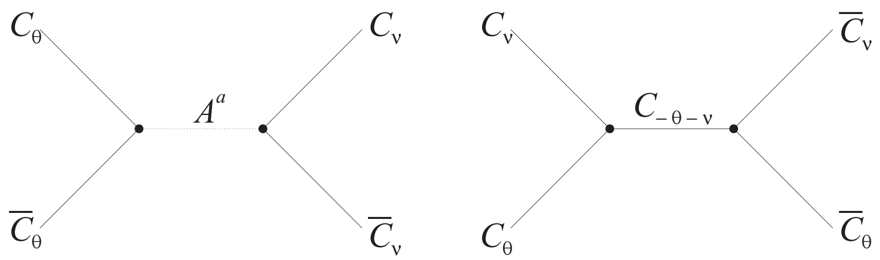

Fig. 11 The factorization limits of the twisted four matter amplitude: On the left side we see the limit of $x \to 0$ which corresponds to and exchange of untwisted particles in the s-channel. On the right side we see the limit of $x \to \infty$ corresponding to the t-channel in which the exchanged particle carries an anti twist. From the latter we can read off the Yukawa coupling.

This resulting amplitude (9.14) cannot be fully integrated but we can investigate it in certain limits, i.e. $x \to 0$ and $x \to \infty$ as depicted in fig. 11. Let us start with the limit of $x \to 0$, where we expect the s-channel. The behavior of the hypergeometric functions in $I_j(x)$ in this limit reads (c.f. appendix B):

$$\begin{aligned} \lim_{x\to 0} {}_2F_1(a,b,c;x) &= 1 \\ \lim_{x\to 0} {}_2F_1(a,b,a+b;1-x) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \ln \frac{\delta}{x}, \end{aligned} \tag{9.15}$$

where $\ln\delta = 2\psi(1) - \psi(a) - \psi(b)$ and $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x)$. The condition $c = a+b$ in the latter formula happens to be fulfilled in the considered cases, while the first formula is true for any parameters a, b, c. For $I_j(x)$ we eventually find:

$$\begin{aligned} \lim_{x\to 0} I_j(x) &= \frac{\sin(p\,\pi\,\theta^j)}{\pi} \Big[\underbrace{2\psi(1) - \psi(\theta^j) - \psi(1-\theta^j)}_{\ln\delta_{\theta^j}} + \underbrace{2\psi(1) - \psi(\nu^j) - \psi(1-\nu^j)}_{\ln\delta_{\nu^j}} - 2\ln x \Big] \\ &= \frac{\sin(p\,\pi\,\theta^j)}{\pi} \ln \frac{\delta_{\theta^j}\,\delta_{\nu^j}}{x^2}. \end{aligned}$$

Neglecting the Chan-Paton factors for now the amplitude reads in the limit of $x \to 0$:

$$
\begin{aligned}
\mathcal{A} &\to t \int_0 \mathrm{d}x \left[\prod_{j=1}^{3} \frac{\sin(p\,\pi\,\theta^j)}{\pi} \ln \frac{\delta_{\theta^j}\,\delta_{\nu^j}}{x^2} \right]^{-\frac{1}{2}} \\
&\sim t \prod_{j=1}^{3} \sqrt{\frac{\pi}{\sin(p\,\pi\,\theta^j)}} \int_0 \mathrm{d}x\; x^{s-1} \ln^{-3/2}(1/x^2).
\end{aligned}
\tag{9.16}
$$

In the last line the numerical factors δ_{θ^j} and δ_{ν^j} are neglected due to simplicity. The limits $x \to 0$ corresponds to two pairs of twist-anti-twist fields forming an intermediate untwisted gauge boson in the s-channel, see fig. 11. To normalize the amplitude we will now compute the three point function of two matter fields C_θ and one such boson A^a. The desired amplitude is:

$$
\mathcal{A}_{C_\theta \overline{C}_\theta A^a} = \langle c(0) V_{C_\theta}^{(-1)}(0,k_1)\; c(1) V_{\overline{C}_\theta}^{(-1)}(1,k_2)\; c(z_\infty) V_{A^a}^{(0)}(z_\infty, k) \rangle, \tag{9.17}
$$

where the gauge vertex operator in the zero picture is given by:

$$
V_{A^a}^{(0)}(z) = \lambda^a \xi_\mu \big[\partial X^\mu(z) + \mathrm{i}k \cdot \psi(z) \psi^\mu(z) \big] \mathrm{e}^{\mathrm{i}k\cdot X(z)}. \tag{9.18}
$$

Here λ^a is a Chan-Paton factor and ξ_μ the polarization vector with the notorious condition $\xi \cdot k = 0$. Note that since we are dealing with three point functions here we can fix all vertex positions which has already been done in (9.17). The only correlator in (9.17) which has not been sufficiently discussed so far is:

$$
\xi_\mu \langle \partial X^\mu(z_3)\, \mathrm{e}^{\mathrm{i}k_1\cdot X(z_1)}\, \mathrm{e}^{\mathrm{i}k_2\cdot X(z_2)}\, \mathrm{e}^{\mathrm{i}k\cdot X(z_3)} \rangle.
$$

In addition to the usual contraction of the exponential factor we receive a contraction of an external bosonic field with the exponentials. Such contractions are in general discussed in [2] and [9]. The result is:

$$
\xi_\mu \langle \partial X^\mu(z_3)\, \mathrm{e}^{\mathrm{i}k_1\cdot X(z_1)}\, \mathrm{e}^{\mathrm{i}k_2\cdot X(z_2)}\, \mathrm{e}^{\mathrm{i}k\cdot X(z_3)} \rangle = \mathcal{E} \cdot \mathrm{i}\, \xi_\mu \left[\frac{k_1^\mu}{z_{13}} + \frac{k_2^\mu}{z_{23}} \right].
$$

After all we find for the amplitude:

$$
\mathcal{A}_{C_\theta \overline{C}_\theta A^a} = \mathrm{i}\, g_o^3 C_{D_2}\, \mathrm{Tr}\,(12)\, \lambda^a\; \xi \cdot k_{12}\; (2\pi)^4 \delta^{(4)}(k_1 + k_2 + k), \tag{9.19}
$$

where we have added a constant factor C_{D_2} for the disk and one g_o for each open string. The delta term accounts for the conservation of momenta. The normalization of (9.14) can now be obtained from the unitarity condition:

$$
A_4(k_1,k_2,k_3,k_4) \to -\mathrm{i} \int \frac{\mathrm{d}^7 k}{(2\pi)^7} \frac{A_3(k_1,k_2,k) A_3(-k,k_3,k_4)}{k^2 - \mathrm{i}\varepsilon}, \tag{9.20}
$$

where A_3 refers to (9.19) and A_4 to (9.16). The boson in the s-channel is an intermediate massless $6'$-$6'$ string. It may carry an arbitrary momentum q along the directions of the $D6'$-brane orthogonal to the intersection, i.e. the three components of q have to be integrated over (see [27]):

$$\int \mathrm{d}^3 q \int_0 \mathrm{d}x\, x^{s-q^2-1} \sim t \prod_{j=1}^{3} \sqrt{\frac{\pi}{\sin(p\,\pi\,\theta_j)}} \int_0 \mathrm{d}x\, x^{s-1} \ln^{-3/2}(1/x^2).$$

From this we can extract the following replacement by performing the x-integration on the l.h.s.:

$$\int \frac{\mathrm{d}^3 q}{s-q^2} \to t \prod_{j=1}^{3} \sqrt{\frac{\pi}{\sin(p\,\pi\,\theta_j)}} \int_0 \mathrm{d}x\, x^{s-1} \ln^{-3/2}(1/x^2). \tag{9.21}$$

Note that (9.21) serves as well as demonstration that the s-channel may actually be found in the limit $x \to 0$ of (9.14). Applying (9.19) to the r.h.s. of (9.20) yields:

$$\begin{aligned} A_4(k_1,k_2,k_3,k_4) \to & 2\pi\mathrm{i}\, g_o^6 C_{D_2}^2 \int \mathrm{d}^7 k \frac{\xi\cdot k_{12}\,\delta(k_1+k_2+k)\,\xi\cdot k_{34}\,\delta(-k+k_3+k_4)}{k^2-\mathrm{i}\varepsilon} \\ = & 2\pi\mathrm{i}\, g_o^6 C_{D_2}^2 \delta(k_1+k_2+k_3+k_4)\, t \int \frac{\mathrm{d}^3 q}{q^2-s}. \end{aligned}$$

This result can now be equated to (9.21) from which we can read off the normalization C of the amplitude:

$$\begin{aligned} C &= 2\pi\mathrm{i}\, g_o^2 C_{D_2} \prod_{j=1}^{3} \sqrt{\frac{\sin(p\,\pi\,\theta_j)}{\pi}} \\ &= 2\pi\mathrm{i} \prod_{j=1}^{3} \sqrt{\frac{\sin(p\,\pi\,\theta_j)}{\pi}}, \end{aligned} \tag{9.22}$$

where we have made use of the standard relation $C_{D_2} = 1/g_o^2$. Note that this normalization cancels off the sine factor from $I_j(x)$. Thus $\prod_j \sqrt{\sin(p\,\pi\,\theta_j)}$ can be regarded as a normalization of $I_j(x)$ itself.
Let us next turn to the limit of $x \to \infty$ which corresponds to the t-channel from which one can extract the Yukawa couplings. To compute the according limit of $I_j(x)$ we need the limit of $x \to \infty$ for the hypergeometric functions, see [28] or appendix B:

$$\begin{aligned} {}_2F_1(a,b,c;x) &\to \mathrm{e}^{\mathrm{i}\pi a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} x^{-a} + \mathrm{e}^{\mathrm{i}\pi b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} x^{-b} \\ {}_2F_1(a,b,c;1-x) &\to \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} x^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} x^{-b}. \end{aligned} \tag{9.23}$$

From this the limit of $I_j(x)$ can be determined:

$$\lim_{x\to\infty} I_j^{-\frac{1}{2}}(x) = \begin{cases} (-1)^{\theta^j-\nu^j} x^{\theta^j+\nu^j}\Gamma_{\theta^j,\nu^j} & 0<\theta_j+\nu_j<1, \\ (-1)^{\theta^j-\nu^j+1} x^{2-\theta^j-\nu^j}\Gamma_{1-\theta^j,1-\nu^j} & 1<\theta_j+\nu_j<2, \end{cases}$$

where:

$$\Gamma_{\theta^j,\nu^j} = \frac{\Gamma(1-\theta^j)\Gamma(1-\nu^j)\Gamma(\theta^j+\nu^j)}{\Gamma(\theta^j)\Gamma(\nu^j)\Gamma(1-\theta^j-\nu^j)}.$$

The amplitude (9.14) then takes the form:

$$C\, t \int^{1} \mathrm{d}x\; x^{t-1} \prod_{j=1}^{3} \sqrt{\frac{\sin(p\,\pi\,\theta^j)}{\pi}}\; \Gamma_{\theta^j,\nu^j}^{-\frac{1}{2}} |Y_j|^2 \to 2\pi \prod_{j=1}^{3} \Gamma_{\theta^j,\nu^j}^{-\frac{1}{2}} |Y_j|^2, \tag{9.24}$$

where the normalization (9.22) was used. The factor $|Y_j|^2$ comes from the contribution of the classical action, i.e. $\mathrm{e}^{-S_{\mathrm{cl}}} \to |Y_j|^2$ in the limit of $x\to\infty$. The integration on the l.h.s. can was done in the following way. We can perform a change of variables, namely $x \to \frac{x-1}{x}$, to map the range of integration from $(1,\infty)$ to $(0,1)$ which takes the limit under survey from $x\to\infty$ to $x\to 1$ which is much easier to handle:

$$\lim_{x\to\infty} t\cdot \int_1^\infty \mathrm{d}x\; x^{s-1}(1-x)^{u-1} = \lim_{x\to 1} t\cdot \int_0^1 \mathrm{d}x\; x^{t-1}(1-x)^{u-1}.$$

Performing the limit on the r.h.s. the integration becomes trivial:

$$t\cdot \int^{1} \mathrm{d}x\; x^{t-1} = t\cdot \frac{x^t}{t}\Big|_{x=1} = 1.$$

From (9.24) we can now read off the Yukawa couplings in the string basis:

$$Y_{C_\theta C_\nu C_{-\theta-\nu}} = \left(\frac{\Gamma(1-\theta^j)\,\Gamma(1-\nu^j)\,\Gamma(\theta^j+\nu^j)}{\Gamma(\theta^j)\,\Gamma(\nu^j)\,\Gamma(1-\theta^j-\nu^j)}\right)^{1/4} Y_j. \tag{9.25}$$

Similarly as in the untwisted case this is independent of momenta. As such this constant part comes from the F-terms of the scalar potential. According to [25] we then have:

$$\mathrm{e}^{\kappa^2\widehat{K}} W_{C_\theta C_\nu C_{-\theta-\nu}} G^{C_{-\theta-\nu}\overline{C}_{\theta+\nu}} \overline{W}_{\overline{C}_\theta\overline{C}_\nu\overline{C}_{-\theta-\nu}} = 2\pi \prod_{j=1}^{3} \Gamma_{\theta^j,\nu^j}^{-\frac{1}{2}} |Y_j|^2 G_{C_\theta\overline{C}_\theta} G_{C_\nu\overline{C}_\nu}, \tag{9.26}$$

where we have again redefined the vertex operators in going to field theory which explains the matter field metrics on the r.h.s. From (9.26) we get for the Yukawa couplings:

$$|W_{C_\theta C_\nu C_{-\theta-\nu}}|^2 = 2\pi \prod_{j=1}^{3} \Gamma^{-\frac{1}{2}}_{\theta^j,\nu^j} |Y_j|^2 \, G_{C_\theta \overline{C}_\theta} \, G_{C_{-\theta-\nu}\overline{C}_{-\theta-\nu}} \, G_{C_\nu \overline{C}_\nu}, \tag{9.27}$$

with the matter metrics (8.34). Inserting those into (9.27) yields:

$$|W_{C_\theta C_\nu C_{-\theta-\nu}}|^2 = 2\pi \prod_{j=1}^{3} \frac{(T^j - \overline{T}^j)^{\theta^j-1}(T^j - \overline{T}^j)^{\nu^j-1}}{(T^j - \overline{T}^j)^{\theta^j+\nu^j+1}} |Y_j|^2. \tag{9.28}$$

Note that the matter metric (8.34) is still preliminary which renders the above finding preliminary as well. A final result will be presented in [1].

10. The Kähler Potential of Moduli Fields

The present chapter is dedicated to the potential $\widehat{K}(M,\overline{M})$ which is the moduli only part of the Kähler potential (6.4). We will determine $\widehat{K}$ by calculating scattering amplitudes of moduli closed strings from the D-brane.

10.1. The General Setup for Closed-Closed String Scattering from D-Branes

Scattering of closed strings amounts to regarding the closed strings as two open strings with their endpoints attached to the brane. Thus we have to carry out a four open string calculation. Similar to our considerations in chapter eight the two closed string momenta q_1 and q_2 split up into four open string momenta $\frac{1}{2}q_1$, $\frac{1}{2}Dq_1$, $\frac{1}{2}q_2$ and $\frac{1}{2}Dq_2$.

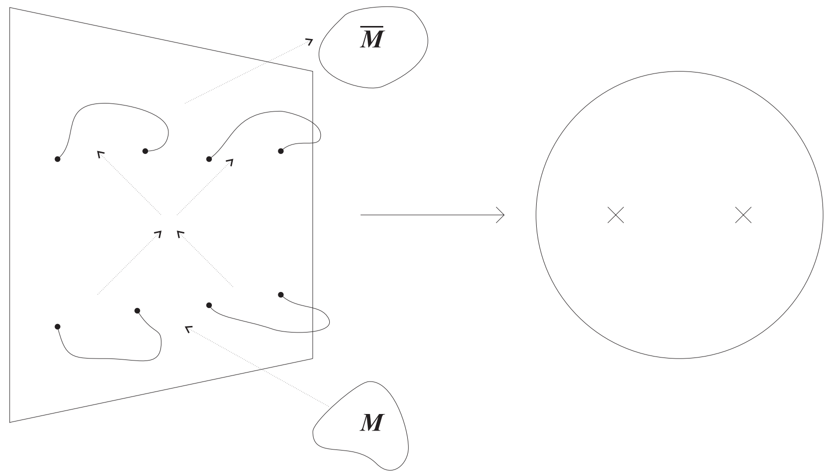

Fig. 12 Scattering of closed strings off a D-brane: The closed string is split into its holomorphic and anti-holomorphic components which enter the scattering process as two independent open strings with their endpoints attached to the brane (left). The closed vertex operators are inserted into the bulk (right).

Note that only momenta parallel to the brane are conserved. Thus momentum conservation expressed in q is given by:

$$q_1^{||} + q_2^{||} = 0. \tag{10.1}$$

With the following choice for momenta k_i entering the Mandelstam momenta (8.10):

$$k_1 = q_1, \quad k_2 = Dq_1, \quad k_3 = q_2, \quad \text{and} \quad k_4 = Dq_2, \tag{10.2}$$

we receive the following constraints:

$$s = 2(q_1^{||})^2 = 2(q_2^{||})^2 = -2q_1^{||} \cdot q_2^{||}, \quad t = q_1 \cdot q_2. \tag{10.3}$$

Contractions are evaluated using (8.1) and (8.2). It is convenient to fix the three vertex position at [24]:

$$z_1 = \mathrm{i}y, \quad \overline{z}_1 = -\mathrm{i}y, \quad z_2 = \mathrm{i} \quad \text{and} \quad \overline{z}_2 = -\mathrm{i},$$

which states that the vertices are inserted into the bulk rather than into the boundary which is the real line.

10.2. Scattering of Moduli Fields from D-branes

We want to determine amplitudes of moduli fields scattered from D-branes. Let us start with the T-$\overline{T}$ amplitude:

$$\mathcal{A}_{T^i\overline{T}^j} = \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{V_{\mathrm{CKG}}} \langle V_{T^i}^{(0,0)}(z_1, \overline{z}_1, k_1, k_2)\, V_{\overline{T}^j}^{(-1,-1)}(z_2, \overline{z}_2, k_3, k_4) \rangle, \tag{10.4}$$

where the according vertex operators are given in eqs. (4.13) and (4.9). On the external side we receive the factor $\mathcal{E}/(\overline{z}_2 - z_2)$, where $\mathcal{E}$ can in the present case be expressed in the following way:

$$\mathcal{E} = \left[\frac{4y}{(1+y)^2}\right]^s \left[\frac{1-y}{1+y}\right]^{2t}.$$

The internal contributions are quiet similar to the ones already discussed in the various amplitudes we have computed. Note that from the vertex operator in the $(0,0)$ -picture we get $\overline{D}^{\,i}$, while in the $(-1,-1)$ -picture we have D^j. The internal contribution reads:

$$\begin{aligned}
&- \frac{\overline{D}^{\,i} D^j}{(\overline{z}_1 - z_1)^2 (\overline{z}_2 - z_2)} + \frac{s}{\overline{z}_1 - z_1} \left(\frac{\overline{D}^{\,i} D^j}{(\overline{z}_1 - z_1)(\overline{z}_2 - z_2)} - \frac{\delta^{ij}}{(\overline{z}_1 - \overline{z}_2)(z_1 - z_2)} \right) \\
&= -\frac{1-s}{8\mathrm{i}y^2}\, \overline{D}^{\,i} D^j + \frac{s}{2\mathrm{i}y(y-1)^2}\, \delta^{ij}.
\end{aligned}$$

Putting this together the amplitude becomes:

$$\begin{aligned}
\mathcal{A}_{T^i\overline{T}^j} = -\frac{1}{4} \frac{C}{(T^i - \overline{T}^{\,i})(T^j - \overline{T}^{\,j})} \int \mathrm{d}y \, &\left[\frac{4y}{(1+y)^2}\right]^s \left[\frac{1-y}{1+y}\right]^{2t} \\
&\times \left[\frac{1-s}{4}\, \frac{1-y^2}{y^2}\, \overline{D}^{\,i} D^j - s\, \frac{1+y}{y(1-y)}\, \delta^{ij}\right].
\end{aligned}$$

In [24] there is a way presented to solve this integral. First one has to perform a change of variables, namely $y \to \frac{1-\sqrt{x}}{1+\sqrt{x}}$ leaving us with:

$$\mathcal{A}_{T^i\overline{T}^j} = -\frac{1}{4}\frac{C}{(T^i-\overline{T}^i)(T^j-\overline{T}^j)}\int \mathrm{d}x\ (1-x)^s x^t \left[\frac{1-s}{(1-x)^2}\,\overline{D}^iD^j - \frac{s}{x(1-x)}\,\delta^{ij}\right].$$

Now we can use the ordinary approach by means of the beta function. That is, the amplitude can be written as:

$$\begin{aligned}\mathcal{A}_{T^i\overline{T}^j} =& \frac{1}{4}\frac{C}{(T^i-\overline{T}^i)(T^j-\overline{T}^j)}\left[(1-s)\cdot B(s-1,t+1)\,\overline{D}^iD^j - s\cdot B(s,t)\,\delta^{ij}\right]\\ =& -\frac{1}{4}\frac{C}{(T^i-\overline{T}^i)(T^j-\overline{T}^j)}(s+t)(t\cdot\overline{D}^iD^j + s\cdot\delta^{ij})\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t+1)}.\end{aligned} \tag{10.5}$$

Let us discuss the case of $i=j$ in more detail. From the identity $D^i\overline{D}^i = 1$ we get for this case:

$$\mathcal{A}_{T^i\overline{T}^i} = -\frac{1}{4}\frac{C}{(T^i-\overline{T}^i)^2}\ (t+s)^2\ \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t+1)}, \tag{10.6}$$

which reads in the limit of low momenta:

$$\mathcal{A}_{T^i\overline{T}^i} = -\frac{1}{4}\frac{C}{(T^i-\overline{T}^i)^2}\left(1-\frac{\pi^2}{6}\,s\,t\right)\frac{(t+s)^2}{s\,t}\ +\ \mathcal{O}(k^6). \tag{10.7}$$

On the other hand we have the case $i \neq j$ in which the amplitude becomes:

$$\mathcal{A}_{T^i\overline{T}^j} = -\frac{1}{4}\frac{C}{(T^i-\overline{T}^i)(T^j-\overline{T}^j)}(s+t)t\cdot\overline{D}^iD^j\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t+1)}, \tag{10.8}$$

where the missing s spoils the low energy expansion such that there is no t-pole:

$$\mathcal{A}_{T^i\overline{T}^j} = -\frac{1}{4}\frac{C}{(T^i-\overline{T}^i)(T^j-\overline{T}^j)}\left(1-\frac{\pi^2}{6}\,s\,t\right)\frac{s+t}{s}\ +\ \mathcal{O}(k^6). \tag{10.9}$$

That means only in (10.7) there is a t-pole from which we can conclude for the moduli metric (see [25]):

$$G_{T^i\overline{T}^j} = -\frac{\delta^{ij}}{(T^i-\overline{T}^i)^2}. \tag{10.10}$$

Note that in (10.7) there are other terms than s/t. These non-trivial contributions have not been sufficiently discussed yet and are subject to future work [29].
Next we shall repeat the above calculation with the U and $\overline{U}$ moduli. The desired amplitude is:

$$\mathcal{A}_{U^i\overline{U}^j} = \int\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{V_{\mathrm{CKG}}}\langle V^{(0,0)}_{U^i}(z_1,\overline{z}_1,k_1,k_2)\ V^{(-1,-1)}_{\overline{U}^j}(z_2,\overline{z}_2,k_3,k_4\rangle,$$

with the vertex operators given in (4.14) and (4.10). The contribution of the external side equals that form the $T-\overline{T}$-scattering. The internal contribution is vastly different though. From the vertex operators of the U-moduli one can easily tell that there is no mixing of holomorphic and anti-holomorphic components of the closed string, which is different than in the case of T-moduli. Furthermore we receive a mixing of holomorphic and anti-holomorphic fields from different planes, i.e. we have to work with D^{ij} instead of D^i and D^j. Taking all this into account the internal contribution reads:

$$
\begin{aligned}
-\frac{s}{\overline{z}_1 - z_1}\left(\frac{\overline{D}^{\,ij}\,D^{ij}}{(\overline{z}_1 - z_2)(\overline{z}_2 - z_1)} - \frac{D^{ij}\,\overline{D}^{\,ij}}{(\overline{z}_1 - \overline{z}_2)(z_1 - z_2)}\right) =& \frac{s}{2\mathrm{i}y}\left(\frac{D^{ij}\,\overline{D}^{\,ij}}{(y-1)^2} - \frac{\overline{D}^{\,ij}\,D^{ij}}{(y+1)^2}\right)\\
=& -2\mathrm{i}\frac{1}{(1-y^2)^2}\cdot s\,\overline{D}^{\,ij}\,D^{ij}.
\end{aligned}
$$

The amplitude as a whole then becomes:

$$
\mathcal{A}_{U^i\overline{U}^{\,j}} = \frac{C}{(U^i - \overline{U}^{\,i})(U^j - \overline{U}^{\,j})}\int \mathrm{d}y\left[\frac{4y}{(1+y)^2}\right]^s\left[\frac{1-y}{1+y}\right]^{2t}\frac{1}{y^2-1}\cdot s\,\overline{D}^{\,ij}\,D^{ij},
$$

which we can again transform using $y \to \frac{1-\sqrt{x}}{1+\sqrt{x}}$ which yields:

$$
\mathcal{A}_{U^i\overline{U}^{\,j}} = -\frac{1}{4}\frac{C}{(U^i - \overline{U}^{\,i})(U^j - \overline{U}^{\,j})}\int \mathrm{d}x\,(1-x)^s x^{t-1}\cdot s\,\overline{D}^{\,ij}\,D^{ij}.
$$

In terms of Beta functions this finally reads:

$$
\begin{aligned}
\mathcal{A}_{U^i\overline{U}^{\,j}} =& -\frac{1}{4}\frac{C}{(U^i - \overline{U}^{\,i})(U^j - \overline{U}^{\,j})}s\,B(s+1,t)\,\overline{D}^{\,ij}\,D^{ij}\\
=& -\frac{1}{4}\frac{C}{(U^i - \overline{U}^{\,i})(U^j - \overline{U}^{\,j})}s^2\,\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t+1)}\,\overline{D}^{\,ij}\,D^{ij}.
\end{aligned}
\tag{10.11}
$$

Again we concentrate on the special case $i = j$ where we find:

$$
\mathcal{A}_{U^i\overline{U}^{\,j}} = -\frac{1}{4}\frac{C}{(U^i - \overline{U}^{\,i})(U^j - \overline{U}^{\,j})}s^2\,\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t+1)}.
\tag{10.12}
$$

The expansion for low momenta is:

$$
\mathcal{A}_{U^i\overline{U}^{\,j}} = -\frac{1}{4}\frac{C}{(U^i - \overline{U}^{\,i})(U^j - \overline{U}^{\,j})}\frac{s}{t}\left(1 - \frac{\pi^2}{6}\,s\,t\right) + \mathcal{O}(k^6),
\tag{10.13}
$$

from which we can again tell the moduli metric [25]:

$$
G_{U^i\overline{U}^{\,j}} = -\frac{\delta^{ij}}{(U^i - \overline{U}^{\,i})^2}.
\tag{10.14}
$$

Finally we are left to survey amplitudes with T and U moduli. These cases are especially simple as the amplitudes turn out to be vanishing. This can be seen either from calculation or from consideration, i.e. mixed T-U amplitudes do not conserve charge. Thus we know:

$$\mathcal{A}_{T^i\overline{U}^j} = 0, \qquad \mathcal{A}_{\overline{T}^iU^j} = 0, \tag{10.15}$$

from which we can conclude for the moduli metric:

$$G_{T^i\overline{U}^j} = 0, \qquad G_{\overline{T}^iU^j} = 0. \tag{10.16}$$

From the found metric entries (10.10), (10.14) and (10.16) we can now derive the Kähler potential for the moduli fields M:

$$\widehat{K}(M,\overline{M}) = -\sum_{j=1}^{3}\ln(T^j - \overline{T}^j) - \sum_{j=1}^{3}\ln(U^j - \overline{U}^j). \tag{10.17}$$

Note that the contribution from the U- and T-moduli are of the same form. This is not surprising as the above computations were independent from the model, i.e. flux or angle. From T-duality we know that it essentially interchanges the moduli. Written in the field theory moduli (6.7), (6.8) and (6.9) the Kähler potential is:

$$\widehat{K}(M,\overline{M}) = -\kappa_4^{-2}\sum_{j=1}^{3}\ln(t^j + \overline{t}^j) - \kappa_4^{-2}\sum_{j=1}^{3}\ln(u^j + \overline{u}^j). \tag{10.18}$$

11. Conclusions

In the previous three chapters we have calculated various string amplitudes to gain information about the ingredients to the $N=1$ supergravity action.

11.1. The Kähler Potential

From scattering of moduli fields and matter fields we found the matter field metrics (8.24) and (8.34). In the flux picture we found:

$$G_{C_i\overline{C}_i} = \left|1 - \frac{f^j f^k}{\mathrm{Im}(T^j)\,\mathrm{Im}(T^k)}\right| \frac{1}{t^i + \overline{t}^{\,i}}, \tag{11.1}$$

where i, j, k label the plane. This translates into the following tree-level Kähler potential for bosonic matter in type IIB super string theory with background flux:

$$K^{IIB}_{\mathrm{matter}} = \sum_{i=1}^{3} \left|1 - \frac{f^j f^k}{\mathrm{Im}(T^j)\,\mathrm{Im}(T^k)}\right| \frac{|C_i|^2}{t^i + \overline{t}^{\,i}} + \ldots \tag{11.2}$$

On the other hand we found for the matter field metric in the untwisted sector of type IIA with 6-branes intersecting at angles (8.25):

$$G_{C_i\overline{C}_i} = \left|1 - \tan(\pi\,\theta^k)\,\tan(\pi\,\theta^l)\right| \frac{1}{u^i + \overline{u}^{\,i}}. \tag{11.3}$$

This gives rise to the untwisted Kähler potential of bosonic matter:

$$K^{IIA}_{\mathrm{untwisted}} = \sum_{i=1}^{3} \left|1 - \tan(\pi\,\theta^k)\,\tan(\pi\,\theta^l)\right| \frac{|C_i|^2}{u^i + \overline{u}^{\,i}} + \ldots \tag{11.4}$$

We have already mentioned that the twisted part of the Kähler potential is not yet fully understood as the according metric suffers from insufficiencies. Anyway, we trust to be able to present a solution in [1].
From scattering moduli fields we determined the Kähler potential for the moduli fields to be (10.18):

$$\widehat{K}(M,\overline{M}) = -\kappa_4^{-2}\ln(s+\overline{s}) - \kappa_4^{-2}\sum_{j=1}^{3}\ln(t^j + \overline{t}^{\,j}) - \kappa_4^{-2}\sum_{j=1}^{3}\ln(u^j + \overline{u}^{\,j}) \tag{11.5}$$

where we have added a Kähler potential for the dilaton as in [23]. The according calculation will be presented in [1]. Thus the full Kähler potential of the type IIB setup with background flux was determined. For the type IIA setup with intersecting branes we have determined the untwisted contribution to the Kähler potential while the twisted one still needs some thoughts.

11.2. The Superpotential

From scattering of four matter fields we found the for Yukawa coupling in the flux picture (9.12):

$$W_{ijk} = \prod_{j=1}^{3} \sqrt{1 + \frac{f^j}{\mathrm{Im}(T^j)}}\, \varepsilon_{ijk}, \tag{11.6}$$

which yields the following trilinear superpotential:

$$W^{IIB} = \sum_{i,j,k} \prod_{j=1}^{3} \sqrt{1 + \frac{f^j}{\mathrm{Im}(T^j)}}\, \varepsilon_{ijk}\, C_i\, C_j\, C_k. \tag{11.7}$$

In the untwisted sector of the angle picture (9.12) took over to (9.13):

$$W_{ijk} = \prod_{j=1}^{3} \sqrt{1 + \tan(\pi\, \theta^j)}\, \varepsilon_{ijk}. \tag{11.8}$$

Thus untwisted superpotential is found to be:

$$W^{IIA}_{\text{untwisted}} = \sum_{i,j,k} \prod_{j=1}^{3} \sqrt{1 + \tan(\pi\, \theta^j)}\, \varepsilon_{ijk}\, C_i\, C_j\, C_k. \tag{11.9}$$

Again the twisted contribution is not yet fully determined. For its determination we would need the correct twisted matter metric. Our result is so far preliminary and not fully understood. The twisted Kähler potential will be presented in [1].

11.3. The Gauge Kinetic Function

The gauge kinetic function f was not calculated in this work. We will however borrow the result from [1], in which f was determined in scattering of gauge fields and moduli. The result is for the flux picture is:

$$f(s, t^j) = |n^1 n^2 n^3| \left(s - \alpha'^{-2} f^1 f^2\, t^3 - \alpha'^{-2} f^1 f^3\, t^2 - \alpha'^{-2} f^2 f^3\, t^1 \right), \tag{11.10}$$

where n^j denotes the winding number of the $D9$-brane around the j-th torus. A similar expression for the angle picture can be found in [1].

In summary we have determined all necessary ingredients to the $N = 1$ supergravity action in four space time dimensions for the setup of type *II*B superstring theory with non-vanishing background flux and a 9-brane. This supergravity action plays the role of a low energy effective action.
On the other side we have the setup in type *II*A string theory where the background fluxes take over to rotating the 6-branes which come from the 9-brane after T-dualizing on radius of each torus of the $T^2 \times T^2 \times T^2$ compactification. There are still open questions in this setup which we hope to answer in [1].

11.4. Limitations and Outlook

Our findings are restricted in several ways. First of all our chosen compactification $T^2 \times T^2 \times T^2$ is only one possible choice. One could as well consider non factorizable T^6 tori, $T^2 \times K3$ or CY3 compactifications.
Secondly, the amplitudes we calculated were done on tree level only. Amplitudes at higher loops are much more difficult to be determined but might yield interesting corrections.
Also we have only considered the scalar component of the chiral fields entering (6.1). Since (6.3) does not hold for the spinor components we would expect very different contributions from the fermionic side.
Furthermore we have not included Wilson line moduli into our picture. Those produce a rather non-trivial field theory with the geometrical moduli. Calculations with Wilson lines turned on is subject to future work [29].
Another interesting subject not addressed in this work are amplitudes involving two moduli and two matter fields which amounts to computing an amplitude with six open vertices. From these amplitudes one would gain conditions on the Riemann tensor of the moduli space [29].

Appendix A. Twist Fields

As we have already depicted open string states can change their boundary conditions at the intersection of branes. This was taken into account by introducing bosonic twist fields known from orbifolding. Supersymmetry requires to introduce fermionic twist fields as well. In this appendix we will study twist fields and their contractions in more detail.
The Z^N twist fields studied in [19] introduce fractional mode expansions $\sim 1/N$. From those one can determine the OPEs of bosonic/fermionic fields with their according twist fields. This generalizes to the notion of intersecting branes where the intersecting angle $\theta \sim 1/N$ introduces fractional exponents in the mode expansion as well. The according OPEs on the bosonic side read [20]:

$$\begin{aligned}
\partial Z(z_1)\,\sigma_{\theta^j}(z_2) &\sim z_{12}^{\theta^j-1}\tau_{\theta^j}(z_2), & \partial Z(z_1)\,\sigma_{-\theta^j}(z_2) &\sim z_{12}^{-\theta^j}\,\widetilde{\tau}_{-\theta^j}(z_2),\\
\partial\overline{Z}(z_1)\,\sigma_{\theta^j}(z_2) &\sim z_{12}^{-\theta^j}\,\widetilde{\tau}_{\theta^j}(z_2), & \partial\overline{Z}(z_1)\,\sigma_{-\theta^j}(z_2) &\sim z_{12}^{\theta^j-1}\tau_{-\theta^j}(z_2),\\
\overline{\partial}Z(z_1)\,\sigma_{\theta^j}(z_2) &\sim -z_{12}^{\theta^j-1}\tau_{\theta^j}(z_2), & \overline{\partial}Z(z_1)\,\sigma_{-\theta^j}(z_2) &\sim -z_{12}^{-\theta^j}\,\widetilde{\tau}_{-\theta^j}(z_2),\\
\overline{\partial Z}(z_1)\,\sigma_{\theta^j}(z_2) &\sim -z_{12}^{-\theta^j}\,\widetilde{\tau}_{\theta^j}(z_2), & \overline{\partial Z}(z_1)\,\sigma_{-\theta^j}(z_2) &\sim -z_{12}^{\theta^j-1}\tau_{-\theta^j}(z_2),
\end{aligned} \tag{A.1}$$

where the cases of negative angles are obtained from those of positive angles by replacing $\theta \to 1-\theta$. These OPEs allows us to determine correlators. Accordingly the OPEs on the fermionic side are:

$$\begin{aligned}
\Psi(z_1)\,s_{\theta^j}(z_2) &\sim z_{12}^{1-\theta^j}\,t_{\theta^j}(z_2), & \Psi(z_1)\,s_{-\theta^j}(z_2) &\sim z_{12}^{\theta^j}\,\widetilde{t}_{-\theta^j}(z_2),\\
\overline{\Psi}(z_1)\,s_{\theta^j}(z_2) &\sim z_{12}^{\theta^j}\,\widetilde{t}_{\theta^j}(z_2), & \overline{\Psi}(z_1)\,s_{-\theta^j}(z_2) &\sim z_{12}^{1-\theta^j}\,t_{-\theta^j}(z_2),\\
\widetilde{\Psi}(z_1)\,s_{\theta^j}(z_2) &\sim -z_{12}^{1-\theta^j}\,t_{\theta^j}(z_2), & \widetilde{\Psi}(z_1)\,s_{-\theta^j}(z_2) &\sim -z_{12}^{\theta^j}\,\widetilde{t}_{-\theta^j}(z_2),\\
\overline{\widetilde{\Psi}}(z_1)\,s_{\theta^j}(z_2) &\sim -z_{12}^{\theta^j}\,\widetilde{t}_{\theta^j}(z_2), & \overline{\widetilde{\Psi}}(z_1)\,s_{-\theta^j}(z_2) &\sim -z_{12}^{1-\theta^j}\,t_{-\theta^j}(z_2).
\end{aligned} \tag{A.2}$$

However, we will rather bosonize the fermionic fields than using these OPEs to determine the correlators.

A.1. Bosonic Twist Field Correlators

In this appendix we want to determine correlators containing bosonic twist fields starting with the two point correlator of two twist fields. From CFT we know that any two point function of primary fields is fully determined by the conformal weight. The conformal weights h and $\overline{h}$ of a primary field $\Phi(w,\overline{w})$ can be obtained from the OPE with the stress tensor $T(z)$ or $\overline{T}(\overline{z})$:

$$\begin{aligned}
T(z)\Phi(w,\overline{w}) &= \frac{h}{(z-w)^2}\,\Phi(w,\overline{w}) + \frac{1}{z-w}\,\partial_w\,\Phi(w,\overline{w}) + ...\\
\overline{T}(\overline{z})\Phi(w,\overline{w}) &= \frac{\overline{h}}{(\overline{z}-\overline{w})^2}\,\Phi(w,\overline{w}) + \frac{1}{\overline{z}-\overline{w}}\,\partial_{\overline{w}}\,\Phi(w,\overline{w}) + ...,
\end{aligned}$$

which serves as well as definition of a primary field. The stress tensor is defined as:

$$T(z) = -\frac{1}{2} : \partial Z(z_1)\, \partial \overline{Z}(z_2) : = -\frac{1}{2} \lim_{z_1 \to z_2} \left[\partial Z(z_1)\, \partial \overline{Z}(z_2) + \frac{1}{(z_1 - z_2)^2} \right]. \qquad \text{(A.3)}$$

This enables us to compute the following correlator where we have divided out the unknown two point function of twist fields:

$$g(z_1, z_2, z_3, z_4) = \frac{\langle \sigma_{\theta^j}(z_1)\, \sigma_{-\theta^j}(z_2)\, \partial Z^j(z_3)\, \partial \overline{Z}^{\,j}(z_4) \rangle}{\langle \sigma_{\theta^j}(z_1)\, \sigma_{-\theta^j}(z_2) \rangle}.$$

This correlator can be determined by examination of all two point limits, where the contractions of bosons and twist fields are determined by their OPEs given in (A.1). We find:

$$g(z_1, z_2, z_3, z_4) = z_{13}^{\theta^j - 1} z_{14}^{-\theta^j}\; z_{23}^{-\theta^j}\; z_{24}^{\theta^j - 1} z_{34}^{-2}\; f(z_1, z_2, z_3, z_4), \qquad \text{(A.4)}$$

where the function $f(z_1, z_2, z_3, z_4)$ accounts for the right singular behavior of the four point function in the limit of two point functions. Hence we are left to determine f. In the limit $z_3 \to z_4$ the four point function g has to show the following divergency:

$$\lim_{z_3 \to z_4} g(z_1, z_2, z_3, z_4) \sim z_{34}^{-2},$$

from which we read off:

$$\lim_{z_3 \to z_4} f(z_1, z_2, z_3, z_4) = \lim_{z_3 \to z_4} z_{13} z_{24}.$$

This condition is fulfilled by the following ansatz for f:

$$f(z_1, z_2, z_3, z_4) = \sum_{k,l=3}^{4} \alpha_{kl} \cdot z_{1k} z_{2l}.$$

Note that in this limit α_{kl} could as well be dependent on z_1 and z_2, i.e. they are demanded to be constant with respect to z_3 and z_4. However, we wish the function f to fix the polar behavior in all limits. Thus we can demand α_{kl} to be independent from all z_i. To determine f further we linearly approximate the z_3 dependence of $g(z_1, z_2, z_3, z_4)$ about z_4 and vice versa, that is:

$$g(z_1, z_2, z_3, z_4) \sim g(z_1, z_2, z_4, z_4) + z_{34} \cdot \partial_{z_3} g(z_1, z_2, z_3, z_4),$$

and similarly for $z_3 \leftrightarrow z_4$. We demand all terms with other divergencies than z_{34}^{-2} to vanish. This leads us to the following conditions for f:

$$\left[(1 - \theta^j) \cdot z_{13}^{-1} + \theta^j \cdot z_{23}^{-1} \right] \cdot f - \partial_{z_3} f = 0$$
$$\left[(1 - \theta^j) \cdot z_{24}^{-1} + \theta^j \cdot z_{14}^{-1} \right] \cdot f - \partial_{z_4} f = 0,$$

or written in one equation:

$$\Big[(1-\theta^j)\cdot z_{13}^{-1}+\theta^j\cdot z_{23}^{-1}\Big]\partial_{z_4}f=\Big[(1-\theta^j)\cdot z_{24}^{-1}+\theta^j\cdot z_{14}^{-1}\Big]\partial_{z_3}f,$$

from which α_{kl} can be read off. We easily find:

$$\alpha_{33}=\alpha_{44}=0,\qquad \alpha_{34}=\theta^j-1,\qquad \text{and}\qquad \alpha_{43}=-\theta^j.$$

Eventually we have determined f to be:

$$f(z_1,z_2,z_3,z_4)=(\theta^j-1)\ z_{13}z_{24}-\theta^j\ z_{14}z_{23}=-z_{13}z_{24}+\theta^j\ z_{12}z_{34}.$$

Note that we could have also determined f in any other two point limit. The result for f would be same. Finally we can write down the complete four point function:

$$g(z_1,z_2,z_3,z_4)=-\frac{1}{z_{34}^2}\left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{\theta^j}\left[1-\theta^j\frac{z_{12}z_{34}}{z_{13}z_{24}}\right].$$

This result enables us to compute the OPE of the stress tensor and twist fields. Using (A.3) we find:

$$\begin{aligned}\frac{\langle\sigma_{\theta^j}(z_1)\,\sigma_{-\theta^j}(z_2)\,T(z_4)\rangle}{\langle\sigma_{\theta^j}(z_1)\,\sigma_{-\theta^j}(z_2)\rangle}&=-\frac{1}{2}\lim_{z_3\to z_4}\left[g(z_1,z_2,z_3,z_4)+z_{34}^{-2}\right]\\&=\frac{1}{2}\theta^j(1-\theta^j)\frac{z_{12}^2}{z_{14}^2\ z_{24}^2},\end{aligned}$$

from which we can read off the conformal weight of the twist fields $h_{\sigma_{\theta^j}}=\frac{1}{2}\theta^j(1-\theta^j)$ by taking the limits $z_1\to z_4$ or $z_2\to z_4$. Hence we have determined the two-twist correlator to be:

$$\langle\sigma_{\theta^j}(z_1)\,\sigma_{-\theta^j}(z_2)\rangle=z_{12}^{-\theta^j(1-\theta^j)}. \tag{A.5}$$

Knowing the two-twist correlator we can now easily determine the full two-twist-two-boson correlators by adding the limit of $z_1\to z_2$ in (A.4). If necessary we have to add the mixing parameter D^j to the correlator of the two internal bosons. We find:

$$\begin{aligned}\langle\sigma_{\theta^j}(z_1)\,\sigma_{-\theta^j}(z_2)\,\partial Z^j(z_3)\,\partial\overline{Z}^j(z_4)\rangle&=-\frac{1}{z_{34}^2}\ z_{12}^{-\theta^j(1-\theta^j)}\left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{\theta^j}\left[1-\theta^j\frac{z_{12}z_{34}}{z_{14}z_{23}}\right]\\\langle\sigma_{\theta^j}(z_1)\,\sigma_{-\theta^j}(z_2)\,\overline{\partial}Z^j(z_3)\,\partial\overline{Z}^j(z_4)\rangle&=-\frac{\overline{D}^j}{z_{34}^2}\ z_{12}^{-\theta^j(1-\theta^j)}\left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{\theta^j}\left[1-\theta^j\frac{z_{12}z_{34}}{z_{13}z_{24}}\right],\end{aligned} \tag{A.6}$$

where the correct polar behavior of the limit $z_1\to z_2$ is ensured by the same function f as in the other limits. The choice of mixing parameters is dictated by (4.17). The other combinations of two bosons and twists follow from similar considerations. These correlators also enables us to compute the correlator between two excited twist fields, where we again refer to the OPEs (A.1). Take:

$$\langle \sigma_{\theta^j}(z_1)\,\sigma_{-\theta^j}(z_2)\,\partial Z^j(z_3)\,\overline{\partial Z}^{\,j}(z_4)\rangle = -\frac{D^j}{z_{34}^2}\; z_{12}^{-\theta^j(1-\theta^j)}\left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{\theta^j}\left[1-\theta^j\,\frac{z_{12}z_{34}}{z_{14}z_{23}}\right].$$

The limit $z_1 \to z_3$ and $z_2 \to z_4$ corresponds to τ_{θ_j} and $\tau_{-\theta_j}$ respectively, that is:

$$\lim_{\substack{z_1\to z_3\\ z_2\to z_4}} z_{12}^{-\theta^j(1-\theta^j)}\,\frac{D^j}{z_{34}^2}\left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{\theta^j}\left[1-\theta^j\,\frac{z_{12}z_{34}}{z_{14}z_{23}}\right] = z_{13}^{\theta^j-1}\tau_{\theta^j}(z_1)\; z_{24}^{\theta^j-1}\tau_{-\theta^j}(z_2).$$

We can read off:

$$\langle \tau_{\theta^j}(z_1)\tau_{-\theta^j}(z_2)\rangle = \theta^j\cdot z_{12}^{-\theta^j(3-\theta^j)}. \qquad \text{(A.7)}$$

Taking the limit of $z_1 \to z_4$ and $z_2 \to z_3$ corresponds to $\widetilde{\tau}_{\theta^j}$ and $\widetilde{\tau}_{-\theta^j}$. We find for their correlator

$$\langle \widetilde{\tau}_{\theta^j}(z_1)\widetilde{\tau}_{-\theta^j}(z_2)\rangle = (\theta^j+1)\cdot z_{12}^{\theta^j(1+\theta^j)-2}. \qquad \text{(A.8)}$$

Finally we will give the four sigma correlator needed in the four matter field amplitude. Note that its derivation is rather lengthy so we will refer to [19] where the basics were lain to compute it. The notions in [19] were refined in heterotic orbifold compactifications in the early 1990ies, see e.g. [28], where the calculations were generalized to more than one independent angle. Recently these rather old calculations were brought into intersecting brane models, see [27] and [26]. The needed correlator can be found in [26]:

$$\langle \sigma_{\theta^j}(z_1)\,\sigma_{-\theta^j}(z_2)\,\sigma_{\nu^j}(z_3)\,\sigma_{-\nu^j}(z_4)\rangle = z_{12}^{-\theta^j(1-\theta^j)}\,z_{34}^{-\nu^j(1-\nu^j)}\times \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{\frac12\theta^j(1-\nu^j)+\frac12\nu^j(1-\theta^j)}\times I_j^{-\frac12}(x), \qquad \text{(A.9)}$$

where x is the cross ratio $x=\frac{z_{12}z_{34}}{z_{13}z_{24}}$ and:

$$I_j(x) = \frac{\sin(p\,\pi\,\theta^j)}{\pi}\Big[B_{1;j}G_{2;j}(x)H_{1;j}(1-x)+B_{2;j}G_{1;j}(x)H_{2;j}(1-x)\Big]. \qquad \text{(A.10)}$$

$B_{1;j}(x)$ and $B_{2;j}(x)$ are abbreviations for:

$$B_{1;j}(x) = \frac{\Gamma(\theta^j)\Gamma(1-\nu^j)}{\Gamma(1+\theta^j-\nu^j)}, \qquad B_{2;j}(x) = \frac{\Gamma(1-\theta^j)\Gamma(\nu^j)}{\Gamma(1-\theta^j+\nu^j)},$$

while $G_{1;j}(x)$ and $G_{2;j}(x)$ stand for:

$$G_{1;j}(x) = {}_2F_1(\theta^j, 1-\nu^j, 1; x), \qquad G_{2;j}(x) = {}_2F_1(1-\theta^j, \nu^j, 1; x),$$

and $H_{1;j}(x)$ and $H_{2;j}(x)$ are given by:

$$H_{1;j}(x) = {}_2F_1(\theta^j, 1-\nu^j, 1+\theta^j-\nu^j; x), \qquad H_{2;j}(x) = {}_2F_1(1-\theta^j, \nu^j, 1-\theta^j+\nu^j; x).$$

A.2. Fermionic Twist Field Correlators

We shall now turn to the fermionic sector where we benefit from the fact that fermionic fields can be bosonized[41], which simplifies the calculations considerably. The two point correlator for fermionic twist fields is found to be:

$$\langle s_{\theta^j}(z_1)\, s_{-\theta^j}(z_2)\rangle = \langle \mathrm{e}^{\mathrm{i}(1-\theta^j)H^j(z_1)}\ \mathrm{e}^{-\mathrm{i}(1-\theta^j)H^j(z_2)}\rangle = z_{12}^{-(1-\theta^j)^2}. \tag{A.11}$$

Again we are interested in four point correlators between two twist fields and two internal fermions. Take e.g.

$$Y(z_1,z_2,z_3,z_4) = \langle s_{\theta^j}(z_1) s_{-\theta^j}(z_2) \Psi^j(z_3) \overline{\widetilde{\Psi}}{}^j(z_4)\rangle.$$

Bosonizing both spin twist fields and internal fermions yields:

$$Y = \langle \mathrm{e}^{\mathrm{i}(1-\theta^j)H_j(z_1)} \mathrm{e}^{-\mathrm{i}(1-\theta^j)H_j(z_2)} \mathrm{e}^{\mathrm{i}H^j(z_3)} \mathrm{e}^{-\mathrm{i}\widetilde{H}^j(z_4)}\rangle = z_{12}^{-(1-\theta^j)^2} \frac{D^j}{z_{34}} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{1-\theta^j}.$$

It is easy to see that this four point functions shows the correct polar behavior in the limit of two point functions, that is we do not need to add further factors. However, note that taking the bar on the other fermion amounts to changing $(1-\theta_j) \to -(1-\theta_j)$. We have:

$$\begin{aligned} \langle s_{\theta^j}(z_1)\, s_{-\theta^j}(z_2)\, \Psi^j(z_3)\, \overline{\widetilde{\Psi}}{}^j(z_4)\rangle &= z_{12}^{-(1-\theta^j)^2} \frac{D^j}{z_{34}} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{(1-\theta^j)} \\ \langle s_{\theta^j}(z_1)\, s_{-\theta^j}(z_2)\, \overline{\Psi}{}^j(z_3)\, \widetilde{\Psi}^j(z_4)\rangle &= z_{12}^{-(1-\theta^j)^2} \frac{\overline{D}{}^j}{z_{34}} \left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{(1-\theta^j)}. \end{aligned} \tag{A.12}$$

Again we could compute the two point functions of excited fermionic twist fields by taking the according limits of above four point correlator. But bosonizing[42] is again much simpler:

$$\langle t_{\theta^j}(z_1)\, t_{-\theta^j}(z_2)\rangle = \langle \mathrm{e}^{-\mathrm{i}\theta^j H^j(z_1)} \mathrm{e}^{\mathrm{i}\theta^j H^j(z_1)}\rangle = z_{12}^{-(\theta^j)^2}. \tag{A.13}$$

[41] Fermionic twist fields are bosonized as follows: $s_{\theta^j}(z) = \mathrm{e}^{\mathrm{i}(1-\theta^j)H^j(z)}$.

[42] Excited fermionic twist fields $t_{\theta^j}(z)$ are bosonized according to non-excited $s_{\theta^j}(z)$ after taking $\theta^j \to \theta^j - 1$, i.e. $t_{\theta^j}(z) = \mathrm{e}^{-\mathrm{i}\theta^j H^j(z)}$.

Accordingly we can state the correlators of two excited fermionic twists and two internal spinors:

$$\begin{aligned}
\langle t_{\theta^j}(z_1)\, t_{-\theta^j}(z_2)\, \Psi^j(z_3)\, \overline{\widetilde{\Psi}}{}^j(z_4)\rangle &= z_{12}^{-(\theta^j)^2}\, \frac{D^j}{z_{34}} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{\theta^j} \\
\langle t_{\theta^j}(z_1)\, t_{-\theta^j}(z_2)\, \overline{\Psi}{}^j(z_3)\, \widetilde{\Psi}^j(z_4)\rangle &= z_{12}^{-(\theta^j)^2}\, \frac{\overline{D}{}^j}{z_{34}} \left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{\theta^j} .
\end{aligned} \tag{A.14}$$

Finally we shall compute correlators involving four twist fields. For four spin twist fields we have:

$$\langle s_{\theta^j}(z_1)\, s_{-\theta^j}(z_2)\, s_{\nu^j}(z_3)\, s_{-\nu^j}(z_4)\rangle = z_{12}^{-(\theta^j-1)^2}\, z_{34}^{-(\nu^j-1)^2} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{(\theta^j-1)(\nu^j-1)} . \tag{A.15}$$

For four excited spin twist fields we have:

$$\langle t_{\theta^j}(z_1)\, t_{-\theta^j}(z_2)\, t_{\nu^j}(z_3)\, t_{-\nu^j}(z_4)\rangle = z_{12}^{-(\theta^j)^2}\, z_{34}^{-(\nu^j)^2} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}}\right)^{\theta^j\nu^j} . \tag{A.16}$$

Appendix B. Special Functions

In this appendix we will have a closer look on the functions occurring in various amplitudes. In nearly all four point string amplitudes we are faced with Beta or Gamma functions which are presented in the first section, while in amplitudes with four twist fields we have to deal with hypergeometric functions as well. Those are presented in the second section.

B.1. Beta and Gamma Functions

Most string amplitudes produce integrals of the following type:

$$B(a,b) = \int_0^1 \mathrm{d}x\; x^{a-1}(1-x)^{b-1},$$

which coincides with the definition of the Euler beta function. Performing a change of variables, namely $x \to \frac{y}{\gamma}$ for fixed γ yields [9]:

$$B(a,b) = \gamma^{1-a-b} \int_0^\gamma \mathrm{d}y\; y^{a-1}(\gamma - y)^{b-1}.$$

Now we can add a factor $\mathrm{e}^{-\gamma}$ to both sides and integrate $\int_0^\infty \mathrm{d}\gamma$:

$$\int_0^\infty \mathrm{d}\gamma\; \mathrm{e}^{-\gamma}\gamma^{a+b-1} B(a,b) = \int_0^\infty \mathrm{d}\gamma\; \mathrm{e}^{-\gamma} \int_0^\gamma \mathrm{d}y\; y^{a-1}(\gamma - y)^{b-1}.$$

Recalling the integral representation of the Euler gamma function:

$$\Gamma(a) = \int_0^\infty \mathrm{d}\gamma\; \mathrm{e}^{-\gamma}\gamma^{a-1}$$

we easily see:

$$\begin{aligned}
\Gamma(a+b)B(a,b) =& \int_0^\infty \mathrm{d}\gamma\; \mathrm{e}^{-\gamma} \int_0^\gamma \mathrm{d}y\; y^{a-1}(\gamma - y)^{b-1} \\
=& \int_0^\infty \mathrm{d}y\; \mathrm{e}^{-y} y^{a-1} \int_0^\infty \mathrm{d}(\gamma - y)\mathrm{e}^{-(\gamma-y)}(\gamma - y)^{b-1} \\
=& \Gamma(a)\Gamma(b).
\end{aligned}$$

Thus we have found a very useful relation:

$$B(a,b) = \int_0^1 \mathrm{d}x\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = B(b,a). \tag{B.1}$$

Gamma functions are in general well known. Especially their polar behavior can be easily obtained. Furthermore they obey certain identities which makes it possible to express different contributions to the amplitude through a single term. Some of the most important gamma identities shall be listed here:

$$\begin{aligned}\Gamma(x+1) &= x\cdot\Gamma(x)\\ \Gamma(x-1) &= \frac{\Gamma(x)}{x-1}\\ \Gamma(x)\cdot\Gamma(1-x) &= \frac{\pi}{\sin\pi x}\\ \Gamma\Big(\frac{x}{2}\Big)\cdot\Gamma\Big(\frac{x}{2}+\frac{1}{2}\Big) &= \frac{\sqrt{\pi}}{2^{x-1}}\,\Gamma(x).\end{aligned} \tag{B.2}$$

The Euler beta function can be easily continued onto the whole complex plane which is necessary for describing closed strings:

$$\begin{aligned}&\int_C \mathrm{d}^2z\, z^{a-1+\alpha}\overline{z}^{a-1+\beta}(1-z)^{b-1+\gamma}(1-\overline{z})^{b-1+\delta}\\ &= \sin\big(\pi(b+\delta)\big)\; B(a+\alpha,b+\gamma)\; B(b+\delta,1-a-b-\beta-\delta).\end{aligned}$$

For the sake of completeness we give another integral yielding gamma functions [24]:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \left(\frac{x^2+1}{2x}\right)^{2t} \frac{A}{x^2+1} = -2tA\cdot\frac{\Gamma(-2t)}{\Gamma^2(1-t)}.$$

This integral is often encountered in open-closed string scattering, while the former integral emerges in open-open and closed-closed string amplitudes.

B.2. The Hypergeometric Function

Bosonic four twist correlators contain the hypergeometric function ${}_2F_1(a,b,c;x)$. We will thus dedicate this appendix to this functions and discuss its limits used in the preceding calculations. The identities and properties being used in this appendix are subject of most mathematical standard literature. We mainly used [30]. First of all we give an integral representation of ${}_2F_1(a,b,c;x)$:

$${}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 \mathrm{d}t\, \frac{t^{b-1}(1-t)^{c-b-1}}{(1-t\cdot x)^a}, \tag{B.3}$$

where a, b and c are parameters while the variable x is the argument. Note that the argument is actually a complex number. For our purposes the argument is real though, because we have fixed all vertex positions on the real line. The parameters are in general complex numbers too. In our cases they consist of angles which makes them real numbers as well. ${}_2F_1(a,b,c;x)$ shows absolute convergence if $|x| = 1$ and $c > a + b$. A remarkable property of the hypergeometric function is that it has very simple derivatives with respect to x, that is:

$$\frac{\mathrm{d}}{\mathrm{d}x}\,{}_2F_1(a,b,c;x) = \frac{a \cdot b}{c}\,{}_2F_1(a+1,b+1,c+1;x). \tag{B.4}$$

From (B.4) we easily get the series expansion of ${}_2F_1(a,b,c;x)$:

$${}_2F_1(a,b,c;x) = 1 + \sum_{j=1}^{\infty} \prod_{k=0}^{j-1} \frac{(a-k)(b-k)}{(c-k)}\,\frac{x^j}{j\,!}, \tag{B.5}$$

which is often abbreviated by:

$${}_2F_1(a,b,c;x) = \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{(c)_j}\,\frac{x^j}{j\,!}, \tag{B.6}$$

where $(a)_j = \Gamma(a+j)\,/\,\Gamma(a)$ are so-called Pochhammer symbols with the following derivative:

$$\frac{\mathrm{d}}{\mathrm{d}a}(a)_n = \big[\psi(a+n) - \psi(a)\big] \cdot (a)_n,$$

where $\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \Gamma(z)$ is the digamma function. The digamma function will as well emerge in the limit of $x \to 1$.

Apparently the hypergeometric function is rather unwieldy especially since it occurs inside integrals. Besides numerically integrating we can as well approximate the hypergeometric function and then integrate analytically. The limits of interest are 0 and $+\infty$. Since in (A.10) we face arguments x and $1-x$ in both terms we have to think about the limits $x \to 1$ and $x \to -\infty$ as well. The limit $x \to 0$ can be easily read off:

$$\begin{aligned}\lim_{x\to 0} {}_2F_1(a,b,c;x) =& \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 \mathrm{d}t\; t^{b-1}(1-t)^{c-b-1} \\ =& \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} B(b, c-b) = 1,\end{aligned}$$

which could of course have as well been extracted form (B.5). Note that this limit is valid for arbitrary parameters a, b and c. The limit of $x \to 1$ is somewhat harder as it depends on the parameters which forces us to think about convergence. For the convergent case $x = \pm 1$ and $c > a + b$ the limit is easy to obtain:

$$\lim_{x\to 1} {}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int\limits_0^1 \mathrm{d}t\, t^{b-1}(1-t)^{c-a-b-1} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \quad \text{(B.7)}$$

Unfortunately this simple case is not applicable for our purposes. We are faced with the special case $c = a + b$, where convergence is not ensured. There is a simple trick to get rid of the limit of the hypergeometric function itself. We will apply a certain identity which transforms the argument x to $x - 1$ reducing the limit to the limit of $x \to 0$ which we have just determined. The needed identity is:

$$\begin{aligned} {}_2F_1(a,b,c;x) =& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\; {}_2F_1(a,b,a+b-c+1;1-x) + \\ & (1-x)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\; {}_2F_1(c-a,c-b,c-a-b+1;1-x). \end{aligned}$$

Note the (B.7) also follows from this formula. For the needed case $c = a + b$ we find:

$${}_2F_1(a,b,a+b;x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\; {}_2F_1(a,b,1;1-x)\cdot\Big[\Gamma(c-a-b)+\Gamma(a+b-c)\Big].$$

The limit of the hypergeometric expression now equals one but on the other hand we have a rather unwieldy expression containing two divergent gamma functions. However, this expression can be approximated:

$$\Gamma(c-a-b)+\Gamma(a+b-c) = 2\psi(1)-\psi(a)-\psi(b)-\ln(1-x),$$

where $\psi(x)$ is again the digamma function. This expression now enables us to write down the desired limit:

$$\begin{aligned} \lim_{x\to 1} {}_2F_1(a,b,a+b;x) = & \lim_{x\to 1}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\; {}_2F_1(a,b,1;1-x)\Big[2\psi(1)-\psi(a)-\psi(b)-\ln(1-x)\Big] \\ =& \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\;\Big[2\psi(1)-\psi(a)-\psi(b)-\ln(x)\Big]. \end{aligned} \quad \text{(B.8)}$$

Another identity allows us to easily determine the asymptotic behavior of the hypergeometric function, that is the limit of $x \to \infty$. The identity of choice is:

$$\begin{aligned} {}_2F_1(a,b,c;x) =& (-x)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}{}_2F_1(a,1-c+a,1-b+a;1/x) + \\ & (-x)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}{}_2F_1(b,1-c+b,1-a+b;1/x). \end{aligned}$$

Taking the limit of $x \to \infty$ on the l.h.s amounts to taking the limit of $x \to 0$ on the r.h.s. This leaves us with:

$$\lim_{x\to\infty} {}_2F_1(a,b,c;x) = (-x)^{-a}\,\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} + (-x)^{-b}\,\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}. \tag{B.9}$$

To complete the menagerie we will present yet another identity which allows us to determine the asymptotic behavior when the argument is $1-x$ which is evidently the case in (A.10). This is essentially equivalent to the limit of $x \to -\infty$. The identity reads:

$$\begin{aligned} {}_2F_1(a,b,c;x) =& (1-x)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}{}_2F_1(a,c-b,a-b+1;1/(1-x)) + \\ & (1-x)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}{}_2F_1(b,c-a,b-a+1;1/(1-x)). \end{aligned}$$

This leads us to:

$$\lim_{x\to\infty} {}_2F_1(a,b,c;1-x) = (-x)^{-a}\,\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} + (-x)^{-b}\,\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}. \tag{B.10}$$

Note that (B.9) and (B.10) are the same. This can be simply explained with the fact that the argument x of (B.3) is actually a complex number. What we just calculated is really $|x| \to \infty$. For our purposes x is a real number though. To distinguish between (B.9) and (B.10) one often introduces phase factors as was done in the first line of (9.23).

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