# Nearly Chebyshev sets are almost convex

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*Dedicated to the memory of Jonathan M. Borwein, a mentor and a dear friend.*

**Abstract.** In this paper we show that nearly Chebyshev sets are almost convex and nearly uniquely remotal sets are almost singleton sets.

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## 1 Introduction

For many years J. M. Borwein had an interest in questions concerning Chebyshev sets and some other related problems to do with proximinal and almost proximinal sets. During his life time Jon published many interesting papers on these topics, including the papers [3–6]. One of Jon's favourite problems in this area was whether every Chebyshev subset of a Hilbert space is convex. In recent years little significant progress has been made on this topic, apart from some counter-examples in (incomplete) inner product spaces. The first was in [14], but this contained several errors, some of which were subsequently corrected in [13]. Later, a geometric construction was given in [2]. Unfortunately, this example is very difficult to read. At Jon's insistence the author of the current paper carefully, along with his masters student James Fletcher, read the proof of the counter-example given in [2]. They then wrote out a long complete proof of the construction of the counter-example and convinced themselves that it was correct. This became the basis of James' masters thesis and later, a survey article [11]. So in a concrete way the paper [11] was initiated by Jon himself.

In this paper, we intend to look further into the problem of the convexity of Chebyshev sets and the related problem of uniquely remotal sets. In keeping with Jon, we will try to obtain a new perspective on this topic rather than just going over old ground again. We will take a "quantitative look" at the Chebyshev problem, with the hope that it might give rise to some new insights that were previously concealed in the classical study of Chebyshev sets. In particular it will reveal the precise relationship between the size of the sets of nearest points relative to the degree of convexity of the underlying set.

We will start by considering a quantitative version of the notion of a uniquely remotal set (called an $\varepsilon$-uniquely remotal set) and show that such sets have a diameter of at most $2\varepsilon$. Uniquely remotal sets are directly related to Chebyshev sets via the paper [1] which shows that there exists a non-convex Chebyshev subset of a Hilbert space if, and only if, there exists a non-trivial set (i.e., a non-singleton set) that is uniquely remotal, also see [11, Theorem 3.22] for details.

Secondly, we will show that under suitable circumstances "nearly" Chebyshev sets (i.e., $\varepsilon$-Chebyshev sets) are "nearly" convex (i.e., $\varepsilon$-convex). This then raises the question of whether there exists an

$\varepsilon$-Chebyshev set that is not $\varepsilon$-convex. Constructing such a counter-example should be easier than constructing a Chebyshev set that is not convex, see [11, Section 4]. However, it may serve as a stepping stone towards the construction of a non-convex Chebyshev subset of a Hilbert space. Finally, we will end with some open problems regarding $\varepsilon$-uniquely remotal sets and $\varepsilon$-Chebyshev sets.

**Notation:** In this paper all normed linear spaces will be over the field of real numbers (denoted $\mathbb{R}$). The *closed unit ball* in a normed linear space $(X, \|\cdot\|)$ will be denoted by $B_X$ and the *norm closed convex hull* of a subset $K$ of a normed linear space $(X, \|\cdot\|)$ will be denoted by, $\overline{\text{co}}(K)$. The set of all subsets of a set $K$ will be denoted by $\mathcal{P}(K)$. If $X$ is a set and $f : X \to \mathbb{R} \cup \{\infty\}$ is a function then $\text{Dom}(f) := \{x \in X : f(x) < \infty\}$ and we say that $f$ is a *proper* function if $\text{Dom}(f) = \varnothing$. We shall call a proper function $f : X \to \mathbb{R} \cup \{\infty\}$, defined on a vector space $X$, (over the real numbers) a *convex* function if for each $x, y \in \text{Dom}(f)$ and $0 < \lambda < 1$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

## 2 Uniquely remotal sets

This paper is a first attempt at considering a quantitative version of the Chebyshev set problem. We have opted for simplicity (of presentation) over full generality, but hope that the ideas introduced here can be extended to more general situations. We also hope that the quantitative approach presented here highlights the role that the geometry of the norm plays in all our consderations.

We begin with some introductory notions.

Let $(X, \|\cdot\|)$ be a normed linear space and $K$ be a nonempty bounded subset of $X$. For any point $x \in X$, we define $r(x, K) := \sup_{y \in K} \|x - y\|$. We refer to the map $x \mapsto r(x, K)$ as the *radial function for* $K$, [11, page 187].

**Proposition 1 ([11, Proposition 3.11])** *Let $K$ be a nonempty bounded subset of a normed linear space $(X, \|\cdot\|)$. Then the radial function for $K$ is convex and nonexpansive (and hence continuous).*

Let $(X, \|\cdot\|)$ be a normed linear space and let $K$ be a subset of $X$. We define a set valued mapping $F_K : X \to \mathcal{P}(K)$ by $F_K(x) := \{y \in K : \|x - y\| = r(x, K)\}$, if $K$ is nonempty and bounded and by $F_K(x) = \emptyset$ otherwise. We refer to the elements of $F_K(x)$ as the *farthest points from $x$ in $K$*.

We say that $K$ is a *remotal set* if $F_K(x)$ is a nonempty for each $x \in X$. Furthermore, we say that $K$ is a *uniquely remotal set* if $F_K(x)$ is a singleton for each $x \in X$, [11, page 187].

**Remark 1 ([11, Remark 3.13])** *It follows from the definition that every remotal set is nonempty and bounded.*

In this section we will also consider the following "quantitative" version of these notions. Let $\varepsilon > 0$, then we shall say that $K$ is an *$\varepsilon$-uniquely remotal set* if $K$ is remotal and $\text{diam}(F_K(x)) \leq \varepsilon$ for each $x \in X$.

A special case of Ekeland's variation principle, [9], which is sufficient for our purposes, is given next.

**Proposition 2 ([11, Theorem 3.33])** *Let $(X, d)$ be a complete metric space and let $f : X \to \mathbb{R}$ be a bounded below, lower semi-continuous function on $X$. If $\varepsilon_0 > 0$ then there exists $x_\infty \in X$ such that:*

$$-\varepsilon_0 < \frac{f(x) - f(x_\infty)}{d(x, x_\infty)} \quad \textit{for all } x \in X \setminus \{x_\infty\}.$$

For our first theorem we will need to call upon the following elementary result that relates the radial function to the farthest point mapping.

**Proposition 3 ([11, Lemma 3.14])** *Let $K$ be a non-trivial (i.e., not a singleton) remotal set in a normed linear space $(X, \|\cdot\|)$. Let $x \in X$ and $z \in F_K(x)$. If for each $\lambda \in (0,1)$, $x_\lambda := x + \lambda(z-x)$, then*

$$\frac{r(x_\lambda, K) - r(x, K)}{\|x_\lambda - x\|} \leq -\left(1 - \frac{\|z - z_\lambda\|}{r(x, K)}\right),$$

*where $z_\lambda \in F_K(x_\lambda)$ for each $\lambda \in (0,1)$.*

The last ingredient that we need in order to state our first theorem is the notion of "metric upper semicontinuity", [15, page 53]. A set-valued mapping $\Phi$ from a topological space $A$ into nonempty subsets of a metric space $(X, d)$ is *metric upper semicontinuous at $t_0 \in A$* if for every $\varepsilon > 0$ there exists an open neighbourhood $U$ of $t_0$ such that $\Phi(U) \subseteq \bigcup\{B(x; \varepsilon) : x \in \Phi(t_0)\}$. If $\Phi$ is metric upper semicontinuous at each point of $A$ then we say that $\Phi$ is *metric upper semicontinuous on $A$*.

**Theorem 1** *Let $(X, \|\cdot\|)$ be a Banach space, $\varepsilon > 0$ and $K$ be an $\varepsilon$-uniquely remotal set in $X$. If the farthest point mapping, $x \mapsto F_K(x)$, is metric upper semicontinuous on $X$, then diam$(K) \leq 2\varepsilon$.*

**Proof:** Suppose, in order to obtain a contradiction, that $2\varepsilon < D := \mathrm{diam}(K)$. Then choose $\varepsilon' > \varepsilon$ such that $2\varepsilon' < D$. By applying Proposition 2 to the mapping, $x \mapsto r(x, K)$, with $\varepsilon_0 := (1 - \frac{2\varepsilon'}{D}) > 0$ we obtain the existence of a point $x_\infty \in X$ such that

$$\left(\frac{2\varepsilon'}{D} - 1\right) < \frac{r(x, K) - r(x_\infty, K)}{\|x - x_\infty\|} \qquad \text{for all } x \in X \setminus \{x_\infty\}. \tag{1}$$

Let us also note that $D \leq 2r(x_\infty, K)$ since, $K \subseteq B[x_\infty; r(x_\infty, K)]$. By Proposition 3, we have that for any $\lambda \in (0,1)$

$$\frac{r(x_\lambda, K) - r(x_\infty, K)}{\|x_\lambda - x_\infty\|} \leq -\left(1 - \frac{\|z_\infty - z_\lambda\|}{r(x_\infty, K)}\right) = \left(\frac{\|z_\infty - z_\lambda\|}{r(x_\infty, K)} - 1\right) \leq \left(\frac{2\|z_\infty - z_\lambda\|}{D} - 1\right),$$

where, $z_\infty \in F_K(x_\infty)$, $x_\lambda := x_\infty + \lambda(z_\infty - x_\infty)$ and $z_\lambda \in F_K(x_\lambda)$. Since $\mathrm{diam}(F_K(x_\infty)) \leq \varepsilon$ and $F_K$ is metric upper semicontinuous at $x_\infty$, there exists $\lambda_0 \in (0,1)$ such that, $\|z_\infty - z_{\lambda_0}\| < \varepsilon'$. To see this, first note that $F_K(x_\infty) \subseteq B[z_\infty; \varepsilon]$. Let $r := (\varepsilon' - \varepsilon)/2 > 0$. Then,

$$\bigcup\{B(z; r) : z \in F_K(x_\infty)\} \subseteq \bigcup\{B(z; r) : z \in B[z_\infty; \varepsilon] \subseteq B(z_\infty; \varepsilon + r) \subseteq B(z_\infty; \varepsilon').$$

Since $F_K$ is metric upper semicontinuous there exists a $0 < \delta$ such that

$$F_K(B(x_\infty; \delta)) \subseteq \bigcup\{B(z; r) : z \in F_K(x_\infty)\} \subseteq B(z_\infty; \varepsilon');$$

and the result follows. Hence,

$$\frac{r(x_{\lambda_0}, K) - r(x_\infty, K)}{\|x_{\lambda_0} - x_\infty\|} < \left(\frac{2\varepsilon'}{D} - 1\right)$$

which contradicts Equation (1) since $x_{\lambda_0} \neq x_\infty$. ☺

In reflexive spaces we can improve upon this result. However, we will need to introduce some more definitions. A set-valued mapping $\Phi$ from a topological space $A$ into subsets of a topological space

$(X, \tau)$ is *$\tau$-upper semicontinuous* if for each $\tau$-open set $W$ in $X$, $\{t \in A : \Phi(t) \subseteq W\}$ is open in $A$. If $\Phi$ also has nonempty compact images then we call $\Phi$ a *$\tau$-usco mapping*. Further, if $(X, \tau)$ is a linear topological space then we call an $\tau$-usco mapping whose values are also convex subsets of $X$ a *$\tau$-cusco mapping*. An important fixed point theorem for $\tau$-cuscos $\Phi : C \to 2^C$ defined on a nonempty compact convex subset $C$ of a Hausdorff locally convex space $(X, \tau)$ is the Kakutani-Glicksberg-Fan fixed point theorem, [10,12], which states that there must exist a point $x_0 \in C$ such that $x_0 \in \Phi(x_0)$. A slightly weaker version of upper semicontinuity is the following, [16, page 218]. A set-valued mapping $\Phi$ from a topological space $A$ into nonempty subsets of a linear topological space $(X, \tau)$ is *$\tau$-Hausdorff upper semicontinuous at $t_0 \in A$* if for every $\tau$-open neighbourhood $W$ of 0 in $X$, there exists a neighbourhood $U$ of $t_0$ such that $\Phi(U) \subseteq \Phi(t_0) + W$. If $\Phi$ is $\tau$-Hausdorff semicontinuous at each point of $A$ then we say that $\Phi$ is *$\tau$-Hausdorff upper semicontinuous on $A$*. In general, $\tau$-Hausdorff upper semicontinuity is a weaker notion than $\tau$-upper semicontinuity, but the notions coincide if the mapping $\Phi$ has nonempty compact images.

A useful result connecting $\tau$-Hausdorff upper semicontinuity to $\tau$-upper semicontinuity is the following.

**Proposition 4 ([17, Lemma 7.12])** *Suppose that $\Phi : A \to 2^X$ is a $\tau$-Hausdorff upper semicontinuous set-valued mapping from a topological space $A$ into nonempty subsets of a Hausdorff locally convex space $(X, \tau)$. If for each $t \in A$, $\overline{co}^\tau \Phi(t)$ is a compact subset of $X$, then the mapping $\Psi : A \to 2^X$ defined by, $\Psi(t) := \overline{co}^\tau \Phi(t)$ for all $t \in A$, is a $\tau$-cusco on $A$.*

We can now prove our second theorem.

**Theorem 2** *Let $(X, \|\cdot\|)$ be a reflexive Banach space, let $\varepsilon > 0$ and let $K$ be an $\varepsilon$-uniquely remotal set in $X$. If the farthest point mapping, $x \mapsto F_K(x)$, is weak-Hausdorff upper semicontinuous on $(\overline{co}(K), weak)$, then $diam(K) \leq 2\varepsilon$.*

**Proof:** Since $K$ is a remotal set, $K$ is nonempty and bounded. Therefore, if $C := \overline{\text{co}}(K)$ then $C$ is weakly compact (and convex). Let $G : C \to 2^C$ be defined by, $G(x) := \overline{\text{co}}(F_K(x))$ for each $x \in C$. Then, by Proposition 4, $G$ is a weak-cusco on $C$. Hence, by the Kakutani-Glicksberg-Fan fixed point theorem, there exists an $x_0 \in C$ such that $x_0 \in G(x_0)$, i.e., $x_0 \in \overline{\text{co}}(F_K(x_0))$. Since

$$\text{diam}(G(x_0)) = \text{diam}(\overline{\text{co}}(F_K(x_0))) = \text{diam}(F_K(x_0)) \leq \varepsilon$$

we have that $F_K(x_0) \subseteq G(x_0) \subseteq B[x_0, \varepsilon]$. Thus, $K \subseteq B[x_0, \varepsilon]$ and so $\text{diam}(K) \leq 2\varepsilon$. ☺

**Corollary 1** *Let $(X, \|\cdot\|)$ be a Banach space, and let $\varepsilon > 0$. If $K$ is a compact $\varepsilon$-uniquely remotal set in $X$, then $diam(K) \leq 2\varepsilon$. In particular, every $\varepsilon$-uniquely remotal subset $K$ of a finite dimensional Banach space has $diam(K) \leq 2\varepsilon$.*

**Proof:** This result follows from the fact that the mapping $F_K$ has a closed graph (and nonempty images, since $K$ is remotal) and the general fact that a set-valued mapping with a closed graph and nonempty images that maps into a compact space is an usco mapping. ☺

## 3 Chebyshev sets

Let $(X, \|\cdot\|)$ be a normed linear space and $K$ be a nonempty subset of $X$. For any point $x \in X$ we define $d(x, K) := \inf_{y \in K} \|x - y\|$ and call this the *distance from $x$ to $K$*. We will also refer to the map $x \mapsto d(x, K)$ as the *distance function for $K$*, [11, page 162].

The key concept underlying the definition of a proximinal set is that of 'nearest points'. The following definition makes this idea precise.

Let $(X, \|\cdot\|)$ be a normed linear space and $K$ be a subset of $X$. We define a set-valued mapping $P_K : X \to \mathcal{P}(K)$ by

$$P_K(x) := \{y \in K : \|x - y\| = d(x, K)\},$$

if $K$ is nonempty and by $P_K(x) := \varnothing$ if $K$ is the empty set. The elements of $P_K(x)$ are said to be the *nearest points to* $x$ *in* $K$).

We say that $K$ is a *proximinal set* if $P_K(x)$ is nonempty for each $x \in X$ and that $K$ is a *Chebyshev set* if $P_K(x)$ is a singleton for each $x \in X$. In the case when $K$ is a Chebyshev set we define the map $p_K : X \to K$ as the map that assigns to each $x \in X$ the unique element of $P_K(x)$. We will refer to both mappings $P_K$ and $p_K$ as the *metric projection mapping for* $K$, [11, page 163].

In this section we will also consider the following "quantitative" version of these notions. Let $\varepsilon > 0$ then we shall say that $K$ is an $\varepsilon$*-Chebyshev set* if $K$ is proximinal and $\mathrm{diam}(P_K(x)) \leq \varepsilon$ for all $x \in X$. Furthermore, we will say that a set $K$ is $\varepsilon$*-convex* if, $\mathrm{co}(K) \subseteq K + \varepsilon B_X$.

In order to state and prove our results in this section of the paper we will need to recall some notions from optimisation theory.

Let $(X, \|\cdot\|)$ be a Banach space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper function (i.e., not identically equal to $\infty$). We say that a sequence $(x_n)_{n=1}^{\infty}$ in $X$ is *minimising* if, $\lim_{n\to\infty} f(x_n) = \inf_{x \in X} f(x)$.

Let $(X, \|\cdot\|)$ be a normed linear space and $f : X \to \mathbb{R} \cup \{\infty\}$. We say that $f$ has a *generalised strong minimum* if $f$ has a global minimum and each minimising sequence $(x_n)_{n=1}^{\infty}$ in $X$ has a subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to an element $x_0 \in X$ satisfying the equation $f(x_0) = \inf_{x \in X} f(x)$.

Let $(X, \|\cdot\|)$ be a normed linear space and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper function that is bounded below by a continuous linear functional. We define a function $\overline{\mathrm{co}}(f)$: $X \to \mathbb{R} \cup \{\infty\}$ by

$$\overline{\mathrm{co}}(f)(x) := \sup\{\psi(x) : \psi : X \to \mathbb{R} \text{ is continuous, convex and } \psi(y) \leq f(y) \text{ for all } y \in X\}.$$

It is immediate from this definition that $\overline{\mathrm{co}}(f)$ is convex, lower semicontinuous and $\overline{\mathrm{co}}(f)(x) \leq f(x)$ for all $x \in X$. Furthermore, if $x^* \in X^*$, then

$$\overline{\mathrm{co}}(f - x^*) = \overline{\mathrm{co}}(f) - x^*. \tag{2}$$

The final piece of notation that we will require from optimisation theory is the following. For a set $X$ and a function $f : X \to \mathbb{R}$, we define

$$\mathrm{argmin}(f) := \{x \in X : f(x) \leq f(y) \text{ for all } y \in X\}.$$

Our first result in this section is an extension of [11, Theorem 3.26].

**Lemma 1** *Let* $(X, \|\cdot\|)$ *be a Banach space. Suppose that* $f : X \to \mathbb{R} \cup \{\infty\}$ *and* $0 < \liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|}$. *If* $f$ *has a generalised strong minimum, then* $\mathrm{argmin}(\overline{\mathrm{co}}(f)) = \overline{\mathrm{co}}(\mathrm{argmin}(f))$.

**Proof:** It is easy to see that, $\mathrm{argmin}(f) \subseteq \mathrm{argmin}(\overline{\mathrm{co}}(f))$ and since $\mathrm{argmin}\,(\overline{\mathrm{co}}(f))$ is closed and convex, it follows that $\overline{\mathrm{co}}(\mathrm{argmin}(f)) \subseteq \mathrm{argmin}(\overline{\mathrm{co}}(f))$. Therefore, it is sufficient to show that

$$\mathrm{argmin}(\overline{\mathrm{co}}(f)) \subseteq \overline{\mathrm{co}}(\mathrm{argmin}(f)).$$

To do this, we will show that for each $\varepsilon > 0$, $\operatorname{argmin}(\overline{\text{co}}(f)) \subseteq \overline{\text{co}}(\operatorname{argmin}(f)) + 2\varepsilon B_X$. To this end, let $\varepsilon > 0$. We claim that for some $n \in \mathbb{N}$, the continuous convex function $c_n : X \to \mathbb{R}$, defined by

$$c_n(x) := \min_{x\in X} f + \frac{1}{n} d(x, \overline{\text{co}}(\operatorname{argmin}(f)) + \varepsilon B_X),$$

satisfies the inequality $c_n(x) \leq f(x)$ for all $x \in X$. Now suppose, in order to obtain a contradiction, that this is not the case. Then, for each $n \in \mathbb{N}$, there will exist an $x_n \in X$ such that $f(x_n) < c_n(x_n)$.

We claim that $(x_n)_{n=1}^\infty$ is a bounded sequence. Indeed, if $(x_n)_{n=1}^\infty$ is not bounded then there exists a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $k \leq \|x_{n_k}\|$ for all $k \in \mathbb{N}$. On the other hand, it follows from the triangle inequality that there exists an $0 < M$ such that

$$0 \leq d(x, \overline{\text{co}}(\operatorname{argmin}(f)) + \varepsilon B_X) \leq \|x\| + M \quad \text{for all } x \in X.$$

Therefore,

$$c_n(x) \leq \min_{x\in X} f + \frac{1}{n}(\|x\| + M) \quad \text{for all } x \in X.$$

In particular,

$$f(x_{n_k}) < c_{n_k}(x_{n_k}) \leq \min_{x\in X} f + \frac{1}{n_k}(\|x_{n_k}\| + M) \quad \text{for all } k \in \mathbb{N}.$$

Thus,

$$\begin{aligned}\limsup_{k\to\infty} \frac{f(x_{n_k})}{\|x_{n_k}\|} \leq \limsup_{k\to\infty} \frac{c_{n_k}(x_{n_k})}{\|x_{n_k}\|} &\leq \limsup_{k\to\infty} \left[\frac{\min_{x\in X} f}{\|x_{n_k}\|} + \frac{1}{n_k}\left(1 + \frac{M}{\|x_{n_k}\|}\right)\right] \\ &\leq \limsup_{k\to\infty} \frac{\min_{x\in X} f}{\|x_{n_k}\|} + \limsup_{k\to\infty} \frac{1}{n_k}\left(1 + \frac{M}{\|x_{n_k}\|}\right) = 0,\end{aligned}$$

which is impossible since $0 < \liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|}$. Thus, the sequence $(x_n)_{n=1}^\infty$ is bounded. Hence,

$$\min_{x\in X} f \leq \liminf_{n\to\infty} f(x_n) \leq \limsup_{n\to\infty} f(x_n) \leq \limsup_{n\to\infty} c_n(x_n) \leq \min_{x\in X} f + \limsup_{n\to\infty} \frac{1}{n}(\|x_n\| + M) = \min_{x\in X} f.$$

That is, $(x_n)_{n=1}^\infty$ is a minimising sequence for $f$. However, as $f$ has a generalised strong minimum the sequence $(x_n)_{n=1}^\infty$ has a convergent subsequence $(x_{n_k})_{k=1}^\infty$ such that $\lim_{k\to\infty} x_{n_k} = x_0$ for some $x_0 \in \operatorname{argmin}(f)$. Therefore, for $k$ sufficiently large,

$$x_{n_k} \in B[x_0, \varepsilon] \subseteq \overline{\text{co}}(\operatorname{argmin}(f)) + \varepsilon B_X),$$

and so

$$f(x_{n_k}) < c_n(x_{n_k}) = \min_{x\in X} f + \frac{1}{n} d(x_{n_k}, \overline{\text{co}}(\operatorname{argmin}(f)) + \varepsilon B_X) = \min_{x\in X} f,$$

which is clearly impossible. This proves the claim. Finally, since $c_n$ is continuous and convex (see, [11, Proposition 2.15]), $c_n(x) \leq \overline{\text{co}}(f)(x)$ for all $x \in X$. This implies that

$$\operatorname{argmin}(\overline{\text{co}}(f)) \subseteq \overline{\text{co}}(\operatorname{argmin}(f)) + 2\varepsilon B_X$$

since, $\overline{\text{co}}(f)(x_0) = f(x_0) = \min_{x\in X} f < c_n(x) \leq \overline{\text{co}}(f)(x)$ for all $x \notin \overline{\text{co}}(\operatorname{argmin}(f)) + 2\varepsilon B_X$. ☺

A subset $K$ of a normed linear space $(X, \|\cdot\|)$ is called *approximatively compact*, [8], if for every $x \in X$ and sequence $(x_n)_{n=1}^\infty$ in $K$ such that $\lim_{n\to\infty} \|x - x_n\| = d(x, K)$ there exists a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ and an element $x_0 \in P_K(x)$ such that $(x_{n_k})_{k=1}^\infty$ converges to $x_0$.

If $f : X \to \mathbb{R} \cup \{\infty\}$ is a convex function defined on a normed linear space $(X, \|\cdot\|)$ and $x \in \mathrm{Dom}(f)$, then we define the *subdifferential of $f$ at $x$* to be the set $\partial f(x)$ of all $x^* \in X^*$ satisfying

$$x^*(y) - x^*(x) \leq f(y) - f(x) \quad \text{for all } y \in \mathrm{Dom}(f).$$

We shall denote by, $\mathrm{Dom}(\partial f(x))$, $\{x \in \mathrm{Dom}(f) : \partial f(x) \neq \varnothing\}$.

The main theorem for this section is given next and is modelled on [11, Theorem 3.27].

**Theorem 3** *Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $K$ be an approximatively compact $\varepsilon$-Chebyshev set in $X$. Then $K$ is $\varepsilon$-convex.*

**Proof:** We begin by considering the auxiliary function $f : X \to \mathbb{R} \cup \{\infty\}$ defined by

$$f(x) := \begin{cases} \|x\|^2 & \text{if } x \in K \\ \infty & \text{if } x \notin K. \end{cases}$$

In order to obtain a contradiction we shall suppose that $K$ is not $\varepsilon$-convex. Then there exists

$$z \in \mathrm{co}(K) \setminus (K + \varepsilon B_X).$$

Now, since $\mathrm{Dom}\,(\overline{\mathrm{co}}(f))$ is convex and contains $\mathrm{Dom}(f) = K$, $\mathrm{co}(K) \subseteq \mathrm{Dom}\,(\overline{\mathrm{co}}(f))$. Furthermore, $K + \varepsilon B_X$ is a closed set (not containing $z$) because $K + \varepsilon B_X = \{x \in X : d(x, K) \leq \varepsilon\}$, since $K$ is proximinal. Therefore, by the Brøndsted-Rockafellar Theorem, [7] there exists a point $z_0 \in \mathrm{Dom}\,(\partial\,(\overline{\mathrm{co}}(f))) \setminus (K + \varepsilon B_X)$. Let $x^* \in \partial\,(\overline{\mathrm{co}}(f))(z_0)$. Now, by Riesz's Representation Theorem, (see [18], or [19, page 248]), there exists $x \in X$ such that $x^*(y) = \langle y, x \rangle$ for all $y \in X$. A simple calculation now reveals that

$$(f - x^*)(k) = f(k) - \langle k, x \rangle = \left\| k - \frac{x}{2} \right\|^2 - \frac{\|x\|^2}{4} \quad \text{for all } k \in K.$$

Thus, $f - x^*$ has a generalised strong minimum and $\mathrm{argmin}(f - x^*) = P_K\left(\frac{x}{2}\right)$.

Furthermore, let us also observe that (i) $\liminf_{\|x\| \to \infty} \frac{(f - x^*)(x)}{\|x\|} = \infty$ and (ii) $z_0 \in \mathrm{argmin}\,(\overline{\mathrm{co}}(f) - x^*)$ since in general, for a convex function $g$, $y^* \in \partial g(z_0)$ if, and only if, $z_0 \in \mathrm{argmin}(g - y^*)$.

Putting all of this together we obtain the following:

$$\begin{aligned} z_0 &\in \mathrm{argmin}\,(\overline{\mathrm{co}}(f) - x^*) \quad \text{since } x^* \in \partial(\overline{\mathrm{co}}(f))(z_0) \\ &= \mathrm{argmin}\,(\overline{\mathrm{co}}(f - x^*)) \quad \text{by Equation (2)} \\ &= \overline{\mathrm{co}}(\mathrm{argmin}(f - x^*)) \quad \text{by Lemma 1} \\ &= \overline{\mathrm{co}}\left\{P_K\left(\frac{x}{2}\right)\right\} \\ &\subseteq K + \varepsilon B_X. \end{aligned}$$

The last line of the expression above follows from the fact that:

$$\mathrm{diam}\left(\overline{\mathrm{co}}\left(P_K\left(\frac{x}{2}\right)\right)\right) = \mathrm{diam}\left(P_K\left(\frac{x}{2}\right)\right) \leq \varepsilon \quad \text{and} \quad \varnothing \neq P_K\left(\frac{x}{2}\right) \subseteq \overline{\mathrm{co}}\left(P_K\left(\frac{x}{2}\right)\right) \cap K.$$

However, this is impossible, since $z_0 \notin K + \varepsilon B_X$. Hence, $K$ is $\varepsilon$-convex. ☺

**Corollary 2** *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $K$ be weakly closed $\varepsilon$-Chebyshev set in $X$, then $K$ is $\varepsilon$-convex.*

**Proof:** By Theorem 3 it is sufficient to show that $K$ is approximatively compact. To this end, let $x \in X$ and let $(x_n)_{n=1}^{\infty}$ be a sequence in $K$ such that $\lim_{n\to\infty} \|x - x_n\| = d(x, K)$. If $x \in K$ then the result is obvious, so we will consider the case when $x \notin K$. Since $X$ is reflexive and the sequence $(x_n)_{n=1}^{\infty}$ is bounded there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ and an element $x_\infty \in X$ such that $(x_{n_k})_{k=1}^{\infty}$ converges to $x_\infty \in X$ with respect to the weak topology on $X$. Since $K$ is weakly closed $x_\infty \in K$ and so $d(x, K) \leq \|x_\infty - x\|$. Furthermore, since the norm is lower semicontinuous with respect to the weak topology on $X$,

$$d(x, K) \leq \|x_\infty - x\| \leq \lim_{k\to\infty} \|x_{n_k} - x\| = d(x, K).$$

In particular, $d(x, K) = \|x_\infty - x\|$, and so $x_\infty \in P_K(x)$. Since the norm on a Hilbert space is a Kadec norm (i.e., the relative weak and norm topologies agree on the unit sphere) we have that $(x_{n_k} - x)_{k=1}^{\infty}$ converges to $x_\infty - x$, with respect to the norm topology, too. Thus, $(x_{n_k})_{k=1}^{\infty}$ converges to $x_\infty$, with respect to the norm topology, and so $K$ is approximatively compact. ☺

# 4 Open problems

We will end this article with some natural questions that have arisen from our study of $\varepsilon$-uniquely remotal sets and $\varepsilon$-Chebyshev sets.

**Question 1** *In Theorem 1 can we replace the conclusion that diam$(K) \leq 2\varepsilon$ by, diam$(K) \leq s\varepsilon$ where, $1 \leq s < 2$ ? Does it depend upon the norm ?*

**Question 2** *In Theorem 2 can we replace the conclusion that diam$(K) \leq 2\varepsilon$ by, diam$(K) \leq s\varepsilon$ where, $1 \leq s < 2$ ? Does it depend upon the norm ?*

**Question 3** *Can we prove something like Theorem 3 for other than Hilbert spaces? Of course, the conclusion that $K$ $\varepsilon$-convex would have to be relaxed to something like "$K$ is $f(\varepsilon)$-convex" for some function $f : (0, \infty) \to (0, \infty)$, that only depends upon the norm.*

**Question 4** *Is it true that every approximatively compact $2\varepsilon$-Chebyshev set in a Hilbert space $X$ is $\varepsilon$-convex?*

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