On a one-sided James' theorem

Warren B. Moors

Abstract. We provide a short proof of the following fact: If X is a Banach space, A and B are bounded, closed and convex sets with $\text{dist}(A, B) > 0$ and every $x^* \in X^*$ with the property that $\sup(x^*, B) < \inf(x^*, A)$ attains its infimum on A and its supremum on B, then both A and B are weakly compact.

AMS (2010) subject classification: Primary 46B20, 46B22.

Keywords: weakly compact sets, James' theorem

The result mentioned in the abstract was first proved in [1, Theorem 2] for Banach spaces whose dual ball is weak* convex block compact. In this short paper we prove this result for arbitrary Banach spaces. For any nonempty bounded subset A of a Banach space X and any $x^* \in X^*$ we shall denote by, $\sup(x^*, A) := \sup\{x^*(a) : a \in A\}$ and by $\inf(x^*, A) := \inf\{x^*(a) : a \in A\}$.

Lemma 1 *Let $(Y, \|\cdot\|)$ be a Banach space and C be a nonempty bounded subset of $Y \times \mathbb{R}$, endowed with the norm $\|(y, r)\|_1 := \|y\| + |r|$. If for every $x^* \in Y^*$, $\max\{(x^*, -1)(y, s) : (y, s) \in C\}$ exists then C is relatively weakly compact.*

Proof: Let $\pi : Y \times \mathbb{R} \to Y$ be defined by $\pi(y, r) := y$, $A := \pi(C)$ and $f : Y \to \mathbb{R} \cup \{\infty\}$ be defined by,

$$f(y) := \begin{cases} \inf\{s \in \mathbb{R} : (y, s) \in C\} & \text{if } y \in A \\ \infty & \text{if } y \notin A. \end{cases}$$

Then f is a proper function on Y and $x^* - f$ attains it maximum for every $x^* \in Y^*$. Therefore, by [2, Theorem 1], (or [4, Theorem 2.4]) for each $a \in \mathbb{R}$, $S(a) := \{(y, s) \in Y \times \mathbb{R} : f(y) \leq s \leq a\}$ is relatively weakly compact. Since C is bounded there exists an $a \in \mathbb{R}$ such that $C \subseteq S(a)$. $\square$

Theorem 1 *Let X be a Banach space and let A and B be bounded, closed and convex sets with dist$(A, B) > 0$. If every $x^* \in X^*$ with $\sup(x^*, B) < \inf(x^*, A)$ attains its infimum on A and its supremum on B, then both A and B are weakly compact.*

Proof: To show that both A and B are weakly compact it is sufficient (and necessary) to show that $B - A$ is weakly compact. This will be our approach. From the hypotheses it follows that if $C := \overline{B - A}$, then C is a bounded nonempty closed and convex subset of X with $0 \notin C$. Furthermore, it follows that each $x^* \in X^*$ with $\sup(x^*, C) < 0$ attains it supremum on C. Choose $y^* \in X^*$ such that $\sup(y^*, C) < 0$. Note that such a functional exists by the Hahn-Banach theorem. Let $Y := \ker(y^*)$ and choose $x_0 \in C$. Define $S : Y \times \mathbb{R} \to X$ by, $S(y, r) := y + rx_0$ and let us consider $Y \times \mathbb{R}$ endowed with the norm $\|(y, r)\|_1 := \|y\| + |r|$. Then S is an isomorphism and there exists an $0 < \varepsilon$ such that $S^{-1}(C) \subseteq \{(y, r) \in Y \times \mathbb{R} : \varepsilon \leq r\}$. Moreover, each $(x^*, r) \in (Y \times \mathbb{R})^*$ with $\sup((x^*, r), S^{-1}(C)) < 0$ attains its supremum over $S^{-1}(C)$. Let $\pi : Y \times \mathbb{R} \to Y$ be defined by $\pi(y, r) := y$, $A := \pi(S^{-1}(C))$ and $f : Y \to \mathbb{R} \cup \{\infty\}$ be defined by,

$$f(y) := \begin{cases} \inf\{s \in \mathbb{R} : (y, s) \in S^{-1}(C)\} & \text{if } y \in A \\ \infty & \text{if } y \notin A. \end{cases}$$

Next, we define $T : Y \times (\mathbb{R} \setminus \{0\}) \to Y \times (\mathbb{R} \setminus \{0\})$ by $T(y, s) := s^{-1}(y, -1)$. Then T is a bijection. In fact, T is a homeomorphism when $Y \times (\mathbb{R} \setminus \{0\})$ is considered with the relative weak topology. Let $p : Y^* \to \mathbb{R}$ be defined by,

$$p(x^*) := \sup_{y \in Y} [x^*(y) - f(y)] = \sup((x^*, -1), S^{-1}(C)).$$

It is routine to check that p is real-valued and convex on Y^*. To show that C is weakly compact it is sufficient to show that $T(S^{-1}(C))$ is a relatively weakly compact subset of $Y \times \mathbb{R}$. To achieve this we appeal to Lemma 1. So let $x^* \in Y^*$. We consider two cases.

Case (I) Suppose that for every $0 < \lambda$, $p(\lambda x^*) \leq -\lambda$. Then $x^*(y) - \lambda^{-1} f(y) \leq -1$ for all $y \in Y$ and all $0 < \lambda$. In particular, $-\lambda^{-1} f(0) \leq -1$ for all $0 < \lambda$, i.e., $\lambda \leq f(0)$ for all $0 < \lambda$. On the other hand, $S(0, 1) = x_0 \in C$, i.e., $(0, 1) \in S^{-1}(C)$ and so $f(0) \leq 1$. Thus, Case (I) does not occur.

Case(II) Suppose that for some $0 < \lambda$, $-\lambda < p(\lambda x^*)$. Then, since the mapping, $\lambda' \mapsto p(\lambda' x^*)$, is real-valued and convex, it is continuous. Furthermore, it follows from the intermediate value theorem applied to the function $g : [0, \lambda] \to \mathbb{R}$, defined by,

$$g(\lambda') := p(\lambda' x^*) + \lambda' \quad \text{for all } \lambda' \in [0, \lambda],$$

that there exists a $0 < \mu < \lambda$ such that $g(\mu) = 0$, i.e., $p(\mu x^*) = -\mu$, since

$$g(0) = p(0x^*) = -\inf_{y \in Y} f(y) \leq -\varepsilon < -0 = 0 < g(\lambda).$$

Thus, $\mu(x^*, -1) = (\mu x^*, p(\mu x^*))$ and so $p(\mu x^*) = \sup((\mu x^*, -1), S^{-1}(C)) = -\mu < 0$.

Choose $(z, s) \in S^{-1}(C)$ such that $(\mu x^*, -1)(z, s) = \sup((\mu x^*, -1), S^{-1}(C)) = p(\mu x^*)$. Note that $z \in A$ and $s = f(z)$. We claim that $(x^*, -1)$ attains its maximum value over $T(S^{-1}(C))$ at $T(z, f(z)) = f(z)^{-1}(z, -1)$. Now,

$$\begin{aligned}(x^*, -1)(T(z, f(z))) &= f(z)^{-1}(x^*(z) + 1) = f(z)^{-1}(x^*(z) - [\mu^{-1} p(\mu x^*)]) \\ &= f(z)^{-1}(x^*(z) - [x^*(z) - \mu^{-1} f(z)]) = \mu^{-1}.\end{aligned}$$

On the other hand, if $(y, s) \in S^{-1}(C)$ then

$$\begin{aligned}(x^*, -1)(T(y, s)) &= s^{-1}(x^*(y) + 1) = s^{-1}(x^*(y) - [\mu^{-1} p(\mu x^*)]) \\ &\leq s^{-1}(x^*(y) - [x^*(y) - \mu^{-1} f(y)]) = s^{-1} f(y) \mu^{-1} \leq \mu^{-1} = (x^*, -1)(T(z, f(z)))\end{aligned}$$

since $f(y) \leq s$. This completes the proof. $\square$

Remark 1 *It might be interesting to note the following: If X is a Banach space, A and B are bounded, closed and convex sets such that every $x^* \in X^*$ with $\inf(x^*, A) < \sup(x^*, B)$ attains its infimum on A and its supremum on B, then both A and B are weakly compact. To see this, note that $C := co[\{0\} \cup \overline{B - A}]$ is a closed and bounded convex subset of X with the property that every continuous linear function attains it supremum over C. Let us also recall that the problem in Theorem 1 was first considered in $L^1(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is a probability space. In this setting there is a very elementary proof of James' theorem, see [3] (which is used within the proof of Lemma 1), since $L^1(\Omega, \mathcal{F}, P)$ is weakly compactly generated.*

References

[1] B. Cascales, J. Orihuela and A. Pérez, One-sided James' compactness theorem, *J. Math. Anal. Appl.* **445** (2017), 1267–1283.

[2] W. B. Moors, Weak compactness of sublevel sets, to appear in *Proc. Amer. Math. Soc.* (2 pages).

[3] W. B. Moors and S. J. White, An elementary proof of James' characterisation of weak compactness II, to appear in *Bull. Aust. Math. Soc.* (4 pages).

[4] J. Saint-Raymond, Weak compactness and variational characterisation of the convexity, *Mediterr. J. Math.* **10** (2013), 927–940.